ON THE VANISHING OF SELF EXTENSIONS OVER COHEN-MACAULAY LOCAL RINGS

TOKUJI ARAYA, OLGUR CELIKBAS, ARASH SADEGHI, AND RYO TAKAHASHI

ABSTRACT. The celebrated Auslander-Reiten Conjecture, on the vanishing of self extensions of a module, is one of the long-standing conjectures in ring theory. Although it is still open, there are several results in the literature that establish the conjecture over Gorenstein rings under certain conditions. The purpose of this article is to obtain extensions of such results over Cohen-Macaulay local rings that admit canonical modules. In particular, our main result recovers theorems of Araya, and Ono and Yoshino simultaneously.

1. INTRODUCTION

Throughout R denotes a commutative Noetherian local ring and $\operatorname{mod} R$ denotes the category of all finitely generated R-modules.

There are various conjectures from the representation theory of algebras that have been transplanted to Commutative Algebra. One of the most important such conjectures is the celebrated *Auslander-Reiten Conjecture* [3], which states that a finitely generated module M over an Artin algebra A satisfying $\operatorname{Ext}_A^i(M,M)=\operatorname{Ext}_A^i(M,A)=0$ for all $i\geq 1$ must be projective. This long-standing conjecture is closely related to other important conjectures such as the *Nakayama's Conjecture* [16] and the *Tachikawa Conjecture* [5, 19]. Although the Auslander-Reiten Conjecture was initially proposed over Artin algebras, it can be stated over arbitrary Noetherian rings; in local algebra, the conjecture is known as follows:

Conjecture 1.1. (Auslander and Reiten [3]) Let R be a local ring and let $M\in \operatorname{mod} R$. If $\operatorname{Ext}_R^i(M,M)=\operatorname{Ext}_R^i(M,R)=0$ for all $i\geq 1$, then M is free.

Recently there have been significant interest and hence some progress towards the Auslander-Reiten conjecture; see, for example, [8, 9, 10, 12, 13, 14]. A particular result worth recording on the Auslander-Reiten Conjecture is due to Huneke and Leuschke: the Auslander-Reiten Conjecture holds over Gorenstein normal domains; see [13, 1.3]. Another result in this direction, which is of interest to us, is due to Araya [1]. For a non-negative integer n and an R-module M, we set $X^n(R)=\{\mathfrak{p}\in \operatorname{Spec}(R) : \operatorname{height}(\mathfrak{p})\leq n\}$ and say M is *locally free on* $X^n(R)$ if $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for each prime ideal $\mathfrak{p}\in X^n(R)$.

Theorem 1.2. *(Araya* [1]*) Let R be a Gorenstein local ring of dimension $d\geq 2$ and let $M\in \operatorname{mod} R$. Then M is free provided that the following hold:*

(i) *M is locally free on $X^{d-1}(R)$.*
(ii) *M is maximal Cohen-Macaulay.*
(iii) $\operatorname{Ext}_R^{d-1}(M,M)=0$.

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Ono and Yoshino [17] relaxed the condition that M is free on $X^{d-1}(R)$ in Araya's theorem under the hypothesis that $\mathsf{Ext}^i_R(M,M)$ vanishes for $i=d-2$ and $i=d-1$. More precisely they proved:

Theorem 1.3. *(Ono and Yoshino* [17]*) Let R be a Gorenstein local ring of dimension $d \geq 3$ and let $M \in \mathsf{mod}\, R$ be a module. Then M is free provided that the following hold:*
- (i) *M is locally free on $X^{d-2}(R)$.*
- (ii) *M is maximal Cohen-Macaulay.*
- (iii) $\mathsf{Ext}^{d-2}_R(M,M) = \mathsf{Ext}^{d-1}_R(M,M) = 0$.

We set $(-)^* = \mathsf{Hom}_R(-,R)$, $(-)^\vee = \mathsf{Hom}(-,E)$, where E is the injective hull of the residue field of R, and $(-)^\dagger = \mathsf{Hom}_R(-,\omega)$ where ω is a canonical module of R. Recall that $M \in \mathsf{mod}\, R$ is said to satisfy (S_n) if $\mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \{n, \mathrm{height}_R(\mathfrak{p})\}$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$. Note, by convention, $\mathsf{depth}(0) = \infty$.

The main aim of this paper is to prove the following result.

Theorem 1.4. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω and let $M \in \mathsf{mod}\, R$. Assume n is an integer with $1 \leq n \leq d-1$. Then M is free provided that the following hold:*
- (i) $\mathsf{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ *for all* $\mathfrak{p} \in X^n(R)$.
- (ii) *M satisfies (S_2) and M^* is maximal Cohen-Macaulay.*
- (iii) $\mathsf{Ext}^i_R(M,(M^*)^\dagger) = 0$ *for all* $i = n, \dots, d-1$.

An immediate consequence of Theorem 1.4 is:

Corollary 1.5. *Let R be a Cohen-Macaulay normal local domain of dimension d with canonical module ω and let $M \in \mathsf{mod}\, R$. Then M is free provided that the following hold:*
- (i) *M satisfies (S_2) and M^* is maximal Cohen-Macaulay.*
- (ii) $\mathsf{Ext}^i_R(M,(M^*)^\dagger) = 0$ *for all* $i = 1, \dots, d-1$.

As maximal Cohen-Macaulay modules satisfy (S_2) over Gorenstein rings, we deduce from Theorem 1.4 that:

Corollary 1.6. *Let R be a Gorenstein local ring of dimension d and let $M \in \mathsf{mod}\, R$. Assume n is an integer with $1 \leq n \leq d-1$. Then M is free provided that the following hold:*
- (i) *M is locally free on $X^n(R)$.*
- (ii) *M is maximal Cohen-Macaulay.*
- (iii) $\mathsf{Ext}^i_R(M,M) = 0$ *for all* $i = n, \dots, d-1$.

Note that we recover theorems 1.2 and 1.3 from Corollary 1.6 by letting $n = d-1$ and $n = d-2$, respectively. Corollary 1.6 especially yields an extension of a result of Huneke and Leuschke [13] mentioned preceding Theorem 1.2. More precisely we obtain the following; see Corollary 2.7.

Corollary 1.7. *Let R be a d-dimensional Gorenstein local normal domain and let M be a maximal Cohen-Macaulay R-module. Then M is free if and only if* $\mathsf{Ext}^i_R(M,M) = 0$ *for all* $i = 1, \dots, d-1$.

Let us remark that Huneke and Leuschke [13, 3.1] obtains the conclusion of Corollary 1.7, when $\mathsf{Ext}^i_R(M,M)$ vanishes for all $i = 1, \dots, d$; see the discussion following Corollary 2.7. Let us also remark that one can in fact prove Theorem 1.3 and Corollary 1.7 by using the proof of Theorem 1.2 given by Araya; see [1]. Our main argument is more general than both of these results; it is quite short and works over Cohen-Macaulay rings that are not necessarily Gorenstein. Hence we will deduce Theorem 1.3 and Corollary 1.7 as immediate corollaries of Theorem 1.4 in the next section without making use of the proof of Theorem 1.2; see also Corollary 2.8.

2. MAIN RESULT AND COROLLARIES

In this section we give a proof of our main result, Theorem 1.4. Following our proof we state two corollaries, one of which extends the result of Huneke and Leuschke [13] mentioned preceding Theorem 1.2. We start with a few notations and preliminary results.

2.1. Let $M,N \in \operatorname{mod} R$. We denote by $\underline{\mathsf{Hom}}_R(M,N)$ the residue of $\mathsf{Hom}_R(M,N)$ by the R-submodule consisting of the R-module homomorphisms from M to N that factor through free modules.

It follows from the definition that M is free if and only if $\underline{\mathsf{Hom}}_R(M,M)=0$. We also remark that $\underline{\mathsf{Hom}}_R(M,N)\cong \mathsf{Tor}_1^R(\mathsf{Tr}M,N)$, where $\mathsf{Tr}M$ is the (Auslander) transpose of M; see [22, 3.9].

2.2. ([6, 1.4.1 and 1.4.19]) Let R be a Cohen-Macaulay local ring and let $M\in \operatorname{mod} R$. Assume $\mathsf{pd}_{R_\mathfrak{p}}(M_\mathfrak{p})<\infty$ for all $\mathfrak{p}\in X^1(R)$. Then M satisfies (S_2) if and only if M is reflexive.

We should point out that there are more than one definition for the (S_n) condition; the one we use in this paper is from [11]. In [6, 5.7.2] a module M is defined to satisfy (S_n) provided that $\mathsf{depth}_{R_p}(M_\mathfrak{p})\geq \min\{n,\dim_{R_p}(M_\mathfrak{p})\}$ for all $p\in \operatorname{Spec}(R)$. This is weaker than the version we use, and indeed, (2.2) does not hold according to this definition of (S_n): let R be a local ring of dimension at least two and let $M\in\operatorname{mod} R$ be a nonzero module of finite length. Then $\mathsf{depth}_{R_p}(M_\mathfrak{p})\geq \min\{n,\dim_{R_p}(M_\mathfrak{p})\}$ for all $p\in\operatorname{Spec}(R)$, i.e., M satisfies (S_n) for each integer $n\geq 0$ (if $\mathfrak{p}\notin \operatorname{Supp}_R(M)$, then $\mathsf{depth}_{R_p}(M_\mathfrak{p})=\infty$ and $\dim_{R_p}(M_\mathfrak{p})=-1$.) Note that $M_\mathfrak{p}=0$ for all $\mathfrak{p}\in X^1(R)$ but M is not reflexive since $\mathsf{Hom}_R(M,R)=0$.

2.3. ([21, 2.2]) Let R be a local ring and let $M,N\in\operatorname{mod} R$. If N is maximal Cohen-Macaulay and $\mathsf{pd}(M)<\infty$, then $\mathsf{Tor}_i^R(M,N)=0$ for all $i\geq 1$.

For our proof of Theorem 1.4, we will make use [7, 3.1], which is well-known over Artinian rings. Since we record an improved version of that result, we give a proof along with its statement.

2.4. ([7, 3.1]) Let R be a d-dimensional Cohen-Macaulay local ring with a canonical module ω and let $M,N\in\operatorname{mod} R$. If $\mathsf{pd}_{R_\mathfrak{p}}(M_\mathfrak{p})<\infty$ for all $\mathfrak{p}\in X^{d-1}(R)$ and N is maximal Cohen-Macaulay, then the following isomorphism holds for all $i\geq 1$:

$$\mathsf{Ext}_R^{d+i}(M,N^\dagger)\cong \mathsf{Ext}_R^d(\mathsf{Tor}_i^R(M,N),\omega).$$

To see this isomorphism, we note, by [18, 10.62], that there is a third quadrant spectral sequence:

$$\mathsf{Ext}_R^p(\mathsf{Tor}_q^R(M,N),\omega)\underset{p}{\Longrightarrow}\mathsf{Ext}_R^n(M,N^\dagger)$$

Let $p\in X^{d-1}(R)$. Then, since $\mathsf{pd}_{R_\mathfrak{p}}(M_\mathfrak{p})<\infty$ and N_p is maximal Cohen-Macaulay over R_p, we conclude that $\mathsf{Tor}_i^R(M,N)_p=0$ for all $i\geq 1$; see 2.3. Therefore $\mathsf{Tor}_q^R(M,N)$ has finite length for all $q>0$. Hence, unless $p=d$, $\mathsf{Ext}_R^p(\mathsf{Tor}_q^R(M,N),\omega)=0$. It follows that the spectral sequence considered collapses and hence gives the desired isomorphism.

We can now prove our main result.

A proof of Theorem 1.4. Note, it follows from our hypotheses and 2.2, that M is reflexive. Set $N = M^*$ and note $M \cong \Omega^2 \mathsf{Tr} N$. Therefore $\mathsf{pd}_{R_\mathfrak{p}}((\mathsf{Tr}N)_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in X^n(R)$.

Let $\mathfrak{p} \in X^n(R)$. Then, since $N_\mathfrak{p}$ is maximal Cohen-Macaulay, it follows from 2.3 that $\mathsf{Tor}_i^R(\mathsf{Tr}N, N)_\mathfrak{p} = 0$ for all $i \geq 1$. In particular $\mathsf{Tor}_1^R(\mathsf{Tr}N, N)_\mathfrak{p} = 0$, and this implies $N_\mathfrak{p}$ is free over $R_\mathfrak{p}$; see 2.1. Since M is reflexive, we conclude that $M_\mathfrak{p}$ is free. Consequently M is a reflexive module that is locally free on $X^n(R)$.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$. We proceed by induction on $\operatorname{height}_R(\mathfrak{p})$ and prove that $M_\mathfrak{p}$ is free over $R_\mathfrak{p}$. If $\operatorname{height}(\mathfrak{p}) \leq n$, then $M_\mathfrak{p}$ is free by the above argument. So we assume $\operatorname{height}(\mathfrak{p}) = t > n$. Localizing at $\mathfrak{p}$, we may assume $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with a canonical module ω, $\dim(R) = t > n \geq 1$, M is reflexive and locally free on $X^{t-1}(R)$, M^* is maximal Cohen-Macaulay and $\mathsf{Ext}_R^{t-1}(M, (M^*)^\dagger) = 0$.

Note that $N^* = M^{**} \cong M$ and $N^\dagger = (M^*)^\dagger$ is maximal Cohen-Macaulay. Moreover $\mathsf{Ext}_R^{t-1}(N^*, N^\dagger) = 0$ by the hypothesis. Hence the following isomorphisms hold:

$$
\begin{aligned}
0 = \mathsf{Ext}_R^{t-1}(N^*, N^\dagger)^\vee &\cong \mathsf{Ext}_R^{t+1}(\mathsf{Tr}N, N^\dagger)^\vee \\
&\cong \mathsf{Ext}_R^{t}(\mathsf{Tor}_1^R(\mathsf{Tr}N, N), \omega)^\vee \\
&\cong \Gamma_\mathfrak{m}(\mathsf{Tor}_1^R(\mathsf{Tr}N, N)) \\
&\cong \mathsf{Tor}_1^R(\mathsf{Tr}N, N) \\
&\cong \underline{\mathsf{Hom}}_R(N, N).
\end{aligned} \tag{1.4.1}
$$

The first isomorphism of (1.4.1) follows from the fact that $N^* \approx \Omega^2 \mathsf{Tr}N$, while the second and third ones follow from 2.4 and the Local Duality Theorem [6, 3.5.9], respectively. The fourth isomorphism is due to the fact that $\mathsf{Tor}_1^R(\mathsf{Tr}N, N)$ has finite length. The fifth isomorphism, and the freeness of N, follows from 2.1. As M is reflexive, we conclude that $M \cong N^*$ is free. $\square$

If R is a Gorenstein local ring and M is a maximal Cohen-Macaulay R-module, then M satisfies (S_2), M^* is maximal Cohen-Macaulay and $(M^*)^\dagger \cong M$. Hence, as an immediate consequence of Theorem 1.4, we obtain Corollary 1.6.

To obtain another corollary of Theorem 1.4, we need the next result.

Lemma 2.5. Let R be a Cohen-Macaulay local ring of dimension d with a canonical module ω and let $M \in \operatorname{mod} R$. If $\mathsf{Ext}_R^i(M, R) = 0$ for all $i = 1, \dots, d$, then $M \otimes_R \omega \cong (M^*)^\dagger$.

Proof. Consider a free resolution $F = (\cdots \to F_2 \to F_1 \to F_0 \to 0)$ of M. As $\mathsf{Ext}^i(M, R) = 0$ for all $i = 1, \dots, d$, applying $\mathsf{Hom}_R(-, R)$ to F, we obtain the following exact sequence: $0 \to M^* \to F_0 \to F_1 \to \cdots \to F_d \to F_{d+1}$. In particular $\mathsf{Tr}M$ is a d-th syzygy module (more precisely d-th syzygy of $\mathsf{Tr}\Omega^d M$), and hence is a maximal Cohen-Macaulay R-module.

Next consider the four term exact sequence from [2, 2.8]:

$$0 \to \mathsf{Ext}_R^1(\mathsf{Tr}M, \omega) \to M \otimes \omega \to \mathsf{Hom}_R(M^*, \omega) \to \mathsf{Ext}_R^2(\mathsf{Tr}M, \omega) \to 0.$$

Note that, since $\mathsf{Tr}M$ is maximal Cohen-Macaulay, we have $\mathsf{Ext}_R^i(\mathsf{Tr}M, \omega) = 0$ for all $i \geq 1$. Consequently $M \otimes_R \omega \cong \mathsf{Hom}_R(M^*, \omega) = M^{*\dagger}$. $\square$

Remark 2.6. Lemma 2.5 is not a new result; it was stated and used previously in the literature; see, for example, [8, 5.3] and [13, 1.4] (see also [5, the remark following 2.1 and the proof of 2.1].) In [13, 1.4] the reader is referred to [5, B4] for a proof of Lemma 2.5. Since [5, B4] contains a typo, we give here a proof for the convenience of the reader. (To be more precise, the second assertion of [5, B4] should be "$\mathsf{Tor}_i^R(L, M) = 0$ for all $i \in [1, m-n]$".)

Corollary 2.7. *Let R be a Cohen-Macaulay local ring of dimension d with a canonical module ω and let $M \in \operatorname{mod} R$. Assume n is an integer with $1 \leq n \leq d-1$. Then M is free provided that the following hold:*

(i) $\operatorname{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ *for all* $\mathfrak{p} \in X^n(R)$.

(ii) *M is reflexive and* $\operatorname{Ext}^i_R(M,R) = 0$ *for all* $i = 1, \dots, d$ *(e.g., M is totally reflexive.)*

(iii) $\operatorname{Ext}^i_R(M, M \otimes_R \omega) = 0$ *for all* $i = n, \dots, d-1$.

Proof. The vanishing of $\operatorname{Ext}^i_R(M,R)$ for all $i = 1, \dots, d$ forces M^* to be a dth syzygy module, i.e., a maximal Cohen-Macaulay module. Therefore the conclusion follows from Theorem 1.4 and 2.5. □

We point out that the case where R is Gorenstein and $n = 1$ of Corollary 2.7 yields Corollary 1.7 advertised in the introduction.

Corollary 2.7 allows us to generalize [17, 5.5], another result of Ono and Yoshino.

Corollary 2.8. *Let R be a Cohen-Macaulay local ring of dimension d with a canonical module ω and let $M \in \operatorname{mod} R$. Assume n is an integer with $1 \leq n \leq d-1$. Then M is free provided that the following hold:*

(i) *M is locally free on $X^{d-n}(R)$.*

(ii) *M is reflexive and* $\operatorname{Ext}^i_R(M,R) = 0$ *for all* $i = 1, \dots, d$.

(iii) $\operatorname{depth}_R(\operatorname{Ext}^{d-i}_R(M, M \otimes_R \omega)) \geq i$ *for all* $i = 1, \dots, n$.

Proof. Suppose M is not free. Then the nonfree locus $\operatorname{NF}(M)$ of M is not empty. Let $\operatorname{NF}(M) = V(I)$ for some ideal I of R with $\operatorname{height}_R(I) = t$. Then it follows from (i) that $t \geq d-n+1$. Setting $i = d-t+1$, we obtain from (iii) that

$$\operatorname{depth}_R(\operatorname{Ext}^{t-1}_R(M, M \otimes_R \omega)) \geq d-t+1. \tag{2.8.1}$$

On the other hand, since $X^{t-1}(R) \nsubseteq V(I)$, we see M is locally free on $X^{t-1}(R)$. This implies

$$\dim_R(\operatorname{Ext}^{t-1}_R(M, M \otimes_R \omega)) \leq d-t. \tag{2.8.2}$$

Consequently, by (2.8.1) and (2.8.2), we conclude $\operatorname{Ext}^{t-1}_R(M, M \otimes_R \omega) = 0$.

Let $\mathfrak{p} \in \operatorname{NF}(M)$ with $\operatorname{height}_R(\mathfrak{p}) = t$. Then $M_{\mathfrak{p}}$ is locally free on $X^{t-1}(R_{\mathfrak{p}})$, $M_{\mathfrak{p}}$ is reflexive over $R_{\mathfrak{p}}$, $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$ for all $i = 1, \dots, t$, and $\operatorname{Ext}^{t-1}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \omega_{\mathfrak{p}}) = 0$. Therefore it follows from Corollary 2.7 that $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$, which is a contradiction. So M is free. □

Huneke and Leuschke [13, 3.1] proved, when R is a d-dimensional complete Cohen-Macaulay local ring such that $R_{\mathfrak{p}}$ is a complete intersection for all $\mathfrak{p} \in X^1(R)$, and R is either Gorenstein or contains $\mathbb{Q}$, a maximal Cohen-Macaulay R-module M of constant rank is free provided $\operatorname{Ext}^i_R(M,R) = \operatorname{Ext}^i_R(M^*,R) = 0$ for all $i = 1, \dots, d$ and $\operatorname{Ext}^i_R(M,M) = 0$ for all $i = 1, \dots, \max\{2,d\}$. Consequently, when R is a Gorenstein normal domain of dimension $d \geq 2$ and M is a maximal Cohen-Macaulay R-module, the result of Huneke and Leuschke [13, 3.1] requires the vanishing of $\operatorname{Ext}^i_R(M,M) = 0$ for all $i = 1, \dots, d$ to conclude that M is free, whilst Corollary 1.7 requires the vanishing of $\operatorname{Ext}^i_R(M,M) = 0$ for all $i = 1, \dots, d-1$ for the same conclusion under the same setup.

One can make use of Corollary 1.7 and obtain the following result whose proof is similar to that of [13, 1.1]; see also [8, 5.12].

Corollary 2.9. *Let $R = S/(x_1, \dots, x_n)$, where S is a Gorenstein normal local domain of dimension $d+n$, and let $x_1, \dots, x_n$ be a regular sequence on S, with $n \geq 0$ and $d \geq 2$.*

If $M \in \operatorname{mod} R$ is a maximal Cohen-Macaulay module such that $\operatorname{Ext}_R^i(M,M) = 0$ for all $i = 1, \dots, d+n-1$, then M is free.

Proof. Corollary 1.7 settles the case where $n = 0$. Hence we assume $n \geq 1$. Then [4, 1.7] gives rise to a module $N \in \operatorname{mod} S$ such that $x = x_1, \dots, x_n$ is N-regular and $N/xN \cong M$. Note that N is maximal Cohen-Macaulay S-module. Moreover, by using [6, 3.1.16], we have $\operatorname{Ext}_S^{i+n}(M,N) = 0$ for all $i = 1, \dots, d+n-1$. Applying Nakayama's lemma to the short exact sequences

$$0 \to N/(x_1, \dots, x_{j-1})N \xrightarrow{x_j} N/(x_1, \dots, x_{j-1})N \to N/(x_1, \dots, x_j)N \to 0,$$

we obtain $\operatorname{Ext}_S^i(N,N) = 0$ for all $i = 1, \dots, d+n-1$. Now Corollary 1.7 shows that N is free over S, and hence M is free over R. □

As Huneke and Leuschke [13] studied the Auslander-Reiten Conjecture for modules that have constant rank, we finish this section with related examples which show that assumption (i) cannot be removed from Corollary 1.6.

Example 2.10. ([1, Example 11]) Let k be a field, $S = k[[x,y,z]]/(xy)$ and $N = S/xS$. Then N is a maximal Cohen-Macaulay S-module such that $\operatorname{Ext}_S^1(N,N) = 0$. However N is not locally free on $X^1(S)$.

Assumption (i) in Corollary 1.6 cannot be removed even if we assume M has rank:

Example 2.11. Let k be a field and $R = k[[x,y,z,u,v]]/(xy-uv)$. Then R is a four dimensional hypersurface domain. In particular any module in $\operatorname{mod} R$ has constant rank. Consider the following minimal free resolution:

$$F = (\cdots \xrightarrow{B} R^2 \xrightarrow{A} R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2 \to 0), \tag{2.10.1}$$

where $A = \begin{pmatrix} x & u \\ v & y \end{pmatrix}$ and $B = \begin{pmatrix} y & -u \\ -v & x \end{pmatrix}$. Set $M = \operatorname{coker} A$.

Applying $\operatorname{Hom}_R(-,M)$ to (2.10.1), we have

$$0 \to M^2 \xrightarrow{{}^tA} M^2 \xrightarrow{{}^tB} M^2 \xrightarrow{{}^tA} M^2 \xrightarrow{{}^tB} \cdots. \tag{2.10.2}$$

Since ${}^tB : M^2 \to M^2$ is injective, we conclude that $\operatorname{Ext}_R^{2i-1}(M,M) = 0$ for all $i \geq 1$.

Let $\mathfrak{p} := (x,y,u,v)$. Then $\mathfrak{p}$ is a prime ideal of R of height three. Localizing (2.10.1) at $\mathfrak{p}$, we get the following exact sequence:

$$\cdots \xrightarrow{B} R_\mathfrak{p}^2 \xrightarrow{A} R_\mathfrak{p}^2 \xrightarrow{B} R_\mathfrak{p}^2 \xrightarrow{A} R_\mathfrak{p}^2 \to M_\mathfrak{p} \to 0. \tag{2.10.3}$$

Since $x,y,u,v \in \mathfrak{p}R_\mathfrak{p}$, (2.10.3) is a minimal free resolution of $M_\mathfrak{p}$. In particular $M_\mathfrak{p}$ is not a free $R_\mathfrak{p}$-module, i.e., M is not locally free on $X^3(R)$.

Remark 2.12. Example 2.11 can also be shown as follows when $\frac{1}{2}, \sqrt{-1} \in S$.

By Knörrer's periodicity (see [20, Theorem B]), the stable categories of maximal Cohen-Macaulay modules over R and S are equivalent as triangulated categories, and M is actually the image of $N = S/xS$ by this functor. As $\operatorname{Ext}_S^1(N,N) = 0$, we have $\operatorname{Ext}_R^1(M,M) = 0$, and hence $\operatorname{Ext}_R^{2i-1}(M,M) = 0$ for all $i \geq 1$ since M is isomorphic to its second syzygy.

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TOKUJI ARAYA, DEPARTMENT OF APPLIED SCIENCE, FACULTY OF SCIENCE, OKAYAMA UNIVERSITY OF SCIENCE, RIDAICHO, KITAKU, OKAYAMA 700-0005, JAPAN.
E-mail address: araya@das.ous.ac.jp

OLGUR CELIKBAS, DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WV 26506-6310, U.S.A
E-mail address: olgur.celikbas@math.wvu.edu

ARASH SADEGHI, SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.O. BOX: 19395-5746, TEHRAN, IRAN
E-mail address: sadeghiarash61@gmail.com

RYO TAKAHASHI, GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FUROCHO, CHIKUSAKU, NAGOYA 464-8602, JAPAN
E-mail address: takahashi@math.nagoya-u.ac.jp
URL: http://www.math.nagoya-u.ac.jp/~takahashi/