RESOLVING SUBCATEGORIES WHOSE FINITELY PRESENTED MODULE CATEGORIES ARE ABELIAN

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Abstract. Let $\mathcal{X}$ be an additive full subcategory of an abelian category. It is a classical fact that if $\mathcal{X}$ is contravariantly finite, then the category $\operatorname{mod}\mathcal{X}$ of finitely presented right $\mathcal{X}$-modules is abelian. In this paper, we consider the question asking when the converse holds true for a resolving subcategory of the category of finitely generated modules over a commutative noetherian henselian local ring. We give both affirmative answers and negative answers to this question.

1. Introduction

Let $\mathcal{A}$ be an abelian category. Let $\mathcal{X}$ be an additive full subcategory of $\mathcal{A}$. It follows from Auslander's 1966 paper [4] that if $\mathcal{X}$ is contravariantly finite, then the category $\operatorname{mod}\mathcal{X}$ of finitely presented $\mathcal{X}$-modules is abelian. It is natural to ask whether the converse holds.

Question 1.1. When $\operatorname{mod}\mathcal{X}$ is abelian, is $\mathcal{X}$ contravariantly finite?

The main purpose of this paper is to study the above Question 1.1 for a resolving subcategory $\mathcal{X}$ of the abelian category $\mathcal{A} = \operatorname{mod} R$ of finitely generated modules over a commutative noetherian ring R. In what follows, we shall explain our main results. For simplicity, from here to the end of this section, we assume that R is a complete local ring with residue field k.

The theorem below gives affirmative answers to Question 1.1. It is included in Corollaries 3.7, 3.10(1).

Theorem 1.2. *Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$ such that $\operatorname{mod}\mathcal{X}$ is an abelian category. Then $\mathcal{X}$ is contravariantly finite if one of the following four conditions is satisfied.*

(1) *The ring R has (Krull) dimension at most one.*
(2) *The ring R is Cohen–Macaulay, and every R-module in $\mathcal{X}$ is maximal Cohen–Macaulay.*
(3) *Every R-module in $\mathcal{X}$ is Gorenstein projective.*
(4) *There is an R-module outside $\mathcal{X}$ that admits a right $\mathcal{X}$-approximation, and one of the following holds.*
    (i) *R is AB.* (ii) *$\mathcal{X}$ contains some syzygy of k.* (iii) *$\mathcal{X}$ is closed under cosyzygies.*

Thus Question 1.1 has an affirmative answer in each of the above four cases.

Here, the notion of a *Gorenstein projective* module has been introduced by Enochs and Jenda [15], which is the same as a totally reflexive module in the sense of Avramov and Martsinkovsky [9], and a module of Gorenstein dimension at most zero in the sense of Auslander and Bridger [5]. The notion of an *AB* ring has been introduced by Huneke and Jorgensen [18], which is a Gorenstein local ring satisfying a certain condition on vanishing of Ext modules. A typical example of an AB ring is a local complete intersection.

Theorem 1.2 would lead us to expect that Question 1.1 always has an affirmative answer, but we shall observe in Corollary 3.13 that it is not true.

Theorem 1.3. *Suppose that R has dimension at least two. Then there exists a proper resolving subcategory $\mathcal{X}$ of $\operatorname{mod} R$ which is closed under subobjects and provides only trivial right approximations. In particular, $\mathcal{X}$ is not contravariantly finite but $\operatorname{mod}\mathcal{X}$ is an abelian category. Thus such an $\mathcal{X}$ gives a negative answer to Question 1.1.*

Finally, we focus in Theorem 4.2 on the full subcategory $\mathsf{GP}(R)$ of $\operatorname{mod} R$ consisting of Gorenstein projective R-modules to relate the abelianity of $\operatorname{mod}\mathsf{GP}(R)$ with the Gorenstein property of the ring R.

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Theorem 1.4. *The following four conditions are equivalent.*

(1) *The ring R is either Gorenstein or G-regular.*
(2) *The category $\mathsf{mod}\,\mathsf{GP}(R)$ is abelian.*
(3) *The subcategory $\mathsf{GP}(R)$ of $\mathsf{mod}\,R$ is contravariantly finite.*
(4) *The evaluation functor $\mathsf{mod}\,\mathsf{GP}(R) \to \mathsf{mod}\,R$ has a right adjoint.*

Here, following [21], we say that R is *G-regular* if every finitely generated Gorenstein projective R-module is projective (hence free). The evaluation functor $\mathsf{mod}\,\mathsf{GP}(R) \to \mathsf{mod}\,R$ means the functor defined by $F \mapsto F(R)$. Note that the abelianity of $\mathsf{mod}\,\mathsf{GP}(R)$ depends only on the structure of $\mathsf{GP}(R)$ as an additive category. Thus Theorem 1.4 in particular says that the Gorensteinness of the ring R is characterized only by the structure of $\mathsf{GP}(R)$ as an additive category in the case where $\mathsf{GP}(R) \neq \mathsf{add}\,R$.

This paper is organized as follows. Section 2 states our convention, basic notions and their basic properties for later use. Section 3 is the main section of this paper, where the proofs of Theorems 1.2 and 1.3 are given. Section 4 is to state applications and more questions, where Theorem 1.4 is proved.

2. Basic definitions and properties

This section is devoted to collecting the background materials of this paper. To be precise, we state in this section the definitions of our basic notions and several known properties of them, which are used in later sections. We begin with our convention.

Convention 2.1. Throughout this paper, we assume the following. All rings are commutative noetherian rings with identity, all modules are finitely generated, and all subcategories are strictly full. We let R be a (commutative noetherian) ring. We denote by $\mathsf{mod}\,R$ the category of (finitely generated) R-modules, and by $\mathsf{CM}(R)$ the (full) subcategory of $\mathsf{mod}\,R$ consisting of maximal Cohen–Macaulay R-modules. For an additive category $\mathcal{E}$, we identify each object $E \in \mathcal{E}$ with the subcategory of $\mathcal{E}$ consisting only of E. We may omit subscripts and superscripts unless there is a danger of confusion.

This paper deals with a lot of closedness properties of subcategories. We state the precise definitions.

Definition 2.2. Let $\mathcal{E}$ be an additive category, and let $\mathcal{X}$ be a subcategory of $\mathcal{E}$. We say that $\mathcal{X}$ is:

(1) *closed under finite direct sums* provided that for any $X_1, \dots, X_n \in \mathcal{X}$ one has $X_1 \oplus \cdots \oplus X_n \in \mathcal{X}$;
(2) *closed under direct summands* provided that for any $A_1, \dots, A_n \in \mathcal{E}$ with $A_1 \oplus \cdots \oplus A_n \in \mathcal{X}$ one has $A_1, \dots, A_n \in \mathcal{X}$.

We denote by $\mathsf{add}_{\mathcal{E}}\,\mathcal{X}$ the *additive closure* of $\mathcal{X}$, that is, the smallest subcategory of $\mathcal{E}$ that contains $\mathcal{X}$ and is closed under finite direct sums and direct summands.

Definition 2.3. Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. We say that $\mathcal{X}$ is:

(1) *closed under subobjects* (resp. *closed under quotient objects*) provided that for every exact sequence $0 \to A \to B$ (resp. $0 \leftarrow A \leftarrow B$) in $\mathcal{A}$ with $B \in \mathcal{X}$ one has $A \in \mathcal{X}$;
(2) *closed under kernels* (resp. *closed under cokernels*) provided that for every exact sequence $0 \to A \to B \to C$ (resp. $0 \leftarrow A \leftarrow B \leftarrow C$) in $\mathcal{A}$ with $B, C \in \mathcal{X}$ one has $A \in \mathcal{X}$;
(3) *closed under kernels of epimorphisms* (resp. *closed under cokernels of monomorphisms*) provided that for every exact sequence $0 \to A \to B \to C \to 0$ (resp. $0 \leftarrow A \leftarrow B \leftarrow C \leftarrow 0$) in $\mathcal{A}$ with $B, C \in \mathcal{X}$ one has $A \in \mathcal{X}$;
(4) *closed under extensions* provided that for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with $A, C \in \mathcal{X}$ one has $B \in \mathcal{X}$. Clearly, when this is the case, $\mathcal{X}$ is closed under finite direct sums.

Remark 2.4. Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. Consider the conditions that $\mathcal{X}$ is closed under

(1) subobjects, (2) kernels, (3) kernels of epimorphisms,
(4) quotient objects, (5) cokernels, (6) cokernels of monomorphisms, and (7) direct summands.

Then one has that (1) $\Rightarrow$ (2) $\Rightarrow$ (3), that (4) $\Rightarrow$ (5) $\Rightarrow$ (6), and that (2) $\Rightarrow$ (7) $\Leftarrow$ (5). Indeed, the only nontrivial implications are the last two. Suppose that (2) or (5) holds. Splicing the split exact sequences $0 \to A \to A \oplus B \to B \to 0$ and $0 \to B \to A \oplus B \to A \to 0$ with $A, B \in \mathcal{A}$, we get an exact sequence $0 \to A \to A \oplus B \to A \oplus B \to A \to 0$. This shows that if $A \oplus B \in \mathcal{X}$, then $A \in \mathcal{X}$. Thus (7) follows.

Next we recall the definitions of a syzygy and a resolving subcategory, the latter of which has been introduced by Auslander and Bridger [5].

Definition 2.5. Let $\mathcal{A}$ be an abelian category with enough projective objects.

(1) We denote by $\mathsf{proj}\,\mathcal{A}$ the subcategory of $\mathcal{A}$ consisting of projective objects.

(2) Let M be an object of $\mathcal{A}$. For an integer $n > 0$, the *nth syzygy* of M is by definition an object N of $\mathcal{A}$ that appears in an exact sequence $0 \to N \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0$ in $\mathcal{A}$ with $P_i \in \mathsf{proj}\,\mathcal{A}$ for all $0 \leqslant i \leqslant n-1$, and it is denoted by $\Omega^n_{\mathcal{A}} M$. We set $\Omega^0_{\mathcal{A}} M = M$. For each $n \geqslant 0$ we denote by $\Omega^n \mathcal{A}$ the subcategory of $\mathcal{A}$ consisting of nth syzygies. The object $\Omega^n_{\mathcal{A}} M$ is uniquely determined up to projective summands, and any projective object is an nth syzygy for all $n \geqslant 0$, i.e., $\mathsf{proj}\,\mathcal{A} \subseteq \Omega^n \mathcal{A}$.

(3) A subcategory $\mathcal{X}$ of $\mathcal{A}$ is said to be *closed under syzygies* if $\Omega_{\mathcal{A}} X \in \mathcal{X}$ for all $X \in \mathcal{X}$, that is to say, if for every exact sequence $0 \to Y \to P \to X \to 0$ in $\mathcal{A}$ with $P \in \mathsf{proj}\,\mathcal{A}$ and $X \in \mathcal{X}$ one has $Y \in \mathcal{X}$.

(4) A subcategory $\mathcal{X}$ of $\mathcal{A}$ is called *resolving* if $\mathcal{X}$ contains $\mathsf{proj}\,\mathcal{A}$ and is closed under direct summands, extensions and kernels of epimorphisms. Here, being closed under kernels of epimorphisms can be replaced with being closed under syzygies, since an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ gives rise to an exact sequence $0 \to \Omega C \to A \oplus P \to B \to 0$ in $\mathcal{A}$ with $P \in \mathsf{proj}\,\mathcal{A}$.

We recall the definitions of a dominant subcategory, a semidualizing module and a Gorenstein projective module over a commutative noetherian ring.

Definition 2.6. (1) A subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$ is called *dominant* if for every prime ideal $\mathfrak{p}$ of R there exists an integer $n \geqslant 0$ such that $\Omega^n_{R_\mathfrak{p}} \kappa(\mathfrak{p})$ belongs to $\mathsf{add}_{\mathsf{mod}\,R_\mathfrak{p}} \mathcal{X}_\mathfrak{p}$. Here, $\kappa(\mathfrak{p})$ denotes the residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ of $R_\mathfrak{p}$, and $\mathcal{X}_\mathfrak{p}$ stands for the subcategory of $\mathsf{mod}\,R_\mathfrak{p}$ consisting of $R_\mathfrak{p}$-modules of the form $X_\mathfrak{p}$ with $X \in \mathcal{X}$. Any subcategory of $\mathsf{mod}\,R$ containing $\Omega^n(\mathsf{mod}\,R)$ for some $n \geqslant 0$ is dominant.

(2) An R-module C is called *semidualizing* if the natural map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^{>0}_R(C, C) = 0$. The R-module R is a typical example of a semidualizing R-module. If R is a Cohen–Macaulay local ring with a canonical module ω, then ω is a semidualizing R-module.

(3) Let C be a semidualizing R-module, and set $(-)^\dagger = \mathrm{Hom}_R(-, C)$. An R-module is called *Gorenstein C-projective* (or *totally C-reflexive*) if the natural map $M \to M^{\dagger\dagger}$ is an isomorphism and $\mathrm{Ext}^{>0}_R(M \oplus M^\dagger, C) = 0$. We denote by $\mathsf{GP}(C)$ the subcategory of $\mathsf{mod}\,R$ consisting of Gorenstein C-projective R-modules. Gorenstein R-projective R-modules are simply called *Gorenstein projective* R-modules. If R is a Cohen–Macaulay local ring with a canonical module ω, then the Gorenstein ω-projective R-modules are precisely the maximal Cohen–Macaulay R-modules, that is to say, $\mathsf{GP}(\omega) = \mathsf{CM}(R)$.

There are indeed a lot of examples of a resolving subcategory. We present here some of them, which appear later. Also, we mention that dominance can be interpreted quite simply in some cases.

Example 2.7. (1) If R is a Cohen–Macaulay local ring, then $\mathsf{CM}(R)$ is a resolving subcategory of $\mathsf{mod}\,R$.

(2) For a semidualizing R-module C the subcategory $\mathsf{GP}(C)$ of $\mathsf{mod}\,R$ is resolving by [3, Theorem 2.1].

(3) Let R be a local ring. Denote by $\mathsf{mod}_0\,R$ the subcategory of $\mathsf{mod}\,R$ consisting of R-modules which are locally free on the punctured spectrum of R. Then $\mathsf{mod}_0\,R$ is a resolving subcategory of $\mathsf{mod}\,R$.

(4) Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$. When R is Cohen–Macaulay, $\mathcal{X}$ is dominant if and only if $\mathcal{X}$ contains $\mathsf{CM}(R)$. When $d = \dim R < \infty$, the dominance of $\mathcal{X}$ is equivalent to saying that $\mathcal{X}$ contains $\Omega^d(\mathsf{mod}\,R)$. These statements are none other than [23, Corollary 1.2].

Now we recall the definitions of a right approximation and a contravariantly finite subcategory, which are introduced by Auslander and Smalø [8].

Definition 2.8. Let $\mathcal{E}$ be an additive category, and let $\mathcal{X}$ be a subcategory of $\mathcal{E}$.

(1) A morphism $f : X \to E$ (in $\mathcal{E}$) with $X \in \mathcal{X}$ is called a *right $\mathcal{X}$-approximation* (of E) if for every morphism $f' : X' \to E$ with $X' \in \mathcal{X}$ there is a morphism $g : X' \to X$ such that $f' = fg$. Note that for each $M \in \mathcal{X}$ the identity morphism of M is a right $\mathcal{X}$-approximation of M. We denote by $\mathsf{rap}_\mathcal{E}\,\mathcal{X}$ the subcategory of $\mathcal{E}$ consisting of objects admitting right $\mathcal{X}$-approximations. There are inclusions

$$\mathcal{X} \subseteq \mathsf{rap}\,\mathcal{X} \subseteq \mathcal{E}. \tag{2.8.1}$$

(2) We say that $\mathcal{X}$ is *contravariantly finite* if every object of $\mathcal{E}$ admits a right $\mathcal{X}$-approximation, that is to say, if the equality $\mathsf{rap}\,\mathcal{X} = \mathcal{E}$ holds, which is the equality of the second inclusion in (2.8.1).

(3) A *left $\mathcal{X}$-approximation* and a *covariantly finite* subcategory are defined dually.

We present two examples of a contravariantly finite subcategory.

Example 2.9. (1) The additive closure $\mathsf{add}\,X$ of an R-module X is a contravariantly finite subcategory of $\mathsf{mod}\,R$. Indeed, for an R-module M, choose a system of generators $f_1, \dots, f_n$ of the R-module $\mathrm{Hom}_R(X, M)$. Then it is easy to see that the map $(f_1, \dots, f_n) : X^{\oplus n} \to M$ is a right $\mathcal{X}$-approximation of M. A dual argument shows that $\mathsf{add}\,X$ is also a covariantly finite subcategory of $\mathsf{mod}\,R$.

(2) If R is a Cohen–Macaulay local ring with a canonical module, then $\mathsf{CM}(R)$ is a contravariantly finite subcategory of $\mathsf{mod}\,R$. This is a consequence of [6, Theorem 1.1].

The following easy lemma becomes necessary once in the next section.

Lemma 2.10. *Let $\mathcal{E}$ be an additive category. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathcal{E}$. If there are inclusions $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathsf{rap}\,\mathcal{X}$, then there is an inclusion $\mathsf{rap}\,\mathcal{Y} \subseteq \mathsf{rap}\,\mathcal{X}$.*

Proof. Let $f : Y \to E$ be a right $\mathcal{Y}$-approximation of an object $E \in \mathcal{E}$. Since $Y \in \mathcal{Y} \subseteq \mathsf{rap}\,\mathcal{X}$, there exists a right $\mathcal{X}$-approximation $g : X \to Y$. We claim that the composition $fg : X \to E$ is a right $\mathcal{X}$-approximation. Indeed, take a homomorphism $a : X' \to E$ with $X' \in \mathcal{X}$. As f is a right $\mathcal{Y}$-approximation and $X' \in \mathcal{X} \subseteq \mathcal{Y}$, there is a homomorphism $b : X' \to Y$ such that $a = fb$. As g is a right $\mathcal{X}$-approximation, there exists a homomorphism $c : X' \to X$ such that $b = gc$. The equality $a = (fg)c$ shows the claim. ■

For a subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$ we denote by $\mathcal{X}^{\perp}$ the subcategory of $\mathsf{mod}\,R$ consisting of R-modules M such that $\mathrm{Ext}_R^{>0}(X, M) = 0$ for all $X \in \mathcal{X}$. It is straightforward that $\mathcal{X}^{\perp}$ is closed under extensions and cokernels of monomorphisms. The following lemma is a fundamental tool throughout the paper, and this is why we need henselianity to obtain our main results.

Lemma 2.11. *Let R be a henselian local ring. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$. Let M be an R-module. Then M possesses a right $\mathcal{X}$-approximation if and only if there exists an exact sequence $0 \to Y \to X \xrightarrow{f} M \to 0$ of R-modules such that $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$.*

Proof. The "only if" part is shown in [22, Lemma 3.8]. To show the "if" part, let $X' \in \mathcal{X}$. The induced sequence $\mathrm{Hom}_R(X', X) \xrightarrow{g} \mathrm{Hom}_R(X', M) \to \mathrm{Ext}_R^1(X', Y)$ is exact, and $\mathrm{Ext}_R^1(X', Y) = 0$ as $X' \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$. Therefore the map g is surjective, which means that the map f is a right $\mathcal{X}$-approximation. ■

Now we recall the definitions of a module over an additive category, and its being finitely generated and finitely presented. These notions have been introduced by Auslander [4].

Definition 2.12. Let $\mathcal{E}$ be an additive category.

(1) We denote by $\mathsf{Mod}\,\mathcal{E}$ the *functor category* of $\mathcal{E}$; recall that the objects of $\mathsf{Mod}\,\mathcal{E}$ are additive contravariant functors from $\mathcal{E}$ to the category of abelian groups, and the morphisms of $\mathsf{Mod}\,\mathcal{E}$ are natural transformations. Note that $\mathsf{Mod}\,\mathcal{E}$ is an abelian category. An object and a morphism of $\mathsf{Mod}\,\mathcal{E}$ are called a *(right) $\mathcal{E}$-module* and an *$\mathcal{E}$-homomorphism*, respectively.

(2) An $\mathcal{E}$-module F is said to be *finitely generated* if there exists an exact sequence $\mathrm{Hom}_{\mathcal{E}}(-, E_0) \to F \to 0$ of $\mathcal{E}$-modules with $E_0 \in \mathcal{E}$. We say that F is *finitely presented* if there exists an exact sequence

(2.12.1) $$\mathrm{Hom}_{\mathcal{E}}(-, E_1) \to \mathrm{Hom}_{\mathcal{E}}(-, E_0) \to F \to 0$$

of $\mathcal{E}$-modules with $E_0, E_1 \in \mathcal{E}$. We call an exact sequence of the form (2.12.1) a *finite presentation* of F. The subcategory of $\mathsf{Mod}\,\mathcal{E}$ consisting of finitely presented $\mathcal{E}$-modules is denoted by $\mathsf{mod}\,\mathcal{E}$. This is called the *Auslander category* of $\mathcal{E}$ in [24, Chapter 4]. However, nowadays, this name is often used to mean a certain different category; see [11, Chapter 3] for instance. Thus, in this paper, we call $\mathsf{mod}\,\mathcal{E}$ the *finitely presented module category* of $\mathcal{E}$ so as not to confuse the reader.

(3) Let $f : X \to Y$ be a morphism in $\mathcal{E}$. A morphism $g : K \to X$ is called a *pseudo-kernel* of f provided that the induced sequence $\mathrm{Hom}_{\mathcal{E}}(-, K) \to \mathrm{Hom}_{\mathcal{E}}(-, X) \to \mathrm{Hom}_{\mathcal{E}}(-, Y)$ of $\mathcal{E}$-homomorphisms is exact. We say that $\mathcal{E}$ *has pseudo-kernels* if every morphism in $\mathcal{E}$ admits a pseudo-kernel.

The existence of right approximations is interpreted in terms of finite generation in the functor category.

Lemma 2.13. *Let $\mathcal{E}$ be an additive category. Let $\mathcal{X}$ be an additive subcategory of $\mathcal{E}$. An object $E \in \mathcal{E}$ admits a right $\mathcal{X}$-approximation if and only if the functor $\mathrm{Hom}_{\mathcal{E}}(-, E)|_{\mathcal{X}}$ is a finitely generated $\mathcal{X}$-module.*

Proof. If $f : X \to E$ is a right $\mathcal{X}$-approximation, then $\mathrm{Hom}_{\mathcal{E}}(-, f)|_{\mathcal{X}}$ is an epimorphism in $\mathsf{Mod}\,\mathcal{X}$. If $\phi : \mathrm{Hom}_{\mathcal{E}}(-, Y)|_{\mathcal{X}} \to \mathrm{Hom}_{\mathcal{E}}(-, E)|_{\mathcal{X}}$ is a surjective $\mathcal{X}$-homomorphism with $Y \in \mathcal{X}$, then Yoneda's lemma gives a morphism $g : Y \to E$ in $\mathcal{E}$ with $\phi = \mathrm{Hom}_{\mathcal{E}}(-, g)|_{\mathcal{X}}$, and g is seen to be a right $\mathcal{X}$-approximation. ■

Remark 2.14. Let $\mathcal{E}$ be an additive category, and let $\mathcal{X}$ be an additive subcategory of $\mathcal{E}$. By Lemma 2.13 the contravariant finiteness of $\mathcal{X}$ means that $\mathrm{Hom}_R(-,E)|_{\mathcal{X}}$ is finitely generated for all $E \in \mathcal{E}$. Thus we may call $\mathcal{X}$ *contravariantly infinite* if the equality of the first inclusion in (2.8.1) holds, that is to say, $\mathcal{X} = \mathsf{rap}\,\mathcal{X}$, because it means that $\mathrm{Hom}_R(-,E)|_{\mathcal{X}}$ is not finitely generated except the trivial case where $E \in \mathcal{X}$. In this paper, we shall consider both contravariant finiteness and contravariant infiniteness. To make it simple and avoid confusion, we often say that $\mathsf{rap}\,\mathcal{X} = \mathcal{E}$ (resp. $\mathcal{X} = \mathsf{rap}\,\mathcal{X}$) rather than that $\mathcal{X}$ is contravariantly finite (resp. $\mathcal{X}$ is contravariantly infinite).

The following lemma yields a criterion for the finitely presented module category to be abelian.

Lemma 2.15. *Let $\mathcal{E}$ be an additive category. Then the following assertions hold true.*
(1) *As a subcategory of $\mathsf{Mod}\,\mathcal{E}$, the category $\mathsf{mod}\,\mathcal{E}$ is closed under cokernels and extensions.*
(2) *The category $\mathsf{mod}\,\mathcal{E}$ is abelian if and only if $\mathcal{E}$ has pseudo-kernels.*

Proof. Let $\mathcal{A} = \mathsf{Mod}\,\mathcal{E}$. Let $\mathcal{P}$ be the subcategory of $\mathcal{A}$ consisting of objects having the form $\mathrm{Hom}_{\mathcal{E}}(-,E)$ with $E \in \mathcal{E}$. Using Yoneda's lemma, we get $\mathcal{P} \subseteq \mathsf{proj}\,\mathcal{A}$. Apply [4, Proposition 2.1(a)(b)] to $\mathcal{A}$ and $\mathcal{P}$. ■

The result below gives sufficient conditions for the abelianity of the finitely presented module category.

Proposition 2.16. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{X}$ be an additive subcategory of $\mathcal{A}$ which is either closed under kernels or contravariantly finite. Then $\mathsf{mod}\,\mathcal{X}$ is an abelian category.*

Proof. According to Lemma 2.15(2), it is enough to prove that each morphism $f : X \to X'$ in $\mathcal{X}$ has a pseudo-kernel. Take an exact sequence $0 \to K \xrightarrow{g} X \xrightarrow{f} X'$ in $\mathcal{A}$. If $\mathcal{X}$ is closed under kernels, then K belongs to $\mathcal{X}$, and the induced exact sequence $0 \to \mathrm{Hom}_{\mathcal{A}}(-,K)|_{\mathcal{X}} \to \mathrm{Hom}_{\mathcal{A}}(-,X)|_{\mathcal{X}} \to \mathrm{Hom}_{\mathcal{A}}(-,X')|_{\mathcal{X}}$ implies that $g : K \to X$ is a pseudo-kernel of f. If $\mathcal{X}$ is contravariantly finite, then there is a right $\mathcal{X}$-approximation $h : X'' \to K$, and the induced exact sequence $\mathrm{Hom}_{\mathcal{A}}(-,X'')|_{\mathcal{X}} \to \mathrm{Hom}_{\mathcal{A}}(-,X)|_{\mathcal{X}} \to \mathrm{Hom}_{\mathcal{A}}(-,X')|_{\mathcal{X}}$ implies that the composition $gh : X'' \to X$ is a pseudo-kernel of f. ■

Next we recall the definitions of the transpose and cosyzygy of a module over the ring R.

Definition 2.17. Let M be an R-module.
(1) Set $(-)^* = \mathrm{Hom}_R(-,R)$. Take an exact sequence $P_1 \xrightarrow{f} P_0 \to M \to 0$ of R-modules with P_0 and P_1 projective. We denote by $\mathrm{Tr}_R\,M$ the cokernel of the map $f^* : P_0^* \to P_1^*$, and call it the *(Auslander) transpose* of M. This is uniquely determined up to projective summands. The Gorenstein projectivity of M is equivalent to the vanishing $\mathrm{Ext}_R^{>0}(M \oplus \mathrm{Tr}\,M, R) = 0$. We refer the reader to [5] for details.
(2) The *(first) cosyzygy* of M is defined as the cokernel of a left $(\mathsf{add}\,R)$-approximation of M (one exists as $\mathsf{add}\,R$ is covariantly finite by Example 2.9(1)) and denoted by $\Omega^{-1}M$. This is uniquely determined up to projective summands. We say that a subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$ is *closed under cosyzygies* provided that $\Omega^{-1}X \in \mathcal{X}$ for all $X \in \mathcal{X}$. There is an isomorphism $\Omega^{-1}M \cong \mathrm{Tr}\,\Omega\,\mathrm{Tr}\,M$ of R-modules (up to projective summands) for every R-module M; see [19, Lemma 4.1] for instance.

We close the section by reminding the reader of a well-known result, which is used several times in this paper. This is a direct consequence of [10, Theorem 3.1.17, Corollary 9.6.2 and Remarks 9.6.4(a)].

Lemma 2.18. *Let R be a local ring. Let n be a nonnegative integer. Suppose that there exists a nonzero R-module M such that $\mathrm{id}_R\,M \leqslant n$. Then R is a Cohen–Macaulay ring with $\dim R \leqslant n$.*

3. Affirmative and negative answers to Question 1.1

In this section we provide several sufficient conditions for Question 1.1 to be affirmative, and present some cases where Question 1.1 is negative. Throughout this section, we fix the following notation.

Notation 3.1. Let $(R, \mathfrak{m}, k)$ be a henselian local ring. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$. Let $\mathcal{C}$ be the subcategory of $\mathsf{mod}\,R$ consisting of modules C such that $\mathrm{Hom}_R(-,C)|_{\mathcal{X}} \in \mathsf{mod}\,\mathcal{X}$. Let $\mathcal{B}$ be the smallest subcategory of $\mathsf{mod}\,R$ which contains $\mathcal{C}$ and is closed under direct summands and extensions.

We make a list of properties of $\mathcal{X}$, $\mathcal{C}$, $\mathcal{B}$ and $\mathsf{rap}\,\mathcal{X}$, some of which are frequently used later.

Proposition 3.2. (1) *The subcategory $\mathsf{rap}\,\mathcal{X}$ of $\mathsf{mod}\,R$ is closed under direct summands and extensions.*
(2) *There are inclusions of subcategories: $\mathcal{X} \subseteq \mathcal{C} \subseteq \mathcal{B} \subseteq \mathsf{rap}\,\mathcal{B} \subseteq \mathsf{rap}\,\mathcal{X} \subseteq \mathsf{mod}\,R$.*

(3) *The subcategory $\mathcal{C}$ of* $\mathsf{mod}\,R$ *is closed under finite direct sums.*
(4) *Suppose that* $\mathsf{mod}\,\mathcal{X}$ *is an abelian category. Then the following statements hold true.*
  (a) *The subcategory $\mathcal{C}$ of* $\mathsf{mod}\,R$ *is closed under kernels. Therefore, $\mathcal{C}$ is closed under direct summands and syzygies, and contains* $\Omega^2(\mathsf{mod}\,R)$. *In particular, $\mathcal{C}$ is dominant, and so are* $\mathcal{B}, \mathsf{rap}\,\mathcal{B}, \mathsf{rap}\,\mathcal{X}$.
  (b) *Let M be an R-module. Let C be an R-module belonging to $\mathcal{C}$. Then* $\mathrm{Hom}_R(M,C)$ *belongs to $\mathcal{C}$.*
  (c) *For any R-module C that belongs to $\mathcal{C}$, the R-module* $\mathrm{Ext}^1_R(\mathrm{Tr}\,C,R)$ *also belongs to $\mathcal{C}$.*
  (d) *The subcategory $\mathcal{B}$ of* $\mathsf{mod}\,R$ *is resolving.*
  (e) *If the equality* $\mathcal{X} = \mathsf{rap}\,\mathcal{X}$ *holds, then the subcategory $\mathcal{X}$ of* $\mathsf{mod}\,R$ *is closed under kernels.*
(5) *If* $\mathsf{rap}\,\mathcal{X}$ *is closed under kernels of epimorphisms (or equivalently, if* $\mathsf{rap}\,\mathcal{X}$ *is resolving), then it holds that* $\mathcal{C} = \mathsf{rap}\,\mathcal{X}$. *In particular, the equality* $\mathsf{rap}\,\mathcal{X} = \mathsf{mod}\,R$ *implies the equality* $\mathcal{C} = \mathsf{mod}\,R$.
(6) *If there is an equality* $\mathcal{X} = \mathsf{rap}\,\mathcal{X}$, *then one has the inclusion* $\mathcal{X}^\perp \cap \Omega(\mathsf{mod}\,R) \subseteq \mathsf{add}\,R$.

Proof. (1) If $\binom{f}{g} : X \to M \oplus N$ is a right $\mathcal{X}$-approximation, then it can directly be verified that f, g are right $\mathcal{X}$-approximations. Hence $\mathsf{rap}\,\mathcal{X}$ is closed under direct summands. We observe from Lemma 2.11 and the proof of [7, Proposition 3.6] that $\mathsf{rap}\,\mathcal{X}$ is closed under extensions.

(2) The only nontrivial inclusion is $\mathsf{rap}\,\mathcal{B} \subseteq \mathsf{rap}\,\mathcal{X}$. By Lemma 2.10, we have only to show $\mathcal{B} \subseteq \mathsf{rap}\,\mathcal{X}$. In view of (1), it suffices to show $\mathcal{C} \subseteq \mathsf{rap}\,\mathcal{X}$. This is a direct consequence of Lemma 2.13.

(3) Let $C, C' \in \mathcal{C}$. Then $\mathrm{Hom}_R(-,C)|_\mathcal{X}$ and $\mathrm{Hom}_R(-,C')|_\mathcal{X}$ are finitely presented $\mathcal{X}$-modules. Taking the direct sum of finite presentations of those two $\mathcal{X}$-modules, we see that the $\mathcal{X}$-module $\mathrm{Hom}_R(-,C)|_\mathcal{X} \oplus \mathrm{Hom}_R(-,C')|_\mathcal{X} = \mathrm{Hom}_R(-,C \oplus C')|_\mathcal{X}$ is also finitely presented. Hence $C \oplus C'$ belongs to $\mathcal{C}$.

(4a) Let $0 \to L \to M \to N$ be an exact sequence of R-modules such that $M, N \in \mathcal{C}$. An exact sequence $0 \to \mathrm{Hom}_R(-,L)|_\mathcal{X} \to \mathrm{Hom}_R(-,M)|_\mathcal{X} \to \mathrm{Hom}_R(-,N)|_\mathcal{X}$ is induced, and $\mathrm{Hom}_R(-,M)|_\mathcal{X}$ and $\mathrm{Hom}_R(-,N)|_\mathcal{X}$ belong to $\mathsf{mod}\,\mathcal{X}$. Since $\mathsf{mod}\,\mathcal{X}$ is abelian, $\mathrm{Hom}_R(-,L)|_\mathcal{X}$ belongs to $\mathsf{mod}\,\mathcal{X}$ as well. Thus L is in $\mathcal{C}$. It follows that $\mathcal{C}$ is closed under kernels. Remark 2.4 implies that $\mathcal{C}$ is closed under direct summands. As $\mathcal{C}$ contains $\mathcal{X}$ and $\mathcal{X}$ is resolving, $\mathcal{C}$ contains $\mathsf{add}\,R = \mathsf{proj}(\mathsf{mod}\,R)$. Combining this with the fact that $\mathcal{C}$ is closed under kernels, we see that $\mathcal{C}$ is closed under syzygies and contains $\Omega^2(\mathsf{mod}\,R)$.

(4b) Take an exact sequence $P_1 \to P_0 \to M \to 0$ with $P_0, P_1 \in \mathsf{add}\,R$. This induces an exact sequence $0 \to \mathrm{Hom}_R(M,C) \to \mathrm{Hom}_R(P_0,C) \to \mathrm{Hom}_R(P_1,C)$. Since the modules $\mathrm{Hom}_R(P_0,C)$ and $\mathrm{Hom}_R(P_1,C)$ belong to $\mathsf{add}\,C$, they are in $\mathcal{C}$. The fact that $\mathcal{C}$ is closed under kernels implies that $\mathrm{Hom}_R(M,C)$ is in $\mathcal{C}$.

(4c) Set $(-)^* = \mathrm{Hom}_R(-,R)$. There is an exact sequence $0 \to \mathrm{Ext}^1_R(\mathrm{Tr}\,C,R) \to C \to C^{**}$ by [5, Proposition 2.6(a)]. Note that M^* is a second syzygy for each R-module M. As $\mathcal{C}$ contains $\Omega^2(\mathsf{mod}\,R)$, we have $C^{**} \in \mathcal{C}$. Since $\mathcal{C}$ is closed under kernels, the module $\mathrm{Ext}^1_R(\mathrm{Tr}\,C,R)$ belongs to $\mathcal{C}$.

(4d) Let $\mathcal{D}$ be the subcategory of $\mathsf{mod}\,R$ consisting of modules M with $\Omega M \in \mathcal{B}$. Then $\mathcal{C}$ is contained in $\mathcal{D}$ since $\mathcal{C}$ is closed under syzygies and contained in $\mathcal{B}$. If N is a direct summand of an R-module M, then ΩN is a direct summand of ΩM. If $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules, then there is an exact sequence $0 \to \Omega L \to \Omega M \to \Omega N \to 0$. Using these facts, we see that $\mathcal{D}$ is closed under direct summands and extensions. The definition of $\mathcal{B}$ implies that $\mathcal{D}$ contains $\mathcal{B}$, which means that $\mathcal{B}$ is closed under syzygies. We conclude that $\mathcal{B}$ is a resolving subcategory of $\mathsf{mod}\,R$.

(4e) As $\mathcal{X} \subseteq \mathcal{C} \subseteq \mathsf{rap}\,\mathcal{X}$, the equality $\mathcal{X} = \mathsf{rap}\,\mathcal{X}$ implies $\mathcal{X} = \mathcal{C}$. Hence $\mathcal{X}$ is closed under kernels.

(5) We have $\mathcal{C} \subseteq \mathsf{rap}\,\mathcal{X}$. Pick an R-module $M \in \mathsf{rap}\,\mathcal{X}$. There is a right $\mathcal{X}$-approximation $f : X \to M$. As $\mathcal{X}$ contains the projective R-modules, we observe that f is surjective. By assumption, the kernel K of f belongs to $\mathsf{rap}\,\mathcal{X}$. There is a right $\mathcal{X}$-approximation $Y \to K$. The induced sequence $\mathrm{Hom}_R(-,Y)|_\mathcal{X} \to \mathrm{Hom}_R(-,X)|_\mathcal{X} \to \mathrm{Hom}_R(-,M)|_\mathcal{X} \to 0$ is seen to be exact, and it follows that M belongs to $\mathcal{C}$.

(6) Let M be an R-module in $\mathcal{X}^\perp \cap \Omega(\mathsf{mod}\,R)$. Then there is an exact sequence $\sigma : 0 \to M \to F \xrightarrow{f} N \to 0$ of R-modules with F free. Since $F \in \mathcal{X}$ and $M \in \mathcal{X}^\perp$, the proof of Lemma 2.11 shows that f is a right $\mathcal{X}$-approximation. Hence $N \in \mathsf{rap}\,\mathcal{X} = \mathcal{X}$. Note that σ corresponds to an element of $\mathrm{Ext}^1_R(N,M)$, which vanishes as $M \in \mathcal{X}^\perp$ and $N \in \mathcal{X}$. Therefore the short exact sequence σ splits, and M is free. ■

To prove our next proposition, we establish a lemma.

Lemma 3.3. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence of R-modules. If* $L \in \mathcal{X}^\perp \cap \mathcal{C}$ *and* $M \in \mathcal{C}$, *then* $N \in \mathcal{C}$. *In particular, the subcategory* $\mathcal{X}^\perp \cap \mathcal{C}$ *of* $\mathsf{mod}\,R$ *is closed under cokernels of monomorphisms.*

Proof. An exact sequence $0 \to \mathrm{Hom}_R(-,L)|_\mathcal{X} \to \mathrm{Hom}_R(-,M)|_\mathcal{X} \to \mathrm{Hom}_R(-,N)|_\mathcal{X} \to \mathrm{Ext}^1_R(-,L)|_\mathcal{X}$ is induced. As the R-module L is in $\mathcal{X}^\perp$, we have $\mathrm{Ext}^1_R(-,L)|_\mathcal{X} = 0$. Since the $\mathcal{X}$-modules $\mathrm{Hom}_R(-,L)|_\mathcal{X}$ and $\mathrm{Hom}_R(-,M)|_\mathcal{X}$ belong to $\mathsf{mod}\,\mathcal{X}$, so does $\mathrm{Hom}_R(-,N)|_\mathcal{X}$ by Lemma 2.15(1), which means $N \in \mathcal{C}$. ■

We provide several sufficient conditions for the residue field k of R to belong to the subcategory $\mathcal{C}$.

Proposition 3.4. *Assume* $\mathsf{mod}\,\mathcal{X}$ *is abelian. Suppose one of the following four conditions is satisfied.*
(1) $\mathcal{C}$ *contains a module of depth* 0. (2) $\operatorname{depth} R = 0$. (3) $\mathcal{C} \cap \mathsf{mod}_0 R \nsubseteq \Omega(\mathsf{mod}\, R)$. (4) $\mathcal{C} \neq \mathcal{X}$.
Then the residue field k *of* R *belongs to* $\mathcal{C}$. *In particular,* k *admits a right* $\mathcal{X}$*-approximation.*

Proof. If k is in $\mathcal{C}$, then there exists a right $\mathcal{X}$-approximation of k by Proposition 3.2(2).

(1) Let C be an R-module in $\mathcal{C}$ of depth 0. Then $\operatorname{Hom}_R(k, C)$ is a nonzero k-vector space, and belongs to $\mathcal{C}$ by Proposition 3.2(4b). As $\mathcal{C}$ is closed under direct summands by Proposition 3.2(4a), we have $k \in \mathcal{C}$.

(2) We have $R \in \mathcal{X}$, while $\mathcal{X} \subseteq \mathcal{C}$ by Proposition 3.2(2). We get $R \in \mathcal{C}$. It follows from (1) that $k \in \mathcal{C}$.

(3) Find an R-module C in $\mathcal{C} \cap \mathsf{mod}_0 R$ which is not a syzygy. Thus $L := \operatorname{Ext}^1_R(\operatorname{Tr} C, R)$ is nonzero by [5, Proposition 2.6(a)] and [16, Lemma 3.4]. As $C \in \mathsf{mod}_0 R$ ($\mathsf{mod}_0 R$ is defined in Example 2.7(3)), the R-module L has finite length and depth 0. Proposition 3.2(4c) implies $L \in \mathcal{C}$. We obtain $k \in \mathcal{C}$ by (1).

(4) We find $C \in \mathcal{C}$ with $C \notin \mathcal{X}$. As C belongs to $\mathsf{rap}\,\mathcal{X}$ by Proposition 3.2(2), there is an exact sequence $0 \to Y \to X \to C \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$ by Lemma 2.11. The subcategory $\mathcal{C}$ contains X, C and is closed under kernels by Proposition 3.2(2)(4a), the module Y is in $\mathcal{C}$, whence $Y \in \mathcal{X}^{\perp} \cap \mathcal{C}$. Take a maximal regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ on Y. There exists a family of exact sequences of R-modules:

$$\{0 \to Y/(x_1, \dots, x_{i-1})Y \xrightarrow{x_i} Y/(x_1, \dots, x_{i-1})Y \to Y/(x_1, \dots, x_i)Y \to 0\}_{i=1}^n.$$

Applying Lemma 3.3 repeatedly, we observe that $Y/\boldsymbol{x}Y \in \mathcal{X}^{\perp} \cap \mathcal{C} \subseteq \mathcal{C}$. It follows from (1) that $k \in \mathcal{C}$. ■

Proposition 3.5. *Assume that* $d = \dim R \geqslant 1$ *and* $\mathsf{mod}\,\mathcal{X}$ *is abelian. Suppose that* k *belongs to* $\mathcal{B}$ *(this holds true under the assumption of Proposition 3.4). Then* $\Omega^{d-1}(\mathsf{mod}\, R)$ *is contained in* $\mathcal{B}$. *Hence any* $(d-1)$*st syzygy has a right* $\mathcal{X}$*-approximation. In particular, the equality* $\mathsf{rap}\,\mathcal{X} = \mathsf{mod}\, R$ *holds when* $d = 1$.

Proof. Taking Proposition 3.2(2) into account, we have only to show that $\Omega^{d-1}M \in \mathcal{B}$ for each R-module M. Note from Proposition 3.2(4d) that $\mathcal{B}$ is a resolving subcategory of $\mathsf{mod}\, R$.

(1) Suppose that M has finite length. Then, since $\mathcal{B}$ is closed under extensions and contains k, we see that $M \in \mathcal{B}$. Since $\mathcal{B}$ is closed under syzygies, we obtain $\Omega^{d-1}M \in \mathcal{B}$.

(2) By (1) we may assume $\dim M > 0$. Then there is an exact sequence $0 \to L \to M \to N \to 0$ of R-modules such that L has finite length and that N is nonzero and has positive depth. Proposition 3.2(4a) says $\mathcal{B}$ is dominant. By [23, Corollary 4.6] we have $\Omega^r N \in \mathcal{B}$, where $r = \sup_{\mathfrak{p} \in \operatorname{Spec} R}\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}}\}$. Note that $0 \leqslant r \leqslant d$. If $r = d$, then $\operatorname{depth} R_{\mathfrak{p}} = d$ and $\operatorname{depth} N_{\mathfrak{p}} = 0$ for some $\mathfrak{p} \in \operatorname{Spec} R$, which implies $\mathfrak{p} = \mathfrak{m}$ and $\operatorname{depth} N = 0$, a contradiction. Hence $r \leqslant d - 1$, and $\Omega^{d-1}N = \Omega^{d-1-r}(\Omega^r N) \in \mathcal{B}$. There is an exact sequence $0 \to \Omega^{d-1}L \to \Omega^{d-1}M \to \Omega^{d-1}N \to 0$, and $\Omega^{d-1}L \in \mathcal{B}$ by (1). Therefore $\Omega^{d-1}M \in \mathcal{B}$. ■

Now we state and prove the theorem below, which is one of the main results of this paper.

Theorem 3.6. *Let* C *be a semidualizing* R*-module with* $\mathcal{X} \subseteq \mathsf{GP}(C)$. *If* $\mathsf{mod}\,\mathcal{X}$ *is abelian,* $\mathsf{rap}\,\mathcal{X} = \mathsf{mod}\, R$.

Proof. Recall by Example 2.7(2) that $\mathsf{GP}(C)$ is a resolving subcategory of $\mathsf{mod}\, R$. We freely use this fact.

We claim that $\mathsf{GP}(C)$ is contained in $\mathsf{rap}\,\mathcal{X}$. Indeed, it follows from Proposition 3.2(2)(4a)(4d) that $\mathcal{B}$ is a dominant resolving subcategory of $\mathsf{mod}\, R$ and contained in $\mathsf{rap}\,\mathcal{X}$. Fix a Gorenstein C-projective R-module M. For each $\mathfrak{p} \in \operatorname{Spec} R$, the localization $C_{\mathfrak{p}}$ is a semidualizing $R_{\mathfrak{p}}$-module, and the localization $M_{\mathfrak{p}}$ is a Gorenstein $C_{\mathfrak{p}}$-projective $R_{\mathfrak{p}}$-module. It holds that $\operatorname{depth} M_{\mathfrak{p}} \geqslant \operatorname{depth} R_{\mathfrak{p}}$ by [17, page 68] (or [12, Theorem (3.14)]). Applying [23, Theorem 1.1], we see that M belongs to $\mathcal{B}$. Now, the claim follows.

It follows by Proposition 3.2(2)(4a) that $\mathsf{rap}\,\mathcal{X}$ contains $\Omega^2(\mathsf{mod}\, R)$. So it suffices to show that for an R-module M with $\Omega M \in \mathsf{rap}\,\mathcal{X}$ one has $M \in \mathsf{rap}\,\mathcal{X}$. Take an exact sequence $0 \to \Omega M \to F \to M \to 0$ with F free. Lemma 2.11 gives an exact sequence $0 \to Y \to X \to \Omega M \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$. As X is in $\mathsf{GP}(C)$, there is an exact sequence $0 \to X \to C' \to G \to 0$ with $C' \in \mathsf{add}\, C$ and $G \in \mathsf{GP}(C)$ (we can get such an exact sequence by applying $(-)^{\dagger} = \operatorname{Hom}_R(-, C)$ to an exact sequence $0 \to \Omega(X^{\dagger}) \to P \to X^{\dagger} \to 0$

with P free). We obtain the left and middle commutative diagrams below, which are pushout diagrams.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & X & \to & \Omega M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & Y & \to & C' & \to & Y' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \Omega M & \to & F & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & Y' & \to & G' & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y'' & = & Y'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & Y''' & \to & X' & \xrightarrow{f} & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & Y' & \to & G' & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The modules Y, C' belong to $\mathcal{X}^{\perp}$, and so does Y' by the middle row in the left diagram. The modules F, G belong to $\mathsf{GP}(C)$, and so does G' by the middle column in the middle diagram. The claim and Lemma 2.11 yield an exact sequence $0 \to Y'' \to X' \to G' \to 0$ with $X' \in \mathcal{X}$ and $Y'' \in \mathcal{X}^{\perp}$. We obtain the right commutative diagram displayed above, which is a pullback diagram. The modules Y', Y'' belong to $\mathcal{X}^{\perp}$, and so does Y''' by the left column in the right diagram. The middle row in the right diagram and the proof of Lemma 2.11 imply that the map f is a right $\mathcal{X}$-approximation of M, and thus $M \in \mathsf{rap}\,\mathcal{X}$. ■

Applying the results stated above, we obtain the corollary below, which includes part of Theorem 1.2.

Corollary 3.7. *Assume* $\mathsf{mod}\,\mathcal{X}$ *is abelian. Then* $\mathsf{rap}\,\mathcal{X} = \mathsf{mod}\,R$ *if one of the following statements holds.*
(1) *The ring R is a homomorphic image of a Gorenstein ring and* $\dim R \leqslant 1$.
(2) *The ring R is a Cohen–Macaulay ring with a canonical module ω, and $\mathcal{X}$ is contained in* $\mathsf{CM}(R)$.
(3) *The subcategory $\mathcal{X}$ is contained in* $\mathsf{GP}(R)$.

Proof. We obtain (3) and (2) by applying Theorem 3.6 to $C = R$ and $C = \omega$, respectively. Let us show (1). By (2) we may assume $\dim R = 1$. By Propositions 3.4(1) and 3.5, we may assume $\operatorname{depth} C > 0$ for all $C \in \mathcal{C}$. Proposition 3.2(2) implies $R \in \mathcal{X} \subseteq \mathcal{C}$. We see that R is Cohen–Macaulay and $\mathcal{X} \subseteq \mathsf{CM}(R)$. As R is a homomorphic image of a Gorenstein ring, it has a canonical module. By (2) we are done. ■

Here we recall a notion introduced by Huneke and Jorgensen [18]. A local ring R is called *AB* if R is Gorenstein and there exists an integer $n \geqslant 0$ such that $\operatorname{Ext}_R^{\gg 0}(M,N) = 0$ with $M, N \in \mathsf{mod}\,R$ implies $\operatorname{Ext}_R^{>n}(M,N) = 0$. We can show the following proposition, which gives a sufficient condition for the resolving subcategory $\mathcal{X}$ to consist of maximal Cohen–Macaulay R-modules. Note that the assumption of the first assertion of the proposition is satisfied if the subcategory $\mathcal{X}$ is dominant.

Proposition 3.8. *Assume that one has $\mathcal{X} \neq \mathsf{rap}\,\mathcal{X}$. Then the following assertions hold true.*
(1) *If $\Omega^n k \in \mathcal{X}$ for some $n \geqslant 0$, then the ring R is Cohen–Macaulay, and $\mathcal{X}$ is contained in* $\mathsf{CM}(R)$.
(2) *If R is an AB ring, then $\mathcal{X}$ is contained in* $\mathsf{CM}(R)$.

Proof. Choose an R-module M such that $M \in \mathsf{rap}\,\mathcal{X}$ and $M \notin \mathcal{X}$. Lemma 2.11 yields an exact sequence $0 \to B \to A \to M \to 0$ of R-modules with $A \in \mathcal{X}$ and $B \in \mathcal{X}^{\perp}$. Since M is not in $\mathcal{X}$, we have $B \neq 0$.

(1) We have $\operatorname{Ext}_R^{>0}(\Omega^n k, B) = 0$. It follows that $\operatorname{Ext}_R^{>n}(k, B) = 0$, which implies $\operatorname{id}_R B \leqslant n$. Lemma 2.18 deduces that R is Cohen–Macaulay. Let $X \in \mathcal{X}$ be a nonzero R-module. Then $\operatorname{Ext}_R^{>0}(X, B) = 0$. It is observed from [10, Exercise 3.1.24] that X is a maximal Cohen–Macaulay R-module. Thus $\mathcal{X} \subseteq \mathsf{CM}(R)$.

(2) Let $0 \neq X \in \mathcal{X}$. Then $\operatorname{Ext}_R^{>0}(X, B) = 0$. By [2, Lemma 2.5] we get $\operatorname{depth} R - \operatorname{depth} X = 0$. Since an AB ring is Cohen–Macaulay, we see that X is maximal Cohen–Macaulay. We obtain $\mathcal{X} \subseteq \mathsf{CM}(R)$. ■

Remark 3.9. The latter half of the proof of Proposition 3.8(1) can be replaced with the following argument using methods in [14]. Suppose $\mathcal{X}$ is not contained in $\mathsf{CM}(R)$. Then there exists $X \in \mathcal{X}$ with $e := \operatorname{depth} X < \dim R =: d$. By [14, Proposition 4.2] we get $\Omega^e k \in \mathcal{X}$. Hence $\operatorname{Ext}_R^{>0}(\Omega^e k, B) = 0$, which implies $\operatorname{id}_R N \leqslant e$. Lemma 2.18 gives $d \leqslant e$. This contradiction shows that $\mathcal{X}$ is contained in $\mathsf{CM}(R)$.

We obtain the following corollary, which includes part of Theorem 1.2.

Corollary 3.10. (1) *Assume that* $\mathsf{mod}\,\mathcal{X}$ *is an abelian category. Suppose* (i) *R is AB, or* (ii) *$\mathcal{X}$ contains $\Omega^n k$ for some $n \geqslant 0$, or* (iii) *$\mathcal{X}$ is closed under cosyzygies. Then either $\mathcal{X} = \mathsf{rap}\,\mathcal{X}$ or $\mathsf{rap}\,\mathcal{X} = \mathsf{mod}\,R$.*

(2) *Consider the following two conditions for a subcategory $\mathcal{Y}$ of* $\mathsf{mod}\,R$.
  (a) *The subcategory $\mathcal{Y}$ is resolving, closed under kernels, and satisfies $\mathcal{Y} \neq \mathsf{rap}\,\mathcal{Y}$.*
  (b) *The ring R is Cohen–Macaulay and has dimension 1 or 2, and $\mathcal{Y} = \mathsf{CM}(R)$.*
  Then (a) *implies* (b). *If R is a homomorphic image of a Gorenstein ring,* (a) *and* (b) *are equivalent.*

Proof. (1) Use Proposition 3.8 and Corollary 3.7(2) for (i),(ii). For (iii), either $k \in \mathcal{X}$ or $\mathcal{X} \subseteq \mathsf{GP}(R)$ holds by [19, Theorem 1.3]. The former case is included in (ii). In the latter case Corollary 3.7(3) applies.

(2) Assume that (a) holds. Since $\mathcal{Y}$ contains $\mathsf{add}\,R$ and is closed under kernels, it contains $\Omega^2(\mathsf{mod}\,R)$. Proposition 3.8(1) implies that R is Cohen–Macaulay and $\mathcal{Y}$ is contained in $\mathsf{CM}(R)$. As $\mathcal{Y}$ is dominant, it contains $\mathsf{CM}(R)$ by [14, Theorem 4.5] or Example 2.7(4). The equality $\mathcal{Y} = \mathsf{CM}(R)$ follows. We thus have $\Omega^2(\mathsf{mod}\,R) \subseteq \mathcal{Y} = \mathsf{CM}(R)$, which implies $\dim R \leqslant 2$. If R is artinian, then $\mathcal{Y} = \mathsf{CM}(R) = \mathsf{mod}\,R$, which contradicts the assumption that $\mathcal{Y} \neq \mathsf{rap}\,\mathcal{Y}$. Therefore R has dimension 1 or 2. Thus (b) holds.

Suppose that R is a homomorphic image of a Gorenstein ring and (b) holds. Then R admits a canonical module. Examples 2.7(1) and 2.9(2) imply that $\mathcal{Y}$ is resolving with $\mathsf{rap}\,\mathcal{Y} = \mathsf{mod}\,R$. Since $\dim R > 0$, we have $\mathcal{Y} \neq \mathsf{rap}\,\mathcal{Y}$. Since $\dim R \leqslant 2$, by the depth lemma $\mathcal{Y}$ is closed under kernels. Therefore (a) holds. ■

Corollary 3.11. *Assume that R is a homomorphic image of a Gorenstein ring and $\mathsf{mod}\,\mathcal{X}$ is abelian.*
(1) *One has $\mathcal{X} = \mathsf{rap}\,\mathcal{X}$ if and only if $\mathcal{X} = \mathcal{C}$.* (2) *If $\mathcal{X} \neq \mathsf{rap}\,\mathcal{X}$, one then has $k \in \mathcal{C} \subseteq \mathcal{B} \subseteq \mathsf{rap}\,\mathcal{X}$.*
(3) *There is an equality $\mathcal{B} = \mathsf{rap}\,\mathcal{B}$.*

Proof. (1) Proposition 3.2(2) gives the inclusions $\mathcal{X} \subseteq \mathcal{C} \subseteq \mathsf{rap}\,\mathcal{X}$, which show the "only if" part. The "if" part will follow if we get a contradiction by assuming $\mathcal{C} = \mathcal{X} \neq \mathsf{rap}\,\mathcal{X}$. Proposition 3.2(4a) says $\mathcal{X} = \mathcal{C}$ is closed under kernels. Corollary 3.10(2) and its proof imply R is Cohen–Macaulay, $\mathcal{X} = \mathsf{CM}(R)$ and $\mathsf{rap}\,\mathcal{X} = \mathsf{mod}\,R$. Proposition 3.2(5) yields $\mathcal{C} = \mathsf{mod}\,R$. Then $\mathcal{X} = \mathsf{mod}\,R$, and $\mathcal{X} = \mathsf{rap}\,\mathcal{X}$, a contradiction.

(2) It follows from (1) that $\mathcal{X} \neq \mathcal{C}$. We get $k \in \mathcal{C} \subseteq \mathcal{B} \subseteq \mathsf{rap}\,\mathcal{X}$ by Propositions 3.2(2) and 3.4(4).

(3) If $\mathcal{X} = \mathsf{rap}\,\mathcal{X}$, then $\mathcal{B} = \mathsf{rap}\,\mathcal{B}$ by Proposition 3.2(2). Let $\mathcal{X} \neq \mathsf{rap}\,\mathcal{X}$. Then $k \in \mathcal{B}$ by (2). We will be done once we derive a contradiction by assuming $\mathcal{B} \neq \mathsf{rap}\,\mathcal{B}$. Choose an R-module $M \in \mathsf{rap}\,\mathcal{B}$ with $M \notin \mathcal{B}$. Lemma 2.11 gives an exact sequence $0 \to N \to B \to M \to 0$ with $B \in \mathcal{B}$ and $0 \neq N \in \mathcal{B}^{\perp}$. As $k \in \mathcal{B}$, we have $\mathrm{Ext}_R^{>0}(k, N) = 0$. Lemma 2.18 shows R is artinian. As $\mathcal{B}$ contains k and is closed under extensions, it coincides with $\mathsf{mod}\,R$. Therefore we have $\mathcal{B} = \mathsf{rap}\,\mathcal{B}$. This gives a desired contradiction. ■

The condition that a subcategory of $\mathsf{mod}\,R$ is both resolving and closed under kernels looks so restrictive that we may wonder if there exists no such example except trivial ones. The following proposition gives rise to such a subcategory, even satisfying more restrictive conditions.

Proposition 3.12. *Let Φ be a subset of $\operatorname{Spec} R$ containing $\operatorname{Ass} R$. Let $\mathcal{Y}$ be the subcategory of $\mathsf{mod}\,R$ consisting of modules M such that $\operatorname{Ass} M$ is contained in Φ.*
(1) *One has that $\mathcal{Y}$ is a resolving subcategory of $\mathsf{mod}\,R$ closed under subobjects. In particular, the subcategory $\mathcal{Y}$ contains $\Omega(\mathsf{mod}\,R)$ and $\mathsf{mod}\,\mathcal{Y}$ is an abelian category.*
(2) *Suppose $\mathcal{Y} \neq \mathsf{rap}\,\mathcal{Y}$. Then R is a Cohen–Macaulay ring of dimension 1, and $\mathcal{Y}$ coincides with $\mathsf{CM}(R)$.*

Proof. (1) Using basic properties of associated prime ideals, we see that $\mathcal{Y}$ is closed under subobjects and extensions. As Φ contains $\operatorname{Ass} R$, we have $R \in \mathcal{Y}$. Thus $\mathcal{Y}$ is resolving. Since a syzygy is a submodule of a projective R-module, $\mathcal{Y}$ contains $\Omega(\mathsf{mod}\,R)$. By Proposition 2.16 (and Remark 2.4), $\mathsf{mod}\,\mathcal{Y}$ is abelian.

(2) It follows by (1) and Corollary 3.10(2) that R is a Cohen–Macaulay ring with dimension 1 or 2 and $\mathcal{Y}$ coincides with $\mathsf{CM}(R)$. As $\mathcal{Y}$ contains $\Omega(\mathsf{mod}\,R)$ by (1) again, the case $\dim R = 2$ does not occur. ■

The above proposition yields the corollary below, which is none other than Theorem 1.3 and gives a negative answer to Question 1.1.

Corollary 3.13. *Assume that R is neither a 1-dimensional Cohen–Macaulay ring nor satisfies $\operatorname{Ass} R = \operatorname{Spec} R$. Then there exists a proper resolving subcategory $\mathcal{Y}$ of $\mathsf{mod}\,R$ which is closed under subobjects and satisfies $\mathcal{Y} = \mathsf{rap}\,\mathcal{Y}$. In particular, one has both that $\mathsf{mod}\,\mathcal{Y}$ is abelian and that $\mathsf{rap}\,\mathcal{Y} \neq \mathsf{mod}\,R$.*

Proof. Choose any subset Φ of $\operatorname{Spec} R$ such that $\operatorname{Ass} R \subseteq \Phi \neq \operatorname{Spec} R$. Let $\mathcal{Y}$ be a subcategory of $\mathsf{mod}\,R$ consisting of modules M with $\operatorname{Ass} M \subseteq \Phi$. Since $\Phi \neq \operatorname{Spec} R$, we see that $\mathcal{Y} \neq \mathsf{mod}\,R$. By Proposition 3.12(1)(2) we have that $\mathcal{Y}$ is resolving and closed under subobjects, $\mathsf{mod}\,\mathcal{Y}$ is abelian, and $\mathcal{Y} = \mathsf{rap}\,\mathcal{Y}$. ■

As an application of the above corollary, we present two examples.

Example 3.14. Suppose that the ring R has dimension at least two.

(1) Assume that R has positive depth. Let $\mathcal{Y}$ be the subcategory of $\operatorname{mod} R$ consisting of R-modules that have positive depth. Then $\mathcal{Y}$ is such a subcategory as in Corollary 3.13 and satisfies $k \notin \mathcal{Y}$. This is observed by taking the punctured spectrum of R as Φ in the proof of the corollary.

(2) Assume R satisfies Serre's condition (S_1). Let $\mathcal{Y}$ be the subcategory of $\operatorname{mod} R$ consisting of R-modules none of whose associated prime ideal has height 1. Then $\mathcal{Y}$ is such a subcategory as in Corollary 3.13 and satisfies $k \in \mathcal{Y}$. This is seen by letting $\Phi = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{ht} \mathfrak{p} \neq 1\}$ in the proof of the corollary.

4. Applications and further questions

In this short section, we first apply our results in the previous sections to the subcategory of Gorenstein projective modules. We then present two questions related to Question 1.1 and give some observations.

Proposition 4.1. *Let $\mathcal{X}$ be an additive subcategory of $\operatorname{mod} R$ containing R. Then $\operatorname{rap} \mathcal{X} = \operatorname{mod} R$ if and only if the functor $\operatorname{ev} : \operatorname{mod} \mathcal{X} \to \operatorname{mod} R$ given by $\operatorname{ev}(F) = F(R)$ has a right adjoint.*

Proof. The "only if" part follows from [1, Theorem 3.4(1)]. We prove the "if" part. Fix an R-module M. Let $\phi : \operatorname{mod} R \to \operatorname{mod} \mathcal{X}$ be a right adjoint to the functor ev. Then there is a functorial isomorphism $\operatorname{Hom}_R(F(R), M) \cong \operatorname{Hom}_{\operatorname{mod} \mathcal{X}}(F, \phi(M))$, where $F \in \operatorname{mod} \mathcal{X}$. For each $X \in \mathcal{X}$, the functor $\operatorname{Hom}_R(-, X)|_{\mathcal{X}}$ belongs to $\operatorname{mod} \mathcal{X}$. Since R is assumed to belong to $\mathcal{X}$, we get functorial isomorphisms

$$\operatorname{Hom}_R(X, M) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(R, X), M) \cong \operatorname{Hom}_{\operatorname{mod} \mathcal{X}}(\operatorname{Hom}_R(-, X)|_{\mathcal{X}}, \phi(M)) \cong \phi(M)(X),$$

where to get the last isomorphism we apply Yoneda's lemma. We thus obtain an isomorphism of functors $\operatorname{Hom}_R(-, M)|_{\mathcal{X}} \cong \phi(M)$. Since $\phi(M)$ belongs to $\operatorname{mod} \mathcal{X}$, it follows from Lemma 2.13 that the R-module M admits a right $\mathcal{X}$-approximation. Consequently, the equality $\operatorname{rap} \mathcal{X} = \operatorname{mod} R$ holds. ■

Using the above proposition, we can get the theorem below. We should remark that condition (2) in the theorem depends only on the structure of $\operatorname{GP}(R)$ as an additive category.

Theorem 4.2. *Let R be a henselian local ring. The following are equivalent.*
(1) *The ring R is Gorenstein or G-regular.* (2) *The category $\operatorname{mod} \operatorname{GP}(R)$ is abelian.*
(3) *One has $\operatorname{rap} \operatorname{GP}(R) = \operatorname{mod} R$.* (4) *The functor $\operatorname{ev} : \operatorname{mod} \operatorname{GP}(R) \to \operatorname{mod} R$ has a right adjoint.*

Proof. First of all, $\operatorname{GP}(R)$ is a resolving subcategory of $\operatorname{mod} R$ by Example 2.7(2). The equivalence (2) $\Leftrightarrow$ (3) (resp. (3) $\Leftrightarrow$ (4)) follows from Proposition 2.16 and Corollary 3.7(3) (resp. Proposition 4.1). If R is Gorenstein (resp. G-regular), then $\operatorname{GP}(R)$ coincides with $\operatorname{CM}(R)$ (resp. $\operatorname{add} R$) and there is an equality $\operatorname{rap} \operatorname{GP}(R) = \operatorname{mod} R$ by Example 2.9. Hence, the implication (1) $\Rightarrow$ (3) holds. The opposite implication (3) $\Rightarrow$ (1) is a consequence of [13, Theorem C] (see also [22, Corollary 1.5]). ■

The following question naturally arises in view of Propositions 3.4, 3.5 and Corollary 3.11(2).

Question 4.3. Let R be a henselian local ring with residue field k. Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. Assume that k is not in $\mathcal{X}$ but admits a right $\mathcal{X}$-approximation. Is then $\mathcal{X}$ contravariantly finite?

Remark 4.4. (1) Question 4.3 has an affirmative answer if R is artinian. Indeed, Proposition 3.2(1) says $\operatorname{rap} \mathcal{X}$ is closed under extensions. If $\operatorname{rap} \mathcal{X}$ contains k and R is artinian, then we have $\operatorname{rap} \mathcal{X} = \operatorname{mod} R$.

(2) The assumption in Question 4.3 that the residue field k of R does not belong to $\mathcal{X}$ is indispensable. In fact, $\mathcal{X} := \operatorname{mod}_0 R$ is a resolving subcategory of $\operatorname{mod} R$ by Example 2.7(3) and we have $k \in \mathcal{X} \subseteq \operatorname{rap} \mathcal{X}$. However, $\mathcal{X}$ is not necessarily contravariantly finite. For example, if the ring R is Gorenstein and $\mathcal{X}$ is contravariantly finite, then $\mathcal{X}$ coincides with $\operatorname{add} R$ or $\operatorname{CM}(R)$ or $\operatorname{mod} R$; see [22, Theorem 1.2].

In view of Corollary 3.13 and Example 3.14, our Question 1.1 is not always affirmative, and we should modify it. It would be reasonable to make the additional assumption that there exists a nontrivial object which admits a right approximation. Thus our modified question is the following.

Question 4.5. Let R be a henselian local ring. Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$ such that the category $\operatorname{mod} \mathcal{X}$ is abelian. Assume that there exists an R-module which does not belong to $\mathcal{X}$ but admits a right $\mathcal{X}$-approximation. Is then $\mathcal{X}$ contravariantly finite?

Remark 4.6. Question 4.5 has an affirmative answer if we replace the abelianity of $\mathsf{mod}\,\mathcal{X}$ with the stronger condition that $\mathcal{X}$ is closed under kernels (see Proposition 2.16) and assume further that R is a homomorphic image of a Gorenstein ring. Indeed, Corollary 3.10(2) shows that R is Cohen–Macaulay and $\mathcal{X} = \mathsf{CM}(R)$. It follows from Example 2.9(2) that $\mathsf{rap}\,\mathcal{X} = \mathsf{mod}\,R$.

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