

# ON THE NUMBER OF INDECOMPOSABLE TOTALLY REFLEXIVE MODULES

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ABSTRACT. In this note, it is proved that over a commutative noetherian henselian non-Gorenstein local ring there are infinitely many isomorphism classes of indecomposable totally reflexive modules, if there is a nonfree cyclic totally reflexive module.

## 1. INTRODUCTION

Throughout this note, we assume that all rings are commutative and noetherian, and that all modules are finitely generated.

In the late 1960s, Auslander [1] introduced a homological invariant for modules which is called Gorenstein dimension, or G-dimension for short. After that, he further developed the theory of G-dimension with Bridger [2]. Many properties enjoyed by G-dimension are analogous to those of projective dimension. An important feature is that G-dimension characterizes Gorenstein local rings exactly as projective dimension characterizes regular local rings. A module of G-dimension zero is called a totally reflexive module. Avramov and Martsinkovsky [3] and Holm [7] proved that over a local ring any module  $M$  of finite G-dimension admits an exact sequence  $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$  such that  $X$  is totally reflexive and  $Y$  is of finite projective dimension. This result says that in the study of modules of finite G-dimension it is essential to consider totally reflexive modules.

On the other hand, Cohen-Macaulay local rings of finite Cohen-Macaulay type, namely Cohen-Macaulay local rings over which there are only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules, have been deeply studied since the work of Herzog [6]. Under a few assumptions, Gorenstein local rings of finite Cohen-Macaulay type are hypersurfaces, and they have been classified completely. Moreover, all isomorphism classes of indecomposable maximal Cohen-Macaulay modules over them are described concretely; see [14] for the details.

Over a Gorenstein local ring, totally reflexive modules are the same as maximal Cohen-Macaulay modules. Hence it is natural to expect that totally reflexive modules over an arbitrary local ring may behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring, and we are interested in local rings over which there are only finitely many isomorphism classes of indecomposable totally reflexive modules; we want to determine all such isomorphism classes. However, we guess that such ring cannot essentially exist in the non-Gorenstein case:

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**Conjecture 1.1.** Let  $R$  be a non-Gorenstein local ring. Suppose that there is a nonfree totally reflexive  $R$ -module. Then there are infinitely many isomorphism classes of indecomposable totally reflexive  $R$ -modules.

In this conjecture, so as to exclude the case where all totally reflexive modules are free, it is assumed that there is a nonfree totally reflexive module. Indeed, for instance, over a Cohen-Macaulay non-Gorenstein local ring with minimal multiplicity, every totally reflexive module is free [15].

The author proved that the above conjecture is true over a henselian local ring of low depth:

**Theorem 1.2.** [11][12][13] *Let  $R$  be a henselian non-Gorenstein local ring of depth at most two. Suppose that there is a nonfree totally reflexive  $R$ -module. Then there are infinitely many isomorphism classes of indecomposable totally reflexive  $R$ -modules.*

The main purpose of this note is to prove that the conjecture is true over a henselian local ring having a nonfree cyclic totally reflexive module.

**Theorem 1.3.** *Let  $R$  be a henselian non-Gorenstein local ring. Suppose that there is a nonfree cyclic totally reflexive  $R$ -module. Then there are infinitely many isomorphism classes of indecomposable totally reflexive  $R$ -modules.*

This theorem says, for example, that if  $R$  is a ring of the form  $S[[X_1, \dots, X_n]]/(f)$  where  $S$  is a complete non-Gorenstein local ring and  $f$  is a monomial, then there are infinitely many isomorphism classes of indecomposable totally reflexive  $R$ -modules.

In the next section, we will prove Theorem 1.3 by using a theorem of Huneke and Leuschke [9] and Theorem 1.2. In the last section, we will give several applications of Theorem 1.3.

## 2. PROOF OF THE THEOREM

In this note,  $(R, \mathfrak{m}, k)$  is always a commutative noetherian local ring, and all  $R$ -modules are finitely generated. We denote by  $\text{mod } R$  the category of finitely generated  $R$ -modules. We begin with recalling the definition of a resolving subcategory.

**Definition 2.1.** A full subcategory  $\mathcal{X}$  of  $\text{mod } R$  is called *resolving* if the following hold.

- (1)  $\mathcal{X}$  contains  $R$ .
- (2)  $\mathcal{X}$  is closed under direct summands: if  $M \in \mathcal{X}$  and  $N$  is a direct summand of  $M$ , then  $N \in \mathcal{X}$ .
- (3)  $\mathcal{X}$  is closed under extensions: if there is an exact sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  in  $\text{mod } R$  with  $L, N \in \mathcal{X}$ , then  $M \in \mathcal{X}$ .
- (4)  $\mathcal{X}$  is closed under kernels of epimorphisms: if there is an exact sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  in  $\text{mod } R$  with  $M, N \in \mathcal{X}$ , then  $L \in \mathcal{X}$ .

In this definition, the condition (3) especially says that  $\mathcal{X}$  is closed under finite direct sums: if  $M, N \in \mathcal{X}$ , then  $M \oplus N \in \mathcal{X}$ . Hence from (1) it follows that  $\mathcal{X}$  contains all free  $R$ -modules. Therefore, by (4),  $\mathcal{X}$  is closed under syzygies: the (first) syzygy of any  $R$ -module in  $\mathcal{X}$  is also in  $\mathcal{X}$ .

For an  $R$ -module  $M$ , we denote by  $e(M)$  ( $\nu(M)$ , respectively) the multiplicity (the minimal number of generators, respectively) of  $M$ , namely,

$$\begin{cases} e(M) = \lim_{n \rightarrow \infty} \frac{d!}{n^d} \ell_R(M/\mathfrak{m}^n M), \\ \nu(M) = \dim_k(M \otimes_R k), \end{cases}$$

where  $d = \dim M$  and  $\ell_R(N)$  denotes the length of an  $R$ -module  $N$ . Huneke and Leuschke essentially proved the following theorem in [9, Theorems 1,3]. (They actually proved the theorem in the case where  $\mathcal{X}$  is the category of maximal Cohen-Macaulay  $R$ -modules.)

**Theorem 2.2** (Huneke-Leuschke). *Let  $\mathcal{X}$  be a full subcategory of  $\text{mod } R$  which is closed under extensions.*

- (1) *Let  $M, N \in \mathcal{X}$ . Assume that there are only finitely many isomorphism classes of  $R$ -modules  $X \in \mathcal{X}$  with  $e(X) = e(M) + e(N)$ , and denote by  $h$  the number of such isomorphism classes. Then  $\mathfrak{m}^h \text{Ext}_R^1(M, N) = 0$ .*
- (2) *Suppose that  $\mathcal{X}$  is resolving. Let  $M \in \mathcal{X}$ . Assume that there are only finitely many isomorphism classes of indecomposable  $R$ -modules  $X \in \mathcal{X}$  with  $e(X) \leq \nu(M) \cdot e(R)$ . Then  $M_{\mathfrak{p}}$  is  $R_{\mathfrak{p}}$ -free for any  $\mathfrak{p} \in \text{Spec } R - \{\mathfrak{m}\}$ .*

As a special case of the second assertion of this theorem, we obtain the following.

**Corollary 2.3.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\text{mod } R$ . Suppose that there are only finitely many isomorphism classes of indecomposable  $R$ -modules in  $\mathcal{X}$ . Then  $M_{\mathfrak{p}}$  is  $R_{\mathfrak{p}}$ -free for any  $M \in \mathcal{X}$  and  $\mathfrak{p} \in \text{Spec } R - \{\mathfrak{m}\}$ .*

Next, we recall the definition of a totally reflexive module. Let  $(-)^*$  be the  $R$ -dual functor  $\text{Hom}_R(-, R)$ .

**Definition 2.4.** We say that an  $R$ -module  $M$  is *totally reflexive* (or  $M$  has *G-dimension zero*) if the natural homomorphism  $M \rightarrow M^{**}$  is an isomorphism and  $\text{Ext}_R^i(M, R) = \text{Ext}_R^i(M^*, R) = 0$  for any  $i > 0$ .

We denote by  $\mathcal{G}$  the full subcategory of  $\text{mod } R$  consisting of all totally reflexive  $R$ -modules. Here, we state the properties of  $\mathcal{G}$  which will be used later.

- Lemma 2.5.**
- (1)  $\mathcal{G}$  is a resolving subcategory of  $\text{mod } R$ .
  - (2)  $\mathcal{G}$  is closed under  $R$ -dual, syzygies and finite direct sums.
  - (3) For any  $M \in \mathcal{G}$ , one has  $\text{depth } M = \text{depth } R$ .

*Proof.* (1) We refer to [3, Lemma 2.3], for example.

(2) It is easy to see from definition that if  $M \in \mathcal{G}$  then  $M^* \in \mathcal{G}$ . The remaining assertions follow from the arguments following Definition 2.1.

(3) See [2, Proposition (4.12)] or [4, Theorem (1.4.8)]. □

**Proposition 2.6.** *Suppose that there is a nonfree cyclic totally reflexive  $R$ -module  $M$  such that  $M_{\mathfrak{p}}$  is  $R_{\mathfrak{p}}$ -free for any  $\mathfrak{p} \in \text{Spec } R - \{\mathfrak{m}\}$ . Then  $\text{depth } R \leq 1$ .*

*Proof.* Suppose that  $\text{depth } R \geq 2$ . We want to derive a contradiction. We may assume  $M = R/I$ , where  $I$  is an ideal of  $R$  with  $0 \neq I \subseteq \mathfrak{m}$ . Setting  $J = (0 : I)$ , we have  $J \neq R$ , hence  $I + J \subseteq \mathfrak{m}$ . Dualizing the natural exact sequence  $0 \rightarrow I \xrightarrow{\theta} R \rightarrow R/I \rightarrow 0$  and using that  $R/I$  is assumed to be totally reflexive, gives an exact sequence

$$0 \rightarrow J \rightarrow R \xrightarrow{\kappa} I^* \rightarrow \text{Ext}^1(R/I, R) = 0,$$

where  $\kappa(1) = \theta$ . Thus we get an isomorphism  $\lambda : R/J \rightarrow I^*$ , where  $\lambda(\bar{1}) = \theta$ . Lemma 2.5(2) says that the  $R$ -modules  $I$  and  $I^*$  are totally reflexive. Hence so is  $R/J$ , and there are isomorphisms

$$I \rightarrow I^{**} \xrightarrow{\lambda^*} (R/J)^* \rightarrow (0 : J).$$

It is easy to check that the composite of these isomorphisms is an identity map; we obtain  $I = (0 : J)$ .

Fix  $\mathfrak{p} \in \text{Spec } R - \{\mathfrak{m}\}$ . Since  $(R/I)_{\mathfrak{p}}$  is  $R_{\mathfrak{p}}$ -free, one has either  $IR_{\mathfrak{p}} = 0$  or  $IR_{\mathfrak{p}} = R_{\mathfrak{p}}$ . If  $IR_{\mathfrak{p}} = 0$ , then  $\mathfrak{p}$  does not belong to  $\text{Supp } I$ . Noting that  $\text{Supp } I = V((0 : I)) = V(J)$ , we see that  $J$  is not contained in  $\mathfrak{p}$ . If  $IR_{\mathfrak{p}} = R_{\mathfrak{p}}$ , then  $I$  is not contained in  $\mathfrak{p}$ . This means that the ideal  $I + J$  is  $\mathfrak{m}$ -primary. There is an exact sequence

$$0 \rightarrow R/(I \cap J) \rightarrow R/I \oplus R/J \rightarrow R/(I + J) \rightarrow 0.$$

Since the  $R$ -module  $R/(I + J)$  has finite length, we have  $\text{depth}(R/(I + J)) = 0$ . According to Lemma 2.5 parts (2) and (3), we get  $\text{depth}(R/I \oplus R/J) = \text{depth } R \geq 2 > 0$ . Hence the depth lemma (cf. [10, Proposition 4.3.1]) yields  $\text{depth}(R/(I \cap J)) = 1$ .

Let  $x \in I \cap J$ . Then, since  $I = (0 : J)$  and  $J = (0 : I)$ , one has  $xJ = xI = 0$ , which implies that  $I + J \subseteq (0 : x)$ . As  $I + J$  is an  $\mathfrak{m}$ -primary ideal, so is  $(0 : x)$ . Hence  $\mathfrak{m}^r x = 0$  for some  $r > 0$ . It follows that  $I \cap J$  is an  $R$ -module of finite length. Noting that  $\text{depth } R \geq 2 > 0$ , one must have  $I \cap J = 0$ . Thus  $2 \leq \text{depth } R = \text{depth}(R/(I \cap J)) = 1$ . This contradiction proves the proposition.  $\square$

Now we can prove our main theorem.

*Proof of Theorem 1.3.* Suppose that  $\mathcal{G}$  has only finitely many isomorphism classes of indecomposable  $R$ -modules. Then Theorem 1.2 implies that  $\text{depth } R \geq 3$ . But  $M_{\mathfrak{p}}$  is  $R_{\mathfrak{p}}$ -free for any  $M \in \mathcal{G}$  and  $\mathfrak{p} \in \text{Spec } R - \{\mathfrak{m}\}$  by Lemma 2.5(1) and Corollary 2.3, hence  $\text{depth } R \leq 1$  by Proposition 2.6. This is a contradiction, which completes the proof of the theorem.  $\square$

### 3. APPLICATIONS

In this section, using Theorem 1.3, we give several results on the number of indecomposable totally reflexive modules.

**Corollary 3.1.** *Let  $(R, \mathfrak{m})$  be a henselian non-Gorenstein local ring. If there exist  $x, y \in \mathfrak{m}$  such that  $(0 : x) = (y)$  and  $(0 : y) = (x)$ , then there exist infinitely many nonisomorphic indecomposable totally reflexive  $R$ -modules.*

*Proof.* Noting that there is an exact sequence  $\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} \cdots$ , we can easily check that the  $R$ -module  $R/(x)$  is nonfree totally reflexive. Hence the assertion follows from Theorem 1.3.  $\square$

**Corollary 3.2.** *Let  $R$  be a complete non-Gorenstein local ring. Then  $S = R[[X_1, X_2, \dots, X_n]]/(X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n})$  admits infinitely many nonisomorphic indecomposable totally reflexive modules.*

*Proof.* Note that  $S$  is faithfully flat over  $R$ . Hence  $S$  is also a complete non-Gorenstein local ring. To show the corollary, we may assume  $a_1 > 0$ . Then it is easily seen that  $(0 :_S X_1) = X_1^{a_1-1} X_2^{a_2} \cdots X_n^{a_n} S$  and  $(0 :_S X_1^{a_1-1} X_2^{a_2} \cdots X_n^{a_n}) = X_1 S$ . Thus we can apply Corollary 3.1.  $\square$

We denote by  $\mathfrak{D}(R)$  the derived category of  $\text{mod } R$ . Recall that an  $R$ -module  $C$  is called *semidualizing* if the natural morphism  $R \rightarrow \mathbf{R}\text{Hom}_R(C, C)$  is an isomorphism in  $\mathfrak{D}(R)$ , equivalently, the natural homomorphism  $R \rightarrow \text{Hom}_R(C, C)$  is an isomorphism and  $\text{Ext}_R^i(C, C) = 0$  for any  $i > 0$ . In the following, we consider the idealization  $S = R \times C$  of a semidualizing module  $C$  over  $R$ . There are two natural homomorphisms  $\phi : R \rightarrow S$  and  $\psi : S \rightarrow R$ , which are given by  $\phi(a) = (a, 0)$  and  $\psi(a, x) = a$ . Through the homomorphism  $\phi$  ( $\psi$ , respectively), one can regard an  $S$ -module ( $R$ -module, respectively) as an  $R$ -module ( $S$ -module, respectively). Note that through the composite of these homomorphisms the  $R$ -module structure is preserved since  $\psi\phi$  is the identity map of  $R$ , but the  $S$ -module structure is not preserved in general.

**Lemma 3.3.** [8, Lemma 3.2] *Let  $C$  be a semidualizing  $R$ -module, and set  $S = R \times C$ . Then there is a natural isomorphism  $\mathbf{R}\text{Hom}_R(-, C) \cong \mathbf{R}\text{Hom}_S(-, S)$  of functors on  $\mathfrak{D}(R)$ .*

Using this lemma, we can get the following result. It says that a non-Gorenstein ring which is the idealization of a semidualizing module over a henselian local ring has infinitely many nonisomorphic totally reflexive modules.

**Corollary 3.4.** *Let  $R$  be a henselian local ring,  $C$  a semidualizing  $R$ -module, and  $S = R \times C$  the idealization. Suppose that there are only finitely many nonisomorphic indecomposable totally reflexive  $S$ -modules. Then  $S$  is Gorenstein. Hence  $R$  is Cohen-Macaulay, and  $C$  is a canonical module of  $R$ .*

*Proof.* The last assertion follows from [5, Theorem 5.6]. (One can also prove it by using the isomorphism  $\mathbf{R}\text{Hom}_R(k, C) \cong \mathbf{R}\text{Hom}_S(k, S)$  induced by Lemma 3.3.) Lemma 3.3 gives isomorphisms  $\mathbf{R}\text{Hom}_S(R, S) \cong C$  and  $\mathbf{R}\text{Hom}_S(C, S) \cong \mathbf{R}\text{Hom}_R(C, C) \cong R$ . We easily see from these isomorphisms that  $R$  is a totally reflexive  $S$ -module. Note that  $R$  is a nonfree cyclic  $S$ -module, and  $S$  is henselian since  $S$  is module-finite over  $R$ . Therefore, Theorem 1.3 implies that  $S$  is Gorenstein.  $\square$

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