

Thick tensor ideals of right bounded derived categories of commutative rings

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ABSTRACT. Let R be a commutative noetherian ring. Denote by $\mathbf{D}^-(R)$ the derived category of cochain complexes X of finitely generated R -modules with $H^i(X) = 0$ for $i \gg 0$. Then $\mathbf{D}^-(R)$ has the structure of a tensor triangulated category with tensor product $\otimes_R^{\mathbf{L}}$ and unit object R . In this article, we study thick tensor ideals of $\mathbf{D}^-(R)$, i.e., thick subcategories closed under the tensor action by each object in $\mathbf{D}^-(R)$, and investigate the Balmer spectrum $\mathrm{Spc} \mathbf{D}^-(R)$ of $\mathbf{D}^-(R)$, i.e., the set of prime thick tensor ideals of $\mathbf{D}^-(R)$. The results of the author given in this article are all taken from [10].

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Introduction

This article makes a report of the series of lectures given by the author at the Workshop and International Conference on Representations of Algebras (ICRA 2016), which was held at Syracuse University in August, 2016. Those lectures are based on joint work with Hiroki Matsui. The complete proofs of the results in this article that are due to Matsui and the author are all stated in [10], together with more detailed information and other related results.

Tensor triangular geometry is a theory established by Balmer [2] at the beginning of the current century. Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbf{1})$ be an (essentially small) *tensor*

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triangulated category, that is, a triangulated category \mathcal{T} equipped with symmetric tensor product \otimes and unit object $\mathbb{1}$. A (thick tensor) *ideal* of \mathcal{T} is by definition a thick subcategory of \mathcal{T} which is closed under the action of \mathcal{T} by \otimes . A proper ideal \mathcal{P} of \mathcal{T} is called *prime* if it satisfies:

$$X \otimes Y \in \mathcal{P} \implies X \in \mathcal{P} \text{ or } Y \in \mathcal{P}.$$

Prime ideals of tensor triangulated categories turn out to behave similarly to prime ideals of commutative rings; both share a lot of analogous properties. Among other things, the *Balmer spectrum* $\mathbf{Spc} \mathcal{T}$ of \mathcal{T} , which is by definition the set of prime ideals of \mathcal{T} , has the structure of a topological space, corresponding to the fact that the Zariski spectrum $\mathrm{Spec} R$ of a commutative ring R has a Zariski topology. Tensor triangular geometry studies Balmer spectra and develops commutative-algebraic and algebro-geometric observations. It is related to a lot of areas of mathematics, including commutative algebra, algebraic geometry, stable homotopy theory, modular representation theory, motivic theory, noncommutative topology and symplectic geometry. As Balmer [4] addressed an invited lecture at the International Congress of Mathematicians (ICM) in 2010, tensor triangular geometry has been attracting a great deal of attention.

Let R be a commutative noetherian ring. Let $\mathbf{D}^-(R)$ be the right bounded derived category of finitely generated R -modules. It is then a routine to verify that

$$(\mathbf{D}^-(R), \otimes_R^{\mathbf{L}}, R)$$

is a tensor triangulated category. The main topics of the series of lectures at ICRA 2016 by the author concern the structure of the ideals of $\mathbf{D}^-(R)$ and the structure of the Balmer spectrum $\mathbf{Spc} \mathbf{D}^-(R)$ of $\mathbf{D}^-(R)$.

The organization of this article is as follows. In Section 1, we make a review of some general properties of tensor triangulated categories and their Balmer spectra, which have been obtained by Balmer. From Section 2 on, we study the tensor triangulated category $\mathbf{D}^-(R)$ for a commutative noetherian ring R . In Section 2, we completely classify the compactly generated thick tensor ideals and the cocompactly ones of $\mathbf{D}^-(R)$. In this section, it turns out that a thick tensor ideal is compactly generated if and only if it is cocompactly generated, and when this is the case, we simply call it compact. In Section 3, we explore the topological structure of the Balmer spectrum of $\mathbf{D}^-(R)$, and make correspondences among certain classes of thick tensor ideals and subsets of the Zariski and Balmer spectra. We introduce the notion of tame thick tensor ideals, and compare those ideals with compact and radical ones. In Section 4, we investigate what is going on in the case where R is a discrete valuation ring. As we observe later, everything is clarified when R is artinian. Hence, the case of discrete valuation rings should be the first nontrivial case, but in fact, it turns out that even in this case the structure of $\mathbf{D}^-(R)$ is highly complicated.

1. Tensor triangulated categories and Balmer spectra

In this section, we introduce some of Balmer's works on general tensor triangulated categories. All the materials in this section are taken from [2, 3, 4]. First of all, we recall the definition of a tensor triangulated category.

DEFINITION 1.1. A *tensor triangulated category* $(\mathcal{T}, \otimes, \mathbb{1})$ is a triangulated category \mathcal{T} equipped with symmetric tensor product \otimes and unit object $\mathbb{1}$. To be more

precise, \mathcal{T} is both a triangulated category and a symmetric monoidal category such that the triangulated and symmetric monoidal structures are compatible.

Here are several examples of a tensor triangulated category. Note that all of them are essentially small.

EXAMPLE 1.2.

- (1) Let X be a (quasi-compact and quasi-separated) scheme. Denote by $\mathbf{D}^{\text{perf}}(X)$ the derived category of perfect complexes of \mathcal{O}_X -modules. Then $(\mathbf{D}^{\text{perf}}(X), \otimes_{\mathcal{O}_X}^{\mathbf{L}}, \mathcal{O}_X)$ is a tensor triangulated category.
- (2) Let R be a commutative ring. Denote by $\mathbf{K}^{\text{b}}(\text{proj } R)$ the homotopy category of bounded complexes of finitely generated projective R -modules. Then $(\mathbf{K}^{\text{b}}(\text{proj } R), \otimes_R, R)$ is a tensor triangulated category. This is nothing but the affine case of (1).
- (3) Let k be a field of positive characteristic, and G a finite group (scheme over k). Denote by $\underline{\text{mod}} kG$ the stable category of finitely generated kG -modules. Then $(\underline{\text{mod}} kG, \otimes_k, k)$ is a tensor triangulated category.
- (4) Let k, G be as in (3). Denote by $\mathbf{D}^{\text{b}}(\text{mod } kG)$ the derived category of bounded complexes of finitely generated kG -modules. Then $(\mathbf{D}^{\text{b}}(\text{mod } kG), \otimes_k, k)$ is a tensor triangulated category.
- (5) Let R be a commutative noetherian ring. Denote by $\mathbf{D}^-(\text{mod } R)$ the derived category of homologically right bounded complexes of finitely generated R -modules. Then $(\mathbf{D}^-(\text{mod } R), \otimes_R^{\mathbf{L}}, R)$ is a tensor triangulated category. This tensor triangulated category plays a main role in this article.

Next, we give the definitions of a (thick tensor) ideal and a Balmer spectrum. We recall here that a *thick* subcategory of a triangulated category is by definition a nonempty full subcategory which is closed under direct summands, shifts and cones.

DEFINITION 1.3. Let \mathcal{T} be an essentially small tensor triangulated category.

- (1) A thick subcategory \mathcal{I} of \mathcal{T} is a (*tensor*) *ideal* if it satisfies the following implication.

$$a \in \mathcal{T}, x \in \mathcal{I} \implies a \otimes x \in \mathcal{I}.$$

This is an analogue of an ideal of a commutative ring.

- (2) An ideal \mathcal{I} of \mathcal{T} is *radical* if $\mathcal{I} = \sqrt{\mathcal{I}}$, where

$$\sqrt{\mathcal{I}} := \{a \in \mathcal{T} \mid \underbrace{a \otimes \cdots \otimes a}_n \in \mathcal{I} \text{ for some } n > 0\}$$

is the *radical* of \mathcal{I} . These are analogues of a radical ideal and the radical of an ideal of a commutative ring, respectively.

- (3) A proper ideal \mathcal{P} of \mathcal{T} is *prime* if it satisfies the following implication.

$$x \otimes y \in \mathcal{P} \implies x \in \mathcal{P} \text{ or } y \in \mathcal{P}.$$

This is an analogue of a prime ideal of a commutative ring.

- (4) The *Balmer spectrum* of \mathcal{T} is defined by:

$$\text{Spc } \mathcal{T} = \{\text{Prime ideals of } \mathcal{T}\}.$$

This corresponds to the Zariski spectrum $\text{Spec } R$ of a commutative ring R .

- (5) The *Balmer support* of an object x of \mathcal{T} is defined by:

$$\mathbf{supp}(x) = \{\mathcal{P} \in \mathbf{Spc} \mathcal{T} \mid x \notin \mathcal{P}\}.$$

This corresponds to the subset $V(f) = \{\mathfrak{p} \in \mathbf{Spec} R \mid f \in \mathfrak{p}\}$ of $\mathbf{Spec} R$ for an element f of R . Note that the containment is opposite.

- (6) We put

$$\mathbf{U}(x) := \mathbf{supp}(x)^{\mathfrak{c}} = \{\mathcal{P} \in \mathbf{Spc} \mathcal{T} \mid x \in \mathcal{P}\}.$$

This corresponds to the subset $D(f) = \{\mathfrak{p} \in \mathbf{Spec} R \mid f \notin \mathfrak{p}\}$ of $\mathbf{Spec} R$.

Throughout the rest of this article, we assume that all tensor triangulated categories are essentially small, so that we can always define their Balmer spectra.

We make the definitions of a maximal ideal and a minimal prime of a tensor triangulated category.

DEFINITION 1.4. Let \mathcal{T} be a tensor triangulated category.

- (1) An ideal of \mathcal{T} is said to be a *maximal ideal* of \mathcal{T} if it is a proper ideal of \mathcal{T} which is maximal with respect to the inclusion relation. We denote by $\max(\mathbf{Spc} \mathcal{T})$ the set of maximal ideals of \mathcal{T} .
- (2) An ideal of \mathcal{T} is said to be a *minimal prime* of \mathcal{T} if it is a prime ideal of \mathcal{T} which is minimal with respect to the inclusion relation. We denote by $\min(\mathbf{Spc} \mathcal{T})$ the set of minimal primes of \mathcal{T} .

Each Balmer spectrum has the structure of a topological space such that the Balmer supports are closed subsets. We state this here together with several fundamental properties which will often be used later.

PROPOSITION 1.5 ([2]). Let \mathcal{T} be a tensor triangulated category.

- (1) $\mathbf{Spc} \mathcal{T}$ is a topological space with an open basis $\{\mathbf{U}(x)\}_{x \in \mathcal{T}}$.
- (2) Every proper ideal of \mathcal{T} is contained in a maximal ideal.
- (3) Maximal ideals of \mathcal{T} are prime.
- (4) Every prime ideal of \mathcal{T} contains a minimal prime.
- (5) For each $\mathcal{P} \in \mathbf{Spc} \mathcal{T}$ the closure $\overline{\{\mathcal{P}\}}$ of $\{\mathcal{P}\}$ is irreducible, and described as follows.

$$(1.5.1) \quad \overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathbf{Spc} \mathcal{T} \mid \mathcal{Q} \subseteq \mathcal{P}\}.$$

Conversely, any nonempty irreducible closed subset of $\mathbf{Spc} \mathcal{T}$ has this form.

- (6) The open subset $\mathbf{U}(x)$ of $\mathbf{Spc} \mathcal{T}$ is quasi-compact for each $x \in \mathcal{T}$. Conversely, any nonempty quasi-compact open subset of $\mathbf{Spc} \mathcal{T}$ has this form.
- (7) For an ideal \mathcal{I} of \mathcal{T} one has

$$\sqrt{\mathcal{I}} = \bigcap_{\mathcal{I} \subseteq \mathcal{P} \in \mathbf{Spc} \mathcal{T}} \mathcal{P}.$$

The equality (1.5.1) corresponds to the equality

$$\overline{\{\mathfrak{p}\}} = \{\mathfrak{q} \in \mathbf{Spec} R \mid \mathfrak{q} \supseteq \mathfrak{p}\}$$

of subsets of $\mathbf{Spec} R$ for a commutative ring R and a prime ideal \mathfrak{p} of R . Again, the containment is opposite.

Thus, ideals of tensor triangulated categories have a lot of similar properties to ideals of commutative rings.

For a full subcategory \mathcal{X} of \mathcal{T} and a subset S of $\mathrm{Spc} \mathcal{T}$, set

$$\begin{aligned} \mathrm{supp} \mathcal{X} &= \bigcup_{x \in \mathcal{X}} \mathrm{supp}(x), \\ \mathrm{supp}^{-1} S &= \{x \in \mathcal{T} \mid \mathrm{supp}(x) \in S\}. \end{aligned}$$

The following theorem is a celebrated result due to Balmer [2, Theorem 4.10].

THEOREM 1.6 (Balmer (2005)). *Let \mathcal{T} be a tensor triangulated category. Then there is a one-to-one correspondence*

$$\{\text{Radical ideals of } \mathcal{T}\} \begin{array}{c} \xrightarrow{\mathrm{supp}} \\ \xleftarrow{\mathrm{supp}^{-1}} \end{array} \{\text{Thomason subsets of } \mathrm{Spc} \mathcal{T}\}.$$

Here, a subset A of a topological space X is said to be *Thomason* if one can write

$$A = \bigcup_{i \in I} B_i$$

for some family $\{B_i\}_{i \in I}$ of subsets of X such that $B_i^c = X \setminus B_i$ is a quasi-compact open subset. A subset C of X is said to be *specialization-closed* if it satisfies the implication

$$x \in C \implies \overline{\{x\}} \subseteq C.$$

We notice that this condition is equivalent to saying that C is a (possibly infinite) union of closed subsets. Therefore, a Thomason subset is always specialization-closed. The name of a Thomason subset comes from the fact that for a quasi-compact quasi-separated scheme X , Thomason [13] gives a complete classification of the ideals of $\mathrm{D}^{\mathrm{perf}}(X)$ in terms of the Thomason subsets of the underlying topological space of X .

Theorem 1.7 says that for a given tensor triangulated category \mathcal{T} the understanding of the structure of the Balmer spectrum of \mathcal{T} provides a complete classification of the radical ideals of \mathcal{T} . Since each ideal of \mathcal{T} is the kernel of some tensor triangulated functor from \mathcal{T} and vice versa, classifying ideals of \mathcal{T} leads us to the understanding of the structure of tensor triangulated functors from \mathcal{T} . In this sense, the above theorem is quite meaningful.

For each tensor triangulated category \mathcal{T} one can define the structure sheaf $\mathcal{O}_{\mathcal{T}}$ on \mathcal{T} , and then the Balmer spectrum $\mathrm{Spc} \mathcal{T}$ has the structure of a locally ringed space [4, Constructions 24 and 29]. More precisely, for each quasi-compact open subset U of $\mathrm{Spc} \mathcal{T}$ we define

$$\mathcal{T}(U) := (\mathcal{T} / \mathrm{supp}^{-1}(U^c))^{\natural},$$

where $(-)^{\natural}$ stands for the idempotent completion. Then it holds that

$$\mathrm{Spc} \mathcal{T}(U) \cong U.$$

The assignment $U \mapsto \mathrm{End}_{\mathcal{T}(U)}(\mathbb{1})$ induces a presheaf of commutative rings, and we define the structure sheaf $\mathcal{O}_{\mathcal{T}}$ on \mathcal{T} as its sheafification. Thus we obtain a locally ringed space

$$\mathrm{Spec} \mathcal{T} := (\mathrm{Spc} \mathcal{T}, \mathcal{O}_{\mathcal{T}}).$$

The following theorem due to Balmer is also well-known. We refer the reader to [2, Theorem 6.3] and [4, Theorem 57]; see also [3, Proposition 6.11].

THEOREM 1.7 (Balmer (2005, 2010)).

- (1) *Let X be a quasi-compact quasi-separated scheme. Then there is an isomorphism*

$$\mathrm{Spec} \mathrm{D}^{\mathrm{perf}}(X) \cong X$$

of locally ringed spaces.

- (2) *Let k be a field of positive characteristic, and G a finite group (scheme over k). Then there are isomorphisms*

$$\begin{aligned} \mathrm{Spec} \mathrm{D}^{\mathrm{b}}(\mathrm{mod} kG) &\cong \mathrm{Spec}^{\mathrm{h}} \mathrm{H}^{\bullet}(G, k), \\ \mathrm{Spec}(\underline{\mathrm{mod}} kG) &\cong \mathrm{Proj} \mathrm{H}^{\bullet}(G, k) \end{aligned}$$

of locally ringed spaces.

Here $\mathrm{H}^{\bullet}(G, k)$ stands for the group cohomology ring. For a graded-commutative ring A , we denote by $\mathrm{Spec}^{\mathrm{h}} A$ the set of homogeneous prime ideals of A , that is, homogeneous ideals \mathfrak{p} of A such that for two homogeneous elements $x, y \in A$ one has

$$xy \in \mathfrak{p} \implies x \in \mathfrak{p} \text{ or } y \in \mathfrak{p}.$$

For a commutative nonnegatively graded ring R we denote by $\mathrm{Proj} R$ the set of homogeneous prime ideals of R that do not contain $R_+ = \bigoplus_{i>0} R_i$. Note that $\mathrm{Proj} \mathrm{H}^{\bullet}(G, k)$ is nothing but the (projective) support variety $\mathcal{V}_G(k)$.

The isomorphism in Theorem 1.7(1) says that a scheme X is reconstructed from its derived category $\mathrm{D}^{\mathrm{perf}}(X)$; see also [1]. This is actually because of the tensor structure of $\mathrm{D}^{\mathrm{perf}}(X)$. Indeed, only from the triangulated structure of $\mathrm{D}^{\mathrm{perf}}(X)$ the original scheme X cannot be reconstructed, since there are a lot of derived equivalences of nonsingular algebraic varieties (e.g. the Fourier–Mukai transformation).

The second isomorphism in Theorem 1.7(2) is obtained by restricting the first one. Key roles in the proof of Theorem 1.7 are played by the classification theorems of ideals due to Hopkins [9], Neeman [11], Thomason [13], Benson–Carlson–Rickard [5] and Friedlander–Pevtsova [8]; see also the works of Benson–Iyengar–Krause [6] and Benson–Iyengar–Krause–Pevtsova [7]. The Balmer spectra are described for some other tensor triangulated categories by several authors; details can be found in [4].

Let $(\mathcal{T}, \otimes, \mathbb{1})$ be a tensor triangulated category. Balmer [3] constructs a continuous map

$$\rho_{\mathcal{T}}^{\bullet} : \mathrm{Spc} \mathcal{T} \rightarrow \mathrm{Spec}^{\mathrm{h}} \mathrm{R}_{\mathcal{T}}^{\bullet},$$

which is given by

$$\rho_{\mathcal{T}}^{\bullet}(\mathcal{P}) := (f \in \mathrm{R}_{\mathcal{T}}^{\bullet} \mid \mathrm{cone}(f) \notin \mathcal{P}).$$

Here,

$$\mathrm{R}_{\mathcal{T}}^{\bullet} = \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Sigma^{\bullet} \mathbb{1})$$

is a graded-commutative ring.

It is seen that for $\mathcal{T} = \mathrm{K}^{\mathrm{b}}(\mathrm{proj} R)$ with R being a commutative ring we have $\mathrm{R}_{\mathcal{T}}^{\bullet} = R$, and it is also observed that for $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(\mathrm{mod} kG)$ with k being a field of positive characteristic and G being a finite group (scheme over k) we have $\mathrm{R}_{\mathcal{T}}^{\bullet} = \mathrm{H}^{\bullet}(G, k)$. It is shown by Balmer [3, Propositions 8.1 and 8.5] that the isomorphism in Theorem 1.7(1) in the affine case, and the first isomorphism in Theorem 1.7(2) are given by the map $\rho_{\mathcal{T}}^{\bullet}$ given above. Thus the following conjecture has been presented by Balmer [4, Conjecture 72] in his invited lecture at the International Congress of Mathematicians (ICM), which was held in 2010 at Hyderabad.

CONJECTURE 1.8 (Balmer, ICM 2010). The map $\rho_{\mathcal{T}}^{\bullet}$ is (locally) injective if \mathcal{T} is algebraic as a triangulated category.

Let $f : X \rightarrow Y$ be a continuous map of topological spaces. We say that f is *locally injective* at a point $x \in X$ if there exists a neighborhood N of x such that the restriction $f|_N$ of f on N is injective. The map f is called *locally injective* if for all points $x \in X$ it is locally injective at x . Also, recall that a triangulated category is called *algebraic* if it is described as the stable category of a Frobenius exact category.

It is known that the conjecture does not hold for a non-algebraic triangulated category; indeed, if \mathcal{T} is the Spanier–Whitehead stable homotopy category $\mathrm{SH}^{\mathrm{fn}}$ of finite pointed CW-complexes, then $\rho_{\mathcal{T}}^{\bullet}$ is not injective; see [4, Theorem 51]. On the other hand, as we have seen above, the conjecture does hold for $\mathrm{K}^{\mathrm{b}}(\mathrm{proj} R)$ and $\mathrm{D}^{\mathrm{b}}(\mathrm{mod} kG)$.

Now we introduce some notation, which will be used throughout the rest of this article.

NOTATION 1.9.

- (1) Let R be a commutative noetherian ring.
- (2) We denote by $\mathrm{Spec} R$ the *Zariski spectrum* of R , namely, the set of prime ideals of R equipped with the Zariski topology.
- (3) For an ideal I of R we define $V(I)$ the set of prime ideals of R containing I , and put $D(I) = V(I)^{\mathrm{c}} = \mathrm{Spec} R \setminus V(I)$.
- (4) The set of maximal ideals (respectively, minimal primes) of R is denoted by $\mathrm{Max} R$ (respectively, $\mathrm{Min} R$).
- (5) We denote by $\mathrm{mod} R$ the category of finitely generated R -modules, and by $\mathrm{proj} R$ the full subcategory of $\mathrm{mod} R$ consisting of finitely generated projective R -modules.
- (6) We denote by $\mathrm{D}^*(R)$ the derived category $\mathrm{D}^*(\mathrm{mod} R)$ of the abelian category $\mathrm{mod} R$, and by $\mathrm{K}^*(R)$ the homotopy category $\mathrm{K}^*(\mathrm{proj} R)$ of the additive category $\mathrm{proj} R$, where $*$ \in $\{-, \mathrm{b}\}$. There are obvious inclusions

$$\mathrm{K}^{\mathrm{b}}(R) \subseteq \mathrm{D}^{\mathrm{b}}(R) \subseteq \mathrm{D}^{-}(R).$$

Taking projective resolutions induces an equivalence

$$\mathrm{D}^{-}(R) \cong \mathrm{K}^{-}(R)$$

of tensor triangulated categories. We will often identify $\mathrm{D}^{-}(R)$ with $\mathrm{K}^{-}(R)$ via this equivalence.

From the next section on, we will investigate the structure of $\mathrm{D}^{-}(R)$ as a tensor triangulated category. We close this section by giving several comments about how hard it is.

DIFFICULTIES FOR $\mathrm{D}^{-}(R)$. The tensor triangulated category $\mathrm{D}^{-}(R)$ possesses a lot of defects on its structure, compared with the other well-established tensor triangulated categories:

- (1) $\mathrm{D}^{-}(R)$ does not have arbitrary products or coproducts. (However, it does have some specific infinite coproducts, which will somehow play a crucial role in the proofs of our results.)
- (2) $\mathrm{D}^{-}(R)$ is not closed under duals. For example, in the case where R is an algebra over a field k , $\mathrm{D}^{-}(R)$ is not closed under k -duals.

- (3) In particular, $D^-(R)$ is never rigid. Recall that a triangulated category \mathcal{T} is called *rigid* if there exist an exact functor $D : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}$ and a functorial isomorphism

$$\text{Hom}_{\mathcal{T}}(a \otimes b, c) \cong \text{Hom}_{\mathcal{T}}(a, D(b) \otimes c)$$

for $a, b, c \in \mathcal{T}$. In fact, $D^-(R)$ is even never closed as a symmetric monoidal category. There are a lot of results on rigid tensor triangulated categories, but we cannot use them for $D^-(R)$.

- (4) One has

$$\text{thick}_{D^-(R)} R \neq D^-(R).$$

Indeed, the left hand side coincides with $K^b(R)$. There are several results on tensor triangulated categories $(\mathcal{T}, \otimes, \mathbb{1})$ satisfying $\text{thick}_{\mathcal{T}} \mathbb{1} = \mathcal{T}$, but they are not available for $D^-(R)$.

Thus, results in the literature are quite limited on tensor triangulated categories that can be applied to our tensor triangulated category $D^-(R)$.

2. Compactly and cocompactly generated thick tensor ideals of $D^-(R)$

In this section, we classify compactly or cocompactly generated ideals of the tensor triangulated category $D^-(R)$. We begin with recalling the definitions of compact and cocompact objects.

DEFINITION 2.1. Let \mathcal{T} be a triangulated category.

- (1) An object $M \in \mathcal{T}$ is called *compact* (respectively, *cocompact*) if the natural morphism

$$\begin{aligned} & \bigoplus_{\lambda \in \Lambda} \text{Hom}_{\mathcal{T}}(M, N_{\lambda}) \rightarrow \text{Hom}_{\mathcal{T}}(M, \bigoplus_{\lambda \in \Lambda} N_{\lambda}) \\ & \left(\text{respectively, } \bigoplus_{\lambda \in \Lambda} \text{Hom}_{\mathcal{T}}(N_{\lambda}, M) \rightarrow \text{Hom}_{\mathcal{T}}\left(\prod_{\lambda \in \Lambda} N_{\lambda}, M\right) \right) \end{aligned}$$

is an isomorphism for all families $\{N_{\lambda}\}_{\lambda \in \Lambda}$ of objects of \mathcal{T} such that the coproduct $\bigoplus_{\lambda \in \Lambda} N_{\lambda}$ (respectively, the product $\prod_{\lambda \in \Lambda} N_{\lambda}$) exists in \mathcal{T} .

- (2) We denote by \mathcal{T}^c (respectively, \mathcal{T}^{cc}) the full subcategory of \mathcal{T} consisting of compact (respectively, cocompact) objects of \mathcal{T} .
- (3) An ideal of \mathcal{T} is said to be *compactly generated* (respectively, *cocompactly generated*) if it is generated by some compact (respectively, cocompact) objects of \mathcal{T} as an ideal.

The following equalities hold for compactly and cocompactly generated ideals of $D^-(R)$.

FACT 2.2. There are equalities

$$\begin{aligned} D^-(R)^c &= K^b(R), \\ D^-(R)^{\text{cc}} &= D^b(R). \end{aligned}$$

The second equality is due to Oppermann–Stovicek [12, Theorem 18]. The first equality in the above fact is well-known, and actually proved along the same lines as in the proof of the fact that the compact objects of the unbounded derived category of all R -modules are the perfect complexes over R . More precisely, take

a complex $X \in K^-(R)^c$ and denote by d^n the n -th differential map for each $n \in \mathbb{Z}$. Let C^n be the cokernel of the map d^{n-1} . There is a natural morphism

$$X \rightarrow \bigoplus_{n \in \mathbb{Z}} C^n[-n],$$

which factors through a finite direct sum by the compactness of X . Thus $X \rightarrow C^n[-n]$ is a zero morphism for all $n \ll 0$, and X is quasi-isomorphic to a truncation of X .

Next, let us recall the definition of the (usual) support of a chain complex. Note that this notion is different from that of a Balmer support introduced in the previous section.

DEFINITION 2.3.

- (1) Let $X \in D^-(R)$ be a complex. The *support* of X is defined to be the union of the supports (as R -modules) of homologies of X . One has equalities

$$\begin{aligned} \text{supph } X &= \bigcup_{i \in \mathbb{Z}} \text{supph } H^i(X) \\ (2.3.1) \quad &= \{\mathfrak{p} \in \text{Spec } R \mid X_{\mathfrak{p}} \neq 0\} \\ &= \{\mathfrak{p} \in \text{Spec } R \mid \kappa(\mathfrak{p}) \otimes_R^{\mathbf{L}} X \neq 0\} \end{aligned}$$

of subsets of $\text{Spec } R$, where $\kappa(\mathfrak{p})$ stands for the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$.

- (2) For a full subcategory \mathcal{X} of $D^-(R)$, set

$$\text{supph } \mathcal{X} = \bigcup_{X \in \mathcal{X}} \text{supph } X.$$

It is easy to see that the following hold.

- $\text{supph } \mathcal{X}$ is a specialization-closed subset of $\text{Spec } R$.
- There is an equality $\text{supph } \mathcal{X} = \text{supph}(\text{thick}^{\otimes} \mathcal{X})$.

Here, $\text{thick}^{\otimes} \mathcal{X}$ stands for the ideal generated by \mathcal{X} , that is, the smallest ideal of $D^-(R)$ containing \mathcal{X} .

- (3) For a subset S of $\text{Spec } R$, set

$$\langle S \rangle = \text{thick}_{D^-(R)}^{\otimes} \{R/\mathfrak{p} \mid \mathfrak{p} \in S\}.$$

The second equality in (2.3.1) holds even for unbounded complexes of non-finitely generated R -modules, while the third equality only holds for complexes in $D^-(R)$. To see this, let X be a complex in $D^-(R)$ and \mathfrak{p} a prime ideal of R . Then, by virtue of Nakayama's lemma, we have

$$\begin{aligned} \sup(\kappa(\mathfrak{p}) \otimes_R^{\mathbf{L}} X) &= \sup(\kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} X_{\mathfrak{p}}) \\ &= \sup \kappa(\mathfrak{p}) + \sup X_{\mathfrak{p}} \\ &= \sup X_{\mathfrak{p}}. \end{aligned}$$

This proves the third equality in (2.3.1). (For a complex X , the *supremum* $\sup X$ of X is by definition the supremum of integers i such that $H^i(X) \neq 0$.)

The following theorem is the first main result of this article.

THEOREM 2.4 ([10, Theorem 2.12]). *There is a one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{Cocompactly generated} \\ \text{ideals of } \mathcal{D}^-(R) \end{array} \right\} \begin{array}{c} \xrightarrow{\text{supph}} \\ \xleftarrow{1-1} \\ \langle \rangle \end{array} \left\{ \begin{array}{l} \text{Specialization-closed} \\ \text{subsets of } \text{Spec } R \end{array} \right\}.$$

Thus the cocompactly generated ideals of $\mathcal{D}^-(R)$ are completely classified.

In fact, this one-to-one correspondence is not just a bijection of sets. For ideals \mathcal{X}, \mathcal{Y} of $\mathcal{D}^-(R)$, define $\mathcal{X} \wedge \mathcal{Y}$ and $\mathcal{X} \vee \mathcal{Y}$ by:

$$\begin{cases} \mathcal{X} \wedge \mathcal{Y} = \text{thick}^\otimes \{X \otimes_R^{\mathbf{L}} Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}, \\ \mathcal{X} \vee \mathcal{Y} = \text{thick}^\otimes (\mathcal{X} \cup \mathcal{Y}). \end{cases}$$

It is then seen that for specialization-closed subsets A, B of $\text{Spec } R$ there are equalities

$$\begin{cases} \langle A \rangle \wedge \langle B \rangle = \langle A \cap B \rangle, \\ \langle A \rangle \vee \langle B \rangle = \langle A \cup B \rangle. \end{cases}$$

Using these equalities, one can show that the set of cocompactly generated ideals of $\mathcal{D}^-(R)$ forms a lattice with join \vee and meet \wedge , and that the bijections in the theorem are lattice isomorphisms; see [10, Proposition 2.17].

On the other hand, using the above theorem, we observe that the assignments $\mathcal{X} \mapsto \mathcal{X} \cap \mathcal{K}^b(R)$ and $\text{thick}^\otimes \mathcal{Y} \leftrightarrow \mathcal{Y}$ make a one-to-one correspondence

$$\{\text{Cocompactly generated ideals of } \mathcal{D}^-(R)\} \leftrightarrow \{\text{Thick subcategories of } \mathcal{K}^b(R)\};$$

see [10, Corollary 2.14].

To prove the theorem, we need to extend the Hopkins–Neeman smash nilpotence theorem as follows; see [10, Theorem 2.7].

LEMMA 2.5 (Generalized smash nilpotence). *Let $f : X \rightarrow Y$ be a morphism in $\mathcal{K}^-(R)$ such that $Y \in \mathcal{K}^b(R)$. If $f \otimes_R \kappa(\mathfrak{p}) = 0$ for all prime ideals \mathfrak{p} of R , then $f^{\otimes t} = 0$ for some integer $t > 0$.*

We do not state the proof of this lemma. Instead, we give several comments on the proof.

REMARK 2.6.

- (1) If we assume further that $X \in \mathcal{K}^b(R)$, then the assertion of the lemma is nothing but the original smash nilpotence due to Hopkins [9, Theorem 10] and Neeman [11, Theorem 1.1]. In the proof of the original smash nilpotence, one can reduce to the case where $X = R$ by replacing the morphism $f : X \rightarrow Y$ with a morphism $f' : R \rightarrow \mathbf{RHom}_R(X, Y)$ via the isomorphism

$$\text{Hom}_{\mathcal{K}^b(R)}(R, \mathbf{RHom}_R(X, Y)) \cong \text{Hom}_{\mathcal{K}^b(R)}(X, Y).$$

Thanks to this reduction, one can identify the morphism $f \in \text{Hom}_{\mathcal{K}^b(R)}(R, Y)$ with the element $f(1) \in H^0 Y$, which plays a key role in the proof of the original smash nilpotence.

- (2) We show and use the following statements; see [10, Lemmas 2.5 and 2.6].
- (a) Let \mathcal{T} be a tensor triangulated category. Let f, g be a morphism in \mathcal{T} , and let \mathcal{X}, \mathcal{Y} be full subcategories of \mathcal{T} . If $f \otimes \mathcal{X} = 0$ and $g \otimes \mathcal{Y} = 0$, then $(f \otimes g) \otimes (\mathcal{X} * \mathcal{Y}) = 0$.
- (b) Let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of R . Let f be a morphism in $\mathcal{K}^-(R)$. If $f \otimes_R R/(\mathbf{x}) = 0$, then $f^{\otimes 2^n} \otimes_R \mathcal{K}(\mathbf{x}) = 0$.

Here, $\mathcal{X} * \mathcal{Y}$ stands for the full subcategory of $\mathcal{K}^-(R)$ consisting of objects E such that there exists an exact triangle

$$X \rightarrow E \rightarrow Y \rightsquigarrow$$

in \mathcal{T} with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and $\mathcal{K}(\mathbf{x})$ stands for the Koszul complex of R with respect to \mathbf{x} . The statement (b) is deduced by using (a).

- (3) We need the assumption that $Y \in \mathcal{K}^b(R)$ to have the equality

$$\text{ann}_{R_{\mathfrak{p}}}(f_{\mathfrak{p}}) = \text{ann}_R(f)_{\mathfrak{p}}$$

for all prime ideals \mathfrak{p} of R . Here, the *annihilator* of a morphism $f : X \rightarrow Y$ in $\mathcal{D}^-(R)$ is defined by

$$\text{ann}_R(f) := \{a \in R \mid af = 0 \text{ in } \mathcal{D}^-(R)\},$$

which is nothing but the kernel of the morphism $R \rightarrow \text{Hom}_{\mathcal{D}^-(R)}(X, Y)$ given by $a \mapsto af$.

By virtue of the generalized smash nilpotence, we can prove the following key proposition. For an object X of $\mathcal{D}^-(R)$ we define the *annihilator* $\text{ann } X$ of X as the annihilator of the identity morphism of X .

PROPOSITION 2.7 ([10, Proposition 2.9]). *Let $X, Y \in \mathcal{D}^-(R)$ be complexes. Then the implication*

$$\text{V}(\text{ann } X) \subseteq \text{supph } Y \implies X \in \text{thick}^{\otimes} Y$$

holds true.

Again, we do not state the proof of this proposition, and instead, we give some comments on it.

REMARK 2.8.

- (1) For every $X \in \mathcal{D}^-(R)$ one has

$$\text{V}(\text{ann } X) \supseteq \text{supph } X.$$

The equality holds if $X \in \mathcal{D}^b(R)$.

- (2) The original statement that is due to Hopkins and Neeman and corresponds to the proposition asserts that for perfect complexes X, Y over R the implication

$$\text{supph } X \subseteq \text{supph } Y \implies X \in \text{thick}^{\otimes} Y$$

holds true; see [11, Lemma 1.2].

- (3) Proposition 2.7 does not hold if $\text{V}(\text{ann } X)$ is replaced with $\text{supph } X$ or if $\text{supph } Y$ is replaced with $\text{V}(\text{ann } Y)$; we will see this in Remarks 3.17 and 4.2.
- (4) In the proof of the proposition, we first take a truncation $Y' \in \mathcal{K}^b(R)$ of Y such that $\text{V}(\text{ann } X)$ is contained in $\text{supph } Y'$. Then we consider the morphism $R \rightarrow \text{Hom}_R(Y', Y)$ sending $1 \in R$ to the inclusion morphism $Y' \rightarrow Y$. The stream of the proof is similar to [11, Lemma 1.2], but we need to make various modifications.
- (5) In the proposition, we can replace the object Y of $\mathcal{D}^-(R)$ with any full subcategory \mathcal{Y} of $\mathcal{D}^-(R)$. Indeed, we find an object $Y \in \mathcal{Y}$ such that $\text{supph } Y$ contains all the prime ideals (minimally) containing $\text{ann } X$. Then $\text{V}(\text{ann } X)$ is contained in $\text{supph } Y$, and we can reduce to the case where the subcategory \mathcal{Y} consists only of Y .

As a corollary of Proposition 2.7 we have the following result. This result will be used in the proof of Theorem 2.4, and several other places.

COROLLARY 2.9 ([10, Corollary 2.11 and Proposition 4.11]).

(1) *Let X be a complex in $\mathcal{D}^-(R)$. Then it holds that*

$$\text{supph } X = \text{Spec } R \iff \text{thick}^{\otimes} X = \mathcal{D}^-(R).$$

(2) *Let I be an ideal of R , and let \mathcal{X} be an ideal of $\mathcal{D}^-(R)$. Take a system of generators $\mathbf{x} = x_1, \dots, x_n$ of I . Then it holds that*

$$V(I) \subseteq \text{supph } \mathcal{X} \iff R/I \in \mathcal{X} \iff K(\mathbf{x}) \in \mathcal{X}.$$

PROOF. (1) The implication (\Leftarrow) follows from the equalities

$$\text{supph } X = \text{supph}(\text{thick}^{\otimes} X) = \text{supph } \mathcal{D}^-(R) = \text{Spec } R.$$

As for the implication (\Rightarrow), for all objects $M \in \mathcal{D}^-(R)$ one has that $V(\text{ann } M)$ is contained in $\text{supph } X$. Hence M belongs to $\text{thick}^{\otimes} X$ by Proposition 2.7.

(2) We have

$$\text{supph } R/I = V(\text{ann } R/I) = V(I) = V(\text{ann } K(\mathbf{x})) = \text{supph } K(\mathbf{x}).$$

Using Proposition 2.7 completes the proof of the assertion. \square

Now we can obtain the proof of the main result of this section.

Proof of Theorem 2.4. Let \mathcal{X} be a cocompactly generated ideal of $\mathcal{D}^-(R)$. Then one can write $\mathcal{X} = \text{thick}^{\otimes} \mathcal{C}$ for some full subcategory \mathcal{C} of $\mathcal{D}^b(R)$. What we want to show is the equality $\mathcal{X} = \langle \text{supph } \mathcal{X} \rangle$. As to the inclusion (\supseteq), Corollary 2.9(2) implies that R/\mathfrak{p} is in \mathcal{X} for all $\mathfrak{p} \in \text{supph } \mathcal{X}$. As for the inclusion (\subseteq), it suffices to show that \mathcal{C} is contained in $\langle \text{supph } \mathcal{X} \rangle = \langle \text{supph } \mathcal{C} \rangle$. Pick an object $M \in \mathcal{C}$. Then M is a bounded complex of finitely generated R -modules, whence it belongs to $\text{thick}\{R/\mathfrak{p} \mid \mathfrak{p} \in \text{supph } M\}$. Now we are done. \square

As a corollary of Theorem 2.4 we have the following result.

COROLLARY 2.10 ([10, Corollary 2.15]). *The following are equivalent for an ideal \mathcal{X} of $\mathcal{D}^-(R)$.*

- (1) *\mathcal{X} is compactly generated.*
- (2) *\mathcal{X} is cocompactly generated.*

When this is the case, we simply say that \mathcal{X} is compact.

PROOF. Since $K^b(R)$ is contained in $\mathcal{D}^b(R)$, compact generation implies cocompact generation. Therefore (1) implies (2). Let us show that (2) implies (1). Let W be a specialization-closed subset of $\text{Spec } R$. Put

$$\begin{aligned} \mathcal{A} &:= \text{thick}^{\otimes}\{R/\mathfrak{p} \mid \mathfrak{p} \in W\}, \\ \mathcal{B} &:= \text{thick}^{\otimes}\{K(\mathbf{x}) \mid \mathbf{x}R \in W\}. \end{aligned}$$

Then \mathcal{A} is cocompactly generated, while \mathcal{B} is compactly generated. We see that $\text{supph } \mathcal{A} = \text{supph } \mathcal{B} = W$. Using Theorem 2.4, we obtain $\mathcal{A} = \mathcal{B}$. \square

As another corollary of Proposition 2.7, we get the following result.

COROLLARY 2.11 ([10, Corollary 2.19]). *If R is artinian, then all ideals of $\mathcal{D}^-(R)$ are compact. Therefore one has a one-to-one correspondence*

$$\{\text{Ideals of } \mathcal{D}^-(R)\} \cong \{\text{Subsets of } \text{Spec } R\}.$$

PROOF. The second assertion is deduced from the first one. To prove that all ideals \mathcal{X} of $D^-(R)$ are compact, it suffices to verify that the equality $\mathcal{X} = \langle \text{supph } \mathcal{X} \rangle$ holds. The inclusion (\supseteq) follows from Corollary 2.9(2). To show the inclusion (\subseteq) , we may assume that \mathcal{X} consists of a single object X of $D^-(R)$. Write the set of maximal ideals of R as follows.

$$\text{Max } R = \underbrace{\{\mathfrak{m}_1, \dots, \mathfrak{m}_s\}}_{\in \text{supph } X}, \underbrace{\{\mathfrak{m}_{s+1}, \dots, \mathfrak{m}_n\}}_{\notin \text{supph } X}.$$

Letting t be the Loewy length of R , we have $\mathfrak{m}_1^t \cdots \mathfrak{m}_n^t = 0$. The Chinese remainder theorem then implies that there is an isomorphism

$$R \cong R/\mathfrak{m}_1^t \oplus \cdots \oplus R/\mathfrak{m}_n^t.$$

Taking the tensor product of X with this, we get an isomorphism

$$\begin{aligned} X \cong (X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_1^t) \oplus \cdots \oplus (X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_s^t) \\ \oplus (X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_{s+1}^t) \oplus \cdots \oplus (X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_n^t). \end{aligned}$$

We have the equality $\text{supph}(X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_i^t) = \text{supph } X \cap \{\mathfrak{m}_i\}$ for each i , which shows that $X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_i^t = 0$ for all $s+1 \leq i \leq n$. Hence X is isomorphic to the direct sum $(X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_1^t) \oplus \cdots \oplus (X \otimes_R^{\mathbf{L}} R/\mathfrak{m}_s^t)$, and it is obtained that

$$\begin{aligned} X \in \text{thick}^{\otimes} \{R/\mathfrak{m}_1^t, \dots, R/\mathfrak{m}_s^t\} \\ \subseteq \text{thick}^{\otimes} \{R/\mathfrak{m}_1, \dots, R/\mathfrak{m}_s\} = \langle \text{supph } X \rangle. \end{aligned}$$

Here, the inclusion relation follows from the fact that for each maximal ideal \mathfrak{m}_i there exists an exact sequence

$$0 \rightarrow \mathfrak{m}_i^k/\mathfrak{m}_i^{k+1} \rightarrow R/\mathfrak{m}_i^{k+1} \rightarrow R/\mathfrak{m}_i^k \rightarrow 0$$

for all $k \geq 0$, and $\mathfrak{m}_i^k/\mathfrak{m}_i^{k+1}$ is a direct sum of copies of R/\mathfrak{m}_i . Now the proof of the corollary is completed. \square

3. The Balmer spectrum of $D^-(R)$ and classifications of thick tensor ideals

In this section, we consider the structure of the Balmer spectrum of $D^-(R)$, and make correspondences among some classes of ideals of $D^-(R)$ and subsets of $\text{Spec } R$ and $\text{Spc } D^-(R)$. The section consists of three subsections.

3.1. The structure of $\text{Spc } D^-(R)$.

We investigate the structure of the Balmer spectrum of $D^-(R)$ as a topological space, comparing it with the Zariski spectrum of R . We start by defining a tame ideal of $D^-(R)$.

DEFINITION 3.1.

- (1) For a subset S of $\text{Spec } R$, we define the full subcategory $\text{supph}^{-1} S$ of $D^-(R)$ by

$$\text{supph}^{-1} S = \{X \in D^-(R) \mid \text{supph } X \subseteq S\}.$$

One easily sees that $\text{supph}^{-1} S$ is an ideal of $D^-(R)$, and furthermore, the following equalities hold.

- $\text{supph}^{-1} S = \text{supph}^{-1} S_{\text{spcl}}$.
- $\text{supph}(\text{supph}^{-1} S) = S_{\text{spcl}}$.

Here, S_{spcl} stands for the largest specialization-closed subset of $\text{Spec } R$ contained in S . (This is the *spcl-interior* of S in $\text{Spec } R$ if we use the terminology in the next Subsection 3.2.)

- (2) An ideal \mathcal{X} of $D^-(R)$ is called *tame* if $\mathcal{X} = \text{supph}^{-1}S$ for some subset S of $\text{Spec } R$. We set

$${}^t\text{Spc } D^-(R) = \{\text{tame prime ideals of } D^-(R)\}.$$

One can construct the following correspondence between $\text{Spec } R$ and $\text{Spc } D^-(R)$; see [10, Propositions 3.4 and 3.7].

PROPOSITION 3.2.

- (1) For $\mathfrak{p} \in \text{Spec } R$, the full subcategory

$$\mathcal{S}(\mathfrak{p}) := \{X \in D^-(R) \mid X_{\mathfrak{p}} = 0\}$$

of $D^-(R)$ is a prime ideal of $D^-(R)$

- (2) For $\mathcal{P} \in \text{Spc } D^-(R)$, the set

$$\{I \subseteq R \mid R/I \notin \mathcal{P}\}$$

of ideals of R has a unique maximal element $\mathfrak{s}(\mathcal{P})$ with respect to the inclusion relation, which is a prime ideal of R

Concerning the correspondence constructed in the above proposition, the following statements hold.

THEOREM 3.3 ([10, Theorems 3.9, 4.5, 4.7, 4.12 and 4.14]).

- (1) One has the order-reversing maps

$$\mathcal{S} : \text{Spec } R \rightleftarrows \text{Spc } D^-(R) : \mathfrak{s}$$

such that

$$\begin{cases} \mathfrak{s} \cdot \mathcal{S} = 1, \\ \mathcal{S} \cdot \mathfrak{s} = \text{supph}^{-1} \text{supph}. \end{cases}$$

In particular, the inequality

$$\dim(\text{Spc } D^-(R)) \geq \dim R$$

between the Krull dimensions holds.

- (2) The subset ${}^t\text{Spc } D^-(R)$ of $\text{Spc } D^-(R)$ is dense. Moreover, there is a commutative diagram

$$\begin{array}{ccccc} \text{Spec } R & \xrightarrow{\mathcal{S}} & \text{Spc } D^-(R) & \xrightarrow{\mathfrak{s}} & \text{Spec } R \\ & \searrow \mathcal{S}' & \uparrow \text{inc} & \swarrow \mathfrak{s}' & \\ & & {}^t\text{Spc } D^-(R) & & \end{array}$$

such that \mathcal{S}' is an open bijection, \mathfrak{s}' is a continuous bijection and \mathfrak{s} is a continuous map. In particular, the image of \mathcal{S} coincides with ${}^t\text{Spc } D^-(R)$.

(3) *There is a commutative diagram*

$$\begin{array}{ccc}
 \text{Min } R & \xrightarrow{\mathcal{S}_{\min}} & \max(\text{Spc } D^-(R)) \\
 \downarrow \text{inc} & & \downarrow \text{inc} \\
 \text{Spec } R & \xrightarrow{\mathcal{S}} & \text{Spc } D^-(R) \\
 \uparrow \text{inc} & & \uparrow \text{inc} \\
 \text{Max } R & \xrightarrow{\mathcal{S}_{\max}} & \min(\text{Spc } D^-(R))
 \end{array}$$

such that \mathcal{S}_{\min} is a homeomorphism, and the injective map \mathcal{S}_{\max} is also a homeomorphism if R is a semilocal ring.

(4) *The following are equivalent.*

- (a) *The map \mathcal{S} is continuous.*
- (b) *The map \mathcal{S}' is homeomorphic.*
- (c) *The map \mathfrak{s}' is homeomorphic.*
- (d) *The set $\text{Spec } R$ is finite.*

Here are several comments on this theorem.

REMARK 3.4.

(1) Recall that for a topological space X the *Krull dimension* $\dim X$ of X is by definition the supremum of the lengths of chains of nonempty irreducible closed subsets of X . For a tensor triangulated category \mathcal{T} we have

$$\begin{aligned}
 \dim(\text{Spc } \mathcal{T}) &= \sup\{n \geq 0 \mid \exists \text{ chain } \overline{\{\mathcal{P}_0\}} \subsetneq \cdots \subsetneq \overline{\{\mathcal{P}_n\}} \text{ of subsets of } \text{Spc } \mathcal{T}\} \\
 &= \sup\{n \geq 0 \mid \exists \text{ chain } \mathcal{P}_0 \subsetneq \cdots \subsetneq \mathcal{P}_n \text{ of points of } \text{Spc } \mathcal{T}\}.
 \end{aligned}$$

(2) Note that $\text{Max } R$, $\text{Min } R$, $\max(\text{Spc } \mathcal{T})$ and $\min(\text{Spc } \mathcal{T})$ are all T_1 -spaces, and that, in general, any finite subset of a T_1 -space is closed. Thus, to show Theorem 3.3(3), it is enough to check that the top and bottom horizontal maps are bijective and injective, respectively (after we verify that they are induced).

(3) The following are equivalent ([10, Lemma 4.6]).

- All specialization-closed subsets of $\text{Spec } R$ are closed.
- There are only finitely many specialization-closed subsets of $\text{Spec } R$.
- There are only finitely many closed subsets of $\text{Spec } R$.
- There are only finitely many prime ideals of R .

Using this equivalences, we can deduce Theorem 3.3(4).

(4) More precisely than Theorem 3.3(1), we actually have a commutative diagram

$$\begin{array}{ccccc}
 & & \text{Spc } D^-(R) & & \\
 & \nearrow \mathcal{S}' & \downarrow \theta \text{ inc} & \searrow \mathfrak{s}' & \\
 \text{Spec } R & \xrightarrow{\mathcal{S}} & \text{Spc } D^-(R) & \xrightarrow{\mathfrak{s}} & \text{Spec } R \\
 & \searrow \tilde{\mathcal{S}} & \downarrow \pi \text{ can} & \nearrow \tilde{\mathfrak{s}} & \\
 & & \text{Spc } D^-(R)/\text{supph} & &
 \end{array}$$

such that $\mathfrak{s}\mathcal{S}$ is identity, \mathcal{S}' , $\tilde{\mathcal{S}}$, \mathfrak{s}' , $\tilde{\mathfrak{s}}$, $\pi\theta$ are bijections, \mathcal{S}' , $\tilde{\mathcal{S}}$ are open and closed, and \mathfrak{s} , \mathfrak{s}' , $\tilde{\mathfrak{s}}$ are continuous ([10, Theorem 4.5]). Here, $\text{Spc } D^-(R)/\text{supph}$ stands

for the quotient topological space by the equivalence relation induced by taking $\text{supph}(-)$, and π the canonical surjection. (To be more precise, we define a relation \sim in $\mathbf{Spc D}^-(R)$ by

$$\mathcal{P} \sim \mathcal{Q} \iff \text{supph } \mathcal{P} = \text{supph } \mathcal{Q}$$

for $\mathcal{P}, \mathcal{Q} \in \mathbf{Spc D}^-(R)$. Then \sim is an equivalence relation in $\mathbf{Spc D}^-(R)$. We denote by $\mathbf{Spc D}^-(R)/\sim$ the set of equivalence classes, and by $\pi : \mathbf{Spc D}^-(R) \rightarrow \mathbf{Spc D}^-(R)/\sim$ the map sending each $\mathcal{P} \in \mathbf{Spc D}^-(R)$ to its equivalence class $[\mathcal{P}] \in \mathbf{Spc D}^-(R)/\sim$. The set $\mathbf{Spc D}^-(R)/\sim$ is a topological space, where a subset S of $\mathbf{Spc D}^-(R)/\sim$ is open if and only if $\pi^{-1}(S)$ is an open subset of $\mathbf{Spc D}^-(R)$.)

(5) More precisely than Theorem 3.3(4), the following assertion holds true ([10, Theorem 4.7]). Consider the three conditions

- (a) The map $\tilde{\mathcal{S}}$ is a homeomorphism,
- (b) The map $\tilde{\mathfrak{s}}$ is a homeomorphism,
- (c) The map $\pi\theta$ is a homeomorphism.

Then (a) is equivalent to (b), and (b) \wedge (c) is equivalent to the four conditions in Theorem 3.3(4).

(6) We do not know whether \mathcal{S}_{\max} is surjective even if R is not semilocal.

Suppose that R is artinian. Then R is semilocal, has only finitely many prime ideals and satisfies $\text{Max } R = \text{Spec } R = \text{Min } R$. Hence, applying Theorem 3.3, we obtain the following corollary.

COROLLARY 3.5. *Let R be an artinian ring. Then the following statements hold true.*

- (1) *The maps $\mathcal{S} : \text{Spec } R \rightleftarrows \mathbf{Spc D}^-(R) : \mathfrak{s}$ are mutually inverse homeomorphisms.*
- (2) *One has $\dim \mathbf{Spc D}^-(R) = \dim R = 0 < \infty$.*
- (3) *All prime ideals of $\mathbf{D}^-(R)$ are tame.*

In fact, a more complete statement than the above corollary holds true; see Theorem 3.14.

Next, let us consider the locality of the Balmer spectrum of $\mathbf{D}^-(R)$ as a topological space.

DEFINITION 3.6.

(1) A topological space X is called *local* if for any open cover

$$X = \bigcup_{i \in I} U_i$$

of X there exists $t \in I$ such that $X = U_t$. In particular, a local topological space is quasi-compact.

(2) A tensor triangulated category \mathcal{T} is said to be *local* if the topological space $\mathbf{Spc } \mathcal{T}$ is local.

Balmer [3, Proposition 4.2] obtains criteria for a given tensor triangulated category to be local.

PROPOSITION 3.7 (Balmer (2010)). *The following are equivalent for a tensor triangulated category \mathcal{T} .*

- (1) *\mathcal{T} is a local tensor triangulated category.*
- (2) *\mathcal{T} has a unique minimal prime.*

(3) $\sqrt{\mathbf{0}}$ is a prime ideal of \mathcal{T} .

If \mathcal{T} is rigid, then (1)–(3) are also equivalent to the following condition.

(4) $\mathbf{0}$ is a prime ideal of \mathcal{T} .

Combining Theorem 3.3(3) and Proposition 3.7 yields that the tensor triangulated category $\mathbf{D}^-(R)$ is local if and only if the ring R is local. We should remark that $\mathbf{D}^-(R)$ is not rigid, as we stated in the previous section. Thus the last assertion of Proposition 3.7 is not available for $\mathcal{T} = \mathbf{D}^-(R)$, but we can directly show the corollary.

COROLLARY 3.8 ([10, Corollary 4.19]). *The following are equivalent.*

- (1) *The tensor triangulated category $\mathbf{D}^-(R)$ is local.*
- (2) *There exists a unique minimal prime of $\mathbf{D}^-(R)$.*
- (3) *The radical $\sqrt{\mathbf{0}}$ of the zero ideal of $\mathbf{D}^-(R)$ is prime.*
- (4) *The zero ideal $\mathbf{0}$ of $\mathbf{D}^-(R)$ is prime.*
- (5) *The ring R is local.*

PROOF. (1) \Leftrightarrow (2) \Leftrightarrow (3): These equivalences follow from Proposition 3.7.

(2) \Rightarrow (5): By Theorem 3.3(3) there is an inequality

$$\#\max(\mathrm{Spec} R) \leq \#\min(\mathrm{Spc} \mathbf{D}^-(R)) = 1,$$

which shows the implication.

(5) \Rightarrow (4): Let \mathfrak{m} be the unique maximal ideal of R . The zero ideal $\mathbf{0}$ of $\mathbf{D}^-(R)$ coincides with the prime ideal $\mathcal{S}(\mathfrak{m})$ of $\mathbf{D}^-(R)$.

(4) \Rightarrow (3): It is easy to see that $\sqrt{\mathbf{0}} = \mathbf{0}$. □

As further observations on the topological structure of $\mathrm{Spc} \mathbf{D}^-(R)$, we can show that the topological space $\mathrm{Spc} \mathbf{D}^-(R)$ is irreducible if and only if so is $\mathrm{Spec} R$. Also, we can describe the Balmer spectrum of $\mathbf{D}^-(R)$ as follows.

$$\mathrm{Spc} \mathbf{D}^-(R) = \begin{cases} \mathrm{U}(R/\mathfrak{m}) \sqcup \{\mathbf{0}\} & \text{if } (R, \mathfrak{m}) \text{ is a local ring,} \\ \bigcup_{\mathfrak{m} \in \mathrm{Max} R} \mathrm{U}(R/\mathfrak{m}) & \text{if } R \text{ is a non-local ring.} \end{cases}$$

More strongly, for any two distinct maximal ideals $\mathfrak{m}, \mathfrak{n}$ of R the equality $\mathbf{D}^-(R) = \mathrm{U}(R/\mathfrak{m}) \cup \mathrm{U}(R/\mathfrak{n})$ holds. We refer the reader to [10, Corollary 4.13 and Proposition 4.20] for the details.

3.2. Classifications of ideals of $\mathbf{D}^-(R)$.

In this subsection, we consider making correspondences among compact, radical and tame ideals of $\mathbf{D}^-(R)$, and specialization-closed subsets of $\mathrm{Spec} R$, $\mathrm{Spc} \mathbf{D}^-(R)$ and ${}^t\mathrm{Spc} \mathbf{D}^-(R)$. First of all, we explore the relationships among these three properties of ideals of $\mathbf{D}^-(R)$.

PROPOSITION 3.9 ([10, Lemma 5.7]). *Let \mathcal{X} be an ideal of $\mathbf{D}^-(R)$.*

(1) *There are equalities*

$$\begin{aligned} \mathcal{X}_{\mathrm{cpt}} &= \langle \mathrm{supph} \mathcal{X} \rangle, \\ \mathcal{X}^{\mathrm{rad}} &= \sqrt{\mathcal{X}}, \\ \mathcal{X}^{\mathrm{tame}} &= \mathrm{supph}^{-1} \mathrm{supph} \mathcal{X} \end{aligned}$$

of ideals of $\mathbf{D}^-(R)$.

(2) *There are inclusions*

$$\mathcal{X}_{\text{cpt}} \subseteq \mathcal{X} \subseteq \mathcal{X}^{\text{rad}} \subseteq \mathcal{X}^{\text{tame}}$$

of ideals of $\mathcal{D}^-(R)$, all of whose supports are equal. In particular, every tame ideal of $\mathcal{D}^-(R)$ is radical.

Here, $\mathcal{X}^{\mathbb{P}}$ (respectively, $\mathcal{X}_{\mathbb{P}}$) stands for the \mathbb{P} -closure (respectively, \mathbb{P} -interior) of \mathcal{X} , namely, the smallest (respectively, largest) \mathbb{P} -ideal containing (respectively, contained in) \mathcal{X} . Also, **cpt** and **rad** denote the compact and radical properties, respectively.

The assertion (1) of the above proposition is seen to hold just by checking the definitions. In relation to (2), the following statement holds: Let W be a specialization-closed subset of $\text{Spec } R$. Then $\langle W \rangle$ (respectively, $\text{supph}^{-1} W$) is the smallest (respectively, largest) ideal of $\mathcal{D}^-(R)$ whose support coincides with W ; see [10, Theorem 6.6(2)].

To state the main result of this section, we introduce notation.

NOTATION 3.10. We use the following sets in the rest of this subsection.

$$\begin{aligned} \mathbf{Rad} &= \{\text{Radical ideals of } \mathcal{D}^-(R)\}, \\ \mathbf{Tame} &= \{\text{Tame ideals of } \mathcal{D}^-(R)\}, \\ \mathbf{Cpt} &= \{\text{Compact ideals of } \mathcal{D}^-(R)\}, \\ \mathbf{Spcl}(\text{Spec}) &= \{\text{Specialization-closed subsets of } \text{Spec } R\}, \\ \mathbf{Spcl}({}^t\text{Spc}) &= \{\text{Specialization-closed subsets of } {}^t\text{Spc } \mathcal{D}^-(R)\}, \\ \mathbf{Thom} &= \{\text{Thomason subsets of } \text{Spc } \mathcal{D}^-(R)\}. \end{aligned}$$

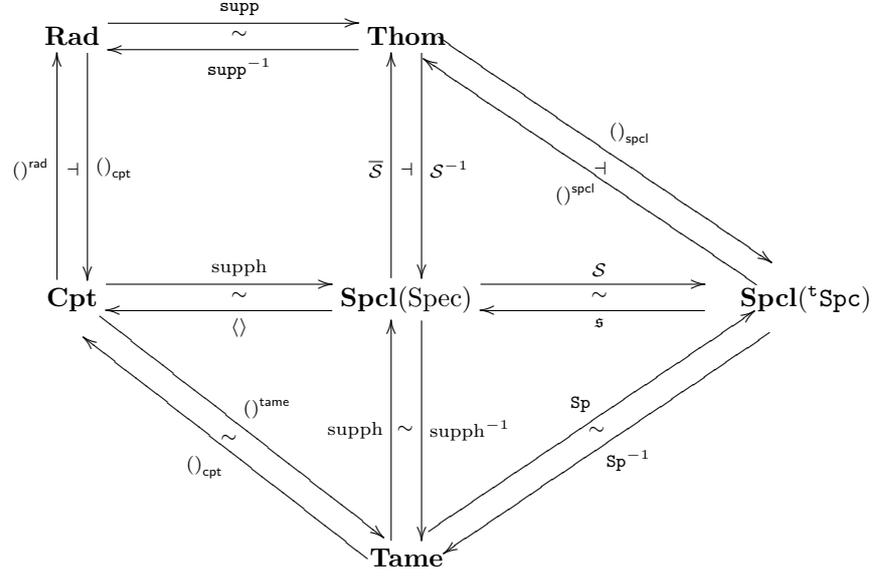
Proposition 3.9(2) implies the inclusion

$$\mathbf{Rad} \supseteq \mathbf{Tame}.$$

The main result of this section makes correspondences among the above six sets.

THEOREM 3.11. [10, Theorems 5.12 and 5.19] *One has the following diagram, which is naturally commutative. (More precisely, the diagram with sections and*

bijections and the diagram with retractions and bijections are commutative.)



Here:

- $\begin{cases} f \sim g \iff gf = 1 \text{ and } fg = 1 \text{ (i.e. } (f, g) \text{ is a bijection pair),} \\ f \dashv g \iff gf = 1 \text{ (i.e. } (f, g) \text{ is a section-retraction pair).} \end{cases}$
- $\begin{cases} A_{\text{spcl}} = \text{the spcl-interior of } A \text{ in } {}^t\text{Spc } D^-(R), \\ B^{\text{spcl}} = \text{the spcl-closure of } B \text{ in } \text{Spc } D^-(R). \end{cases}$
- $\begin{cases} \bar{\mathcal{S}}(W) = \bigcup_{\mathfrak{p} \in W} \overline{\{\mathcal{S}(\mathfrak{p})\}}, \\ \mathcal{S}^{-1}(A) = \{\mathfrak{p} \in \text{Spec } R \mid \mathcal{S}(\mathfrak{p}) \in A\}, \\ \text{Sp}(-) = \text{supp}(-) \cap {}^t\text{Spc } D^-(R), \\ \text{Sp}^{-1}(B) = \{M \in D^-(R) \mid \text{Sp } M \subseteq B\}, \\ \mathcal{S}(W) = \{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in W\}, \\ \mathfrak{s}(B) = \{\mathfrak{s}(\mathcal{P}) \mid \mathcal{P} \in B\}. \end{cases}$

Moreover, the following are equivalent.

- (1) The pair $\mathcal{S} : \text{Spec } R \rightleftarrows \text{Spc } D^-(R) : \mathfrak{s}$ of maps is a one-to-one correspondence.
- (2) The pair $((\)^{\text{rad}}, (\)_{\text{cpt}})$ of maps is a one-to-one correspondence.
- (3) The pair $(\bar{\mathcal{S}}, \mathcal{S}^{-1})$ of maps is a one-to-one correspondence.
- (4) The pair $((\)^{\text{spcl}}, (\)_{\text{spcl}})$ of maps is a one-to-one correspondence.
- (5) The equality $\mathbf{Rad} = \mathbf{Tame}$ holds.

The **spcl-interior** A_{spcl} is the largest specialization-closed subset of ${}^t\text{Spc } D^-(R)$ contained in A , while **spcl-closure** B^{spcl} is the smallest specialization-closed subset of $\text{Spc } D^-(R)$ containing B .

Here are some comments on the above theorem.

REMARK 3.12.

- (a) The one-to-one correspondence $\mathbf{Rad} \cong \mathbf{Thom}$ in the diagram of Theorem 3.11 is nothing but Theorem 1.6 due to Balmer, while the one-to-one correspondence

$\mathbf{Cpt} \cong \mathbf{Spcl}(\text{Spec})$ is nothing but Theorem 2.4. Thus this diagram connects Theorems 1.6 and 2.4, and gives rise to several related correspondences.

- (b) The proof of Theorem 3.11 proceeds step by step; for example, we show and use the equalities

$$\begin{cases} A_{\text{spcl}} = A \cap {}^t\mathbf{Spc} D^-(R), \\ B^{\text{spcl}} = \{\mathcal{P} \in \mathbf{Spc} D^-(R) \mid \mathcal{P}^{\text{tame}} \in B\} = \bigcup_{\mathcal{P} \in B^{\text{spcl}}} \text{supp}(R/\mathfrak{s}(\mathcal{P})). \end{cases}$$

- (c) Theorem 3.11 yields a commutative diagram

$$\begin{array}{ccccccc} & & & \mathbf{Rad} & & & \\ & & & \swarrow & & \searrow & \\ & & & \text{supph} & & & \\ & & & \swarrow & & \searrow & \\ & & & \mathbf{Spcl}(\text{Spec}) & & \mathbf{Tame} & \\ & & & \swarrow & & \searrow & \\ \mathbf{Cpt} & \xleftarrow{\sim} & & & \xrightarrow{\sim} & & \mathbf{Spcl}({}^t\mathbf{Spc}) \end{array}$$

$(\)_{\text{cpt}}$ $(\)^{\text{tame}}$ Sp

where the bottom bijections are the ones in the diagram of Theorem 3.11. Furthermore, the conditions (1)–(5) in Theorem 3.11 are also equivalent to the following three conditions.

- (6) The map $\text{supph} : \mathbf{Rad} \rightarrow \mathbf{Spcl}(\text{Spec})$ is a bijection.
- (7) The map $(\)^{\text{tame}} : \mathbf{Rad} \rightarrow \mathbf{Tame}$ is a bijection.
- (8) The map $\text{Sp} : \mathbf{Rad} \rightarrow \mathbf{Spcl}({}^t\mathbf{Spc})$ is a bijection.

For the details, we refer the reader to [10, Corollary 5.20].

The corollary below is immediately obtained from the above theorem.

COROLLARY 3.13. *If every radical ideal of $D^-(R)$ is compact, then $\mathbf{Rad} = \mathbf{Tame}$.*

PROOF. For each radical ideal \mathcal{X} of $D^-(R)$ one has $\mathcal{X} = \mathcal{X}_{\text{cpt}} = (\mathcal{X}_{\text{cpt}})^{\text{rad}}$. Hence

$$(\)^{\text{rad}} : \mathbf{Cpt} \rightleftarrows \mathbf{Rad} : (\)_{\text{cpt}}$$

is a one-to-one correspondence. Theorem 3.11 implies $\mathbf{Rad} = \mathbf{Tame}$. \square

We are interested in what rings R are characterized by the eight conditions (1)–(8) appearing in Theorem 3.11 and Remark 3.12.

THEOREM 3.14 ([10, Theorem 6.5]). *The equivalent conditions (1)–(8) are also equivalent to the condition that*

- (9) *the ring R is artinian.*

Furthermore, when this is the case, every ideal of $D^-(R)$ is compact, tame and radical.

The most difficult part of the proof of this theorem is to show the necessity of the condition (9). Here, let us only check the last assertion of the theorem. Suppose that R is artinian, and pick any ideal \mathcal{X} of $D^-(R)$. Then Corollary 3.5(3) implies that \mathcal{X} is compact, and that taking $\text{supph}(-)$ makes an injective map. Hence the equality

$$\text{supph}(\mathcal{X}) = \text{supph}(\text{supph}^{-1} \text{supph} \mathcal{X})$$

implies that \mathcal{X} coincides with $\text{supph}^{-1} \text{supph} \mathcal{X}$, which shows that \mathcal{X} is tame. In general, a tame ideal of $D^-(R)$ is radical, and hence \mathcal{X} is radical.

Using Theorem 3.14 and Corollary 3.13, we immediately obtain the following.

COROLLARY 3.15. *Suppose that R is not artinian. Then there exists a non-compact radical ideal of $\mathbf{D}^-(R)$.*

In relation to this result, we consider the existence of compact prime ideals of $\mathbf{D}^-(R)$. If R is a local ring with maximal ideal \mathfrak{m} , then

$$\mathbf{Cpt} \cap \mathfrak{s}^{-1}(\mathfrak{m}) = \{\mathbf{0}\}.$$

In particular, there exists a compact prime ideal of $\mathbf{D}^-(R)$. On the other hand, if R is a non-local semi-local domain of positive dimension, then $\mathbf{D}^-(R)$ admits no compact prime ideal. We refer the reader to [10, Proposition 6.2]. If R is a discrete valuation ring, we can show

$$\mathbf{Cpt} \cap \mathbf{Spc} \mathbf{D}^-(R) = \{\mathbf{0}, \text{thick}^{\otimes} k\},$$

where k is the residue field of R . This follows from Theorems 3.11 and 4.1. See also [10, Example 7.16].

3.3. On Balmer's conjecture for $\mathbf{D}^-(R)$.

From now on, we consider Balmer's conjecture stated in Section 1 for our tensor triangulated category $\mathbf{D}^-(R)$. First of all, we investigate the difference between radical and tame ideals of $\mathbf{D}^-(R)$. We have already learned that the following holds.

$$\mathcal{X}^{\text{rad}} \subseteq \mathcal{X}^{\text{tame}}.$$

The following theorem says that if \mathcal{X} is compact, then the equality does not hold under mild assumptions.

THEOREM 3.16 ([10, Theorem 6.6]). *Let W be a nonempty proper specialization-closed subset of $\text{Spec } R$, and put $\mathcal{X} = \langle W \rangle$. Assume that R is either a domain or a local ring. Then*

$$\mathcal{X}^{\text{rad}} \subsetneq \mathcal{X}^{\text{tame}}.$$

PROOF. Since W is nonempty, it contains a prime ideal P . Take a system of generators $\mathbf{x} = x_1, \dots, x_r$ of P . It is essential to think of the following complex.

$$C := \bigoplus_{i>0} \mathbf{K}(\mathbf{x}^i)[i].$$

Thanks to the shifts, this infinite direct sum exists in our tensor triangulated category $\mathbf{D}^-(R)$. Since $\text{supp } C = \mathbf{V}(P)$ is contained in W , the complex C is in $\text{supp}^{-1} W = \mathcal{X}^{\text{tame}}$ by Proposition 3.9(1).

Suppose $\mathcal{X}^{\text{rad}} = \mathcal{X}^{\text{tame}}$. Then C belongs to $\mathcal{X}^{\text{rad}} = \sqrt{\mathcal{X}}$. Hence there is an integer $n > 0$ such that

$$C' := \underbrace{C \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} C}_n$$

belongs to \mathcal{X} . Note that C' contains

$$D := \bigoplus_{i>0} \mathbf{K}(\mathbf{x}^i)[ni]$$

as a direct summand. Therefore D is in $\mathcal{X} = \langle W \rangle = \text{thick}^{\otimes} \{R/\mathfrak{p} \mid \mathfrak{p} \in W\}$, and we find a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ in W such that

$$\text{ann } D \supseteq (\text{ann } R/\mathfrak{p}_1) \cdots (\text{ann } R/\mathfrak{p}_m) = \mathfrak{p}_1 \cdots \mathfrak{p}_m.$$

Krull's intersection theorem implies

$$\text{ann } D = \bigcap_{i>0} \mathbf{x}^i R = 0,$$

and we have $\mathfrak{p}_1 \cdots \mathfrak{p}_m = 0$. Thus for every prime ideal \mathfrak{p} of R there exists an integer $1 \leq t \leq m$ such that \mathfrak{p} contains \mathfrak{p}_t . Since W is specialization-closed and contains \mathfrak{p}_t , the prime ideal \mathfrak{p} belongs to W . This shows that $W = \text{Spec } R$, contrary to the assumption on W . \square

Using the above proof, we have an observation related to Proposition 2.7.

REMARK 3.17. We use the same notation as in the proof of Theorem 3.16.

- (1) It holds that $\text{supph } C$ is contained in $\text{supph } R/P$, but C does not belong to $\text{thick}^{\otimes} R/P$. Indeed, we have $\text{supph } C = V(P) = \text{supph } R/P$. Assume that C is in $\text{thick}^{\otimes} R/P$. Then

$$0 = \text{ann } C \supseteq (\text{ann } R/P)^u = P^u$$

for some integer $u > 0$. Hence the equality $\text{Spec } R = V(P)$ holds, which is contained in W since W is specialization-closed. Therefore W coincides with $\text{Spec } R$, which is a contradiction.

- (2) It holds that $V(\text{ann } R)$ is contained in $V(\text{ann } C)$, but R does not belong to $\text{thick}^{\otimes} C$. In fact, we have $V(\text{ann } R) = V(0) = V(\text{ann } C)$. As $\text{supph } C = V(P)$ is a proper subset of $\text{Spec } R$, it is observed from Corollary 2.9(1) that R is not in $\text{thick}^{\otimes} C$.

Now, we consider Balmer's conjecture (Conjecture 1.8) for our tensor triangulated category $\mathbf{D}^-(R)$. First of all, let us check that the triangulated category $\mathbf{D}^-(R)$ is algebraic. The category $\mathbf{C}^-(R)$ of right bounded complexes of finitely generated R -modules is a Frobenius exact category with respect to the split short exact sequences of complexes in $\mathbf{C}^-(R)$, and $\mathbf{K}^-(R)$ is the stable category of $\mathbf{C}^-(R)$. Thus $\mathbf{K}^-(R)$ is an algebraic triangulated category.

Recall that Conjecture 1.8 concerns the continuous map

$$\rho_{\mathbf{D}^-(R)}^{\bullet} : \text{Spc } \mathbf{D}^-(R) \rightarrow \text{Spec}^h \mathbf{R}_{\mathbf{D}^-(R)}^{\bullet}.$$

One can actually observe that

- (a) $\mathbf{R}_{\mathbf{D}^-(R)}^{\bullet} = \mathbf{R}_{\mathbf{D}^-(R)}^0 = R$,
- (b) $\text{Spec}^h \mathbf{R}_{\mathbf{D}^-(R)}^{\bullet} = \text{Spec } R$, and
- (c) $\rho_{\mathbf{D}^-(R)}^{\bullet} = \mathfrak{s}$.

Thus, Conjecture 1.8 for $\mathbf{D}^-(R)$ just claims the local injectivity of the map \mathfrak{s} .

We can show that under quite mild assumptions the algebraic tensor triangulated category $\mathbf{D}^-(R)$ does not satisfy Balmer's conjecture.

COROLLARY 3.18 ([10, Corollary 6.10]). *Assume that $\dim R > 0$, and that R is either a domain or a local ring. Then \mathfrak{s} is not locally injective. Hence, Balmer's Conjecture 1.8 does not hold true for $\mathbf{D}^-(R)$.*

PROOF. By assumption we find a nonunit $x \in R$ such that the principal ideal xR of R has positive height. We apply Theorem 3.16 to $\mathcal{X} = \langle V(x) \rangle$ to get

$$\bigcap_{\mathcal{X} \subseteq \mathcal{P} \in \text{Spc } \mathbf{D}^-(R)} \mathcal{P} = \mathcal{X}^{\text{rad}} \subsetneq \mathcal{X}^{\text{tame}} = \bigcap_{\mathcal{X} \subseteq \mathcal{P} \in {}^v\text{Spc } \mathbf{D}^-(R)} \mathcal{P}.$$

Hence we can choose a prime ideal \mathcal{P} of $\mathbf{D}^-(R)$ such that $\mathcal{X} \subseteq \mathcal{P} \subsetneq \mathcal{P}^{\text{tame}}$.

Assume that \mathfrak{s} is locally injective at the point \mathcal{P} . Then there exists an object $M \in \mathbf{D}^-(R)$ with $\mathcal{P} \in \mathcal{U}(M)$ such that $\mathfrak{s}|_{\mathcal{U}(M)}$ is injective. Then $\mathcal{U}(M)$ contains two distinct points \mathcal{P} and $\mathcal{P}^{\text{tame}}$, which are sent by \mathfrak{s} to the same point in $\text{Spec } R$. This contradicts the injectivity of the map $\mathfrak{s}|_{\mathcal{U}(M)}$. \square

4. The case of discrete valuation rings and applications

In this section, we mainly consider the case where R is a discrete valuation ring, and make an application for the higher-dimensional case.

For a local ring R with maximal ideal \mathfrak{m} , we denote by $\mathbf{D}_{\mathfrak{H}}^-(R)$ the full subcategory of $\mathbf{D}^-(R)$ consisting of complexes with finite length homologies. Note that this is nothing but $\text{supph}^{-1}\{\mathfrak{m}\}$, and in particular, it is an ideal of $\mathbf{D}^-(R)$.

When R is a discrete valuation ring, we can construct certain concrete prime ideals of $\mathbf{D}^-(R)$ by making use of the growth of the Loewy lengths of homologies, and moreover we can obtain their generators.

THEOREM 4.1 ([10, Propositions 7.7, 7.17 and Theorems 7.11, 7.14]). *Let R be a discrete valuation ring with maximal ideal \mathfrak{m} . For an integer $n \geq 0$, we denote by \mathcal{P}_n the full subcategory of $\mathbf{D}_{\mathfrak{H}}^-(R)$ consisting of complexes X such that there exists an integer $t \geq 0$ with*

$$\ell(\mathbf{H}^{-i}X) \leq ti^n$$

for all $i \gg 0$, where $\ell(-)$ stands for the Loewy length. Then the following assertions hold true.

(1) One has $\mathcal{P}_n = \text{thick}^{\otimes} G_n$, where

$$G_n = (\cdots \xrightarrow{0} R/x^{3^n}R \xrightarrow{0} R/x^{2^n}R \xrightarrow{0} R/x^{1^n}R \rightarrow 0).$$

In particular, \mathcal{P}_n is an ideal of $\mathbf{D}^-(R)$.

(2) The ideal \mathcal{P}_n is prime and non-tame. It is non-compact if $n \geq 1$.

(3) There is a strictly ascending chain

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \cdots$$

of prime ideals of $\mathbf{D}^-(R)$. Therefore $\text{Spc } \mathbf{D}^-(R)$ has infinite Krull dimension.

Using this theorem, we have an observation related to Proposition 2.7, whose context is the same as Remark 3.17.

REMARK 4.2. We use the same notation as in Theorem 4.1. We denote by k the residue field of R , that is, $k = R/\mathfrak{m}$.

(1) It holds that $\text{supph } G_n$ is contained in $\text{supph } k$, but G_n is not in $\text{thick}^{\otimes} k$ for all $n \geq 1$. In fact, we have $\text{supph } G_n = \mathbf{V}(x) = \text{supph } k$. Assume that G_n belongs to $\text{thick}^{\otimes} k$. Then $\mathcal{P}_n = \text{thick}^{\otimes} G_n$ is contained in $\text{thick}^{\otimes} k = \mathcal{P}_0$, which gives rise to a contradiction.

(2) It holds that $\mathbf{V}(\text{ann } R)$ is contained in $\mathbf{V}(\text{ann } G_n)$, but R does not belong to $\text{thick}^{\otimes} G_n$. Indeed, we have $\mathbf{V}(\text{ann } R) = \mathbf{V}(0) = \mathbf{V}(\text{ann } G_n)$. As $\text{supph } G_n = \mathbf{V}(x)$ is a proper subset of $\text{Spec } R$, we see from Corollary 2.9(1) that R is not in $\text{thick}^{\otimes} C$.

The proof of Theorem 4.1 is rather technical, which uses the following fact and lemma. See [10, Propositions 7.1, 7.2, Corollary 7.3 and Lemma 7.8].

FACT 4.3. Let R be a discrete valuation ring.

- (1) For each complex $M \in \mathcal{D}^-(R)$ there is an isomorphism

$$M \cong \mathbf{H}(M) = \bigoplus_{i \in \mathbb{Z}} \mathbf{H}^i(M)[-i],$$

since the ring R is hereditary. In particular, we may assume that each object of $\mathcal{D}^-(R)$ is a complex with zero differentials.

- (2) The nonisomorphic indecomposable R -modules of finite length are precisely given by the list

$$\{R/x^i R\}_{i>0},$$

which is a consequence of the structure theorem for finitely generated modules over a principal ideal domain.

LEMMA 4.4. *Let $\mathcal{X} = \mathbf{thick}^{\otimes}(\cdots \xrightarrow{0} X_2 \xrightarrow{0} X_1 \xrightarrow{0} X_0 \rightarrow 0)$ be an ideal of $\mathcal{D}^-(R)$.*

- (1) *One has*

$$\mathcal{X} = \mathbf{thick}^{\otimes}(\cdots \xrightarrow{0} X_2 \oplus X_1 \oplus X_0 \xrightarrow{0} X_1 \oplus X_0 \xrightarrow{0} X_0 \rightarrow 0).$$

- (2) *The ideal \mathcal{X} contains the complex*

$$(\cdots \rightarrow 0 \rightarrow X_2^{\oplus a_2} \rightarrow 0 \rightarrow X_1^{\oplus a_1} \rightarrow 0 \rightarrow X_0^{\oplus a_0} \rightarrow 0)$$

for all integers $a_i \geq 0$.

- (3) *The complex*

$$(\cdots \xrightarrow{0} X_3^{\oplus a_3} \xrightarrow{0} X_2^{\oplus a_2} \xrightarrow{0} X_1^{\oplus a_1} \xrightarrow{0} X_0^{\oplus a_0} \rightarrow 0)$$

is in $\mathbf{thick}^{\otimes}\{X_{\text{even}}, X_{\text{odd}}\}$, where

$$\begin{cases} X_{\text{even}} = (\cdots \xrightarrow{0} X_6 \xrightarrow{0} X_4 \xrightarrow{0} X_2 \xrightarrow{0} X_0 \rightarrow 0), \\ X_{\text{odd}} = (\cdots \xrightarrow{0} X_7 \xrightarrow{0} X_5 \xrightarrow{0} X_3 \xrightarrow{0} X_1 \rightarrow 0). \end{cases}$$

Here are some questions naturally arisen from the above lemma.

QUESTION. Let k be a field, and let $R = k[[x]]$ be a formal power series ring.

- (1) Does the ideal $\mathbf{thick}^{\otimes}(\cdots \xrightarrow{0} X_2 \xrightarrow{0} X_1 \xrightarrow{0} X_0 \rightarrow 0)$ contain the complex $(\cdots \xrightarrow{0} X_2^{a_2} \xrightarrow{0} X_1^{a_1} \xrightarrow{0} X_0^{a_0} \rightarrow 0)$ for all integers $a_i \geq 0$?
- (2) Does the ideal $\mathbf{thick}^{\otimes}(\cdots \rightarrow 0 \rightarrow X_1 \rightarrow 0 \rightarrow X_0 \rightarrow 0)$ contain the complex $(\cdots \xrightarrow{0} X_1 \xrightarrow{0} X_0 \rightarrow 0)$?
- (3) Can we characterize the complexes C belonging to the ideal

$$\mathbf{thick}^{\otimes}(\cdots \xrightarrow{0} R/x^{2!} \xrightarrow{0} R/x^{1!} \xrightarrow{0} R/x^{0!} \rightarrow 0)$$

in terms of the Loewy lengths of homologies of C ?

Recall that a triangulated functor $F : \mathcal{T} \rightarrow \mathcal{T}'$ of tensor triangulated categories is called *tensor triangulated* if $F(\mathbb{1}) = \mathbb{1}$ and $F(x \otimes y) = F(x) \otimes F(y)$ for all $x, y \in \mathcal{T}$. A tensor triangulated functor $F : \mathcal{T} \rightarrow \mathcal{T}'$ induces a continuous map

$$\mathbf{Spc} F : \mathbf{Spc} \mathcal{T}' \rightarrow \mathbf{Spc} \mathcal{T},$$

which is given by $\mathcal{P} \mapsto \mathcal{P} \cap \mathcal{T} := F^{-1}(\mathcal{P})$. This is injective if F is essentially surjective; see [2, Proposition 3.6 and Corollary 3.8].

Note that if $\mathbf{Spc} F$ is injective, then it sends a chain

$$\cdots \subsetneq \mathcal{P} \subsetneq \mathcal{Q} \subsetneq \cdots$$

of prime ideals of \mathcal{T}' to a chain

$$\cdots \subsetneq \mathcal{P} \cap \mathcal{T} \subsetneq \mathcal{Q} \cap \mathcal{T} \subsetneq \cdots$$

of prime ideals of \mathcal{T} . Hence the inequality

$$\dim(\mathrm{Spc} \mathcal{T}') \leq \dim(\mathrm{Spc} \mathcal{T})$$

of Krull dimensions holds.

This observation and Theorem 4.1(3) yield the following result; see [10, Corollary 7.13].

COROLLARY 4.5. *Assume that $R_{\mathfrak{p}}$ is regular for some \mathfrak{p} with positive height (e.g. R is normal). Then*

$$\dim(\mathrm{Spc} D^-(R)) = \infty.$$

PROOF. We may assume that \mathfrak{p} has height 1. Localization at \mathfrak{p} induces an essentially surjective tensor triangulated functor

$$D^-(R) \rightarrow D^-(R_{\mathfrak{p}}).$$

The above argument implies that

$$\dim \mathrm{Spc} D^-(R_{\mathfrak{p}}) \leq \dim \mathrm{Spc} D^-(R).$$

Combining this with Theorem 4.1(3), we see that $\mathrm{Spc} D^-(R)$ has infinite Krull dimension. \square

Recall that a topological space is said to be *noetherian* if any descending chain of closed subsets stabilizes. Corollary 4.5 naturally makes us to have the following question.

QUESTION 4.6. Let R satisfy the same assumption as in Corollary 4.5. Is then the Balmer spectrum $\mathrm{Spc} D^-(R)$ non-noetherian?

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