ON THE CATEGORICAL ENTROPY OF THE FROBENIUS PUSHFORWARD FUNCTOR

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Abstract. In this paper, we consider the Frobenius pushforward endofunctor F_* of the bounded derived category of finitely generated modules over an F-finite noetherian local ring. We completely determine the categorical entropy of F_* in the sense of Dimitrov, Haiden, Katzarkov, and Kontsevich.

1. Introduction

For a *categorical dynamical system*, namely, a pair $(\mathcal{T}, \Phi)$ of a triangulated category $\mathcal{T}$ and an exact endofunctor $\Phi : \mathcal{T} \to \mathcal{T}$, Dimitrov, Haiden, Katzarkov, and Kontsevich [1] have introduced an invariant $\mathbf{h}_t^{\mathcal{T}}(\Phi)$ called the *categorical entropy* of Φ, which is a categorical analogue of the topological entropy. The categorical entropy $\mathbf{h}_t^{\mathcal{T}}(\Phi)$ is a function in one real variable with values in $\mathbb{R} \cup \{-\infty\}$ and measures the complexity of the exact endofunctor Φ.

In this paper, we consider the Frobenius endomorphism $F : R \to R$ of a commutative noetherian local ring R with prime characteristic $p > 0$, assuming that R is *F-finite*, that is to say, the map F is (module-)finite. The Frobenius endomorphism F induces two exact endofunctors. One is called the *Frobenius pushforward* F_* on the bounded derived category $\mathsf{D}^{\mathsf{b}}(R)$ of finitely generated R-modules and the other is called the *Frobenius pullback* $\mathbb{L}F^*$ on the derived category $\mathsf{D}^{\mathsf{perf}}(R)$ of perfect R-complexes. As to the latter, Majidi-Zolbanin and Miasnikov [11] considered the full subcategory $\mathsf{D}^{\mathsf{perf}}_{\mathsf{fl}}(R)$ of $\mathsf{D}^{\mathsf{perf}}(R)$ consisting of perfect complexes with finite length cohomologies, and computed the categorical entropy $\mathbf{h}_t^{\mathsf{D}^{\mathsf{perf}}_{\mathsf{fl}}(R)}(\mathbb{L}F^*)$.

The aim of this paper is to study the Frobenius pushforward F_* on $\mathsf{D}^{\mathsf{b}}(R)$ and compute its categorical entropy. The main result of this paper is the following theorem.

Theorem 1.1 (Corollary 4.3)**.** *Let $(R, \mathfrak{m}, k)$ be a d-dimensional F-finite noetherian local ring with prime characteristic p. Then there is an equality*

$$\mathbf{h}_t^{\mathsf{D}^{\mathsf{b}}(R)}(F_*) = d \log p + \log[F_*(k) : k].$$

For an arbitrary finite local endomorphism $\phi : R \to R$ of an arbitrary noetherian local ring R, one can take its pushforward $\phi_* : \mathsf{D}^{\mathsf{b}}(R) \to \mathsf{D}^{\mathsf{b}}(R)$. In such a general setting, one can still obtain the following weaker result on the categorical entropy.

Theorem 1.2 (Theorem 3.2)**.** *Let $(R, \mathfrak{m}, k)$ be a d-dimensional noetherian local ring and $\phi : R \to R$ a local ring endomorphism of finite length. Then there is an inequality*

$$\mathbf{h}_t^{\mathsf{D}^{\mathsf{b}}(R)}(\phi_*) \geq \mathbf{h}_{\mathrm{loc}}(\phi) + \log[\phi_*(k) : k].$$

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Here, $\mathbf{h}_{\mathrm{loc}}(\phi)$ is the *local entropy* of ϕ, which has been introduced by Majidi-Zolbanin, Miasnikov, and Szpiro [12].

The organization of this paper is as follows. Section 2 is devoted to giving basic definitions including that of the categorical entropy. In Section 3, as a consequence of Theorem 1.2, we prove one inequality of the equality given in Theorem 1.1. In Section 4, we prove the opposite inequality, so that the the proof of Theorem 1.1 is completed.

Convention. Throughout the present paper, we assume that all rings are commutative and noetherian, and all subcategories are strictly full.

2. Preliminaries

In this section, we recall the notions of the *categorical entropy* of an exact endofunctor of a triangulated category and the *local entropy* of an endomorphism of local ring. First of all, let us fix some notations.

Notation 2.1. (1) For a ring R, denote by $\mathsf{D^b}(R) = \mathsf{D^b}(\operatorname{mod} R)$ the category of bounded complexes of finitely generated R-modules, and by $\mathsf{D^{perf}}(R)$ the full subcategory of perfect complexes. Here, a complex X of R-modules is called *perfect* if there exists a bounded complex P of finitely generated projective R-modules such that $X \cong P$ in $\mathsf{D^b}(R)$.

(2) Let $f : R \to S$ be a ring homomorphism.
  - (a) The *pullback functor* $f^* : \operatorname{mod} R \to \operatorname{mod} S$ is defined by $f^*(M) = M \otimes_R S$ for $M \in \operatorname{mod} R$. The left derived functor of f^* is an exact functor $\mathbb{L}f^* : \mathsf{D^{perf}}(R) \to \mathsf{D^{perf}}(S)$.
  - (b) If f is finite, then the *pushforward functor* $f_* : \operatorname{mod} S \to \operatorname{mod} R$ is defined by the abelian group $f_*(M) := M$ for $M \in \operatorname{mod} S$ together with the R-module structure via f. The pushforward functor f_* is exact, whence its right derived functor $f_* = \mathbb{R}f_* : \mathsf{D^b}(\operatorname{mod} S) \to \mathsf{D^b}(\operatorname{mod} R)$ is defined by the degreewise application of f_*.

(3) For a triangulated category $\mathcal{T}$ and an object X of $\mathcal{T}$, we denote by $\operatorname{thick} X$ the smallest thick subcategory of $\mathcal{T}$ that contains X.

2.1. **Categorical entropy.** We recall the definition and basic properties of the categorical entropy of an exact endofunctor of a triangulated category. The following notation is useful.

Notation 2.2. Let $\mathcal{T}$ be a triangulated category.

(1) For two full subcategories $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$, we denote by $\mathcal{X} * \mathcal{Y}$ the full subcategory of $\mathcal{T}$ consisting of objects Z which fit into an exact triangle $X \to Z \to Y \to X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. By the octahedral axiom, this symbol turns out to be associative: the equality $(\mathcal{X} * \mathcal{Y}) * \mathcal{Z} = \mathcal{X} * (\mathcal{Y} * \mathcal{Z})$ holds for full subcategories $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ of $\mathcal{T}$.

(2) For an integer $r \geq 2$ and full subcategories $\mathcal{E}_1, \dots, \mathcal{E}_r \subseteq \mathcal{T}$, we inductively define $\mathcal{E}_1 * \cdots * \mathcal{E}_r = (\mathcal{E}_1 * \cdots * \mathcal{E}_{r-1}) * \mathcal{E}_r$. If $\mathcal{E}_i = \{X_i\}$ for all i, then we write $X_1 * \cdots * X_r$ for $\mathcal{E}_1 * \cdots * \mathcal{E}_r$. If $X_i = X$ for all i, then we put $X^{*r} = \overbrace{X * \cdots * X}^{r \text{ times}}$.

Definition 2.3. ([1, Definition 2.1]) Let X, Y be objects in $\mathcal{T}$. For a real number t, define the *complexity* $\delta_t^{\mathcal{T}}(X, Y) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ of Y with respect to X by

$$\delta_t^{\mathcal{T}}(X, Y) = \inf \left\{ \sum_{i=1}^{r} e^{n_i t} \;\middle|\; \begin{array}{c} Y \oplus Y' \in X[n_1] * \cdots * X[n_r] \\ \text{for some } Y' \in \mathcal{T} \text{ and } n_1, \dots, n_r \in \mathbb{Z} \end{array} \right\}.$$

We will drop the superscript $\mathcal{T}$ when there is no possibility of confusion. By definition, $\delta_t(X, Y) < \infty$ if and only if $Y \in \operatorname{thick} X$.

We list several fundamental properties of the complexity.

Lemma 2.4. *Let $\mathcal{T}$ be a triangulated category.*

(1) *For $X, Y, Z \in \mathcal{T}$ with $Z \in \mathsf{thick}\, Y \subseteq \mathsf{thick}\, X$, one has $\delta_t(X, Z) \le \delta_t(X, Y)\delta_t(Y, Z)$.*
(2) *For $X, Y, Z \in \mathcal{T}$, one has $\delta_t(X, Y) \le \delta_t(X, Y \oplus Z) \le \delta_t(X, Y) + \delta_t(X, Z)$.*
(3) *For $X, Y, Z \in \mathcal{T}$, one has $\delta_t(X \oplus Y, Z) \le \delta_t(X, Z)$.*
(4) *For $X, Y \in \mathcal{T}$, one has $\delta_t(X, Y[n]) = \delta_t(X, Y)e^{nt}$.*
(5) *For $X, Y, Y_1, \dots, Y_r \in \mathcal{T}$ with $Y \in Y_1 * \cdots * Y_r$, one has $\delta_t(X, Y) \le \sum_{i=1}^{r} \delta_t(X, Y_i)$.*
(6) *For an exact functor $\Phi : \mathcal{T} \to \mathcal{T}'$ and $X, Y \in \mathcal{T}$, one has $\delta_t^{\mathcal{T}'}(\Phi(X), \Phi(Y)) \le \delta_t^{\mathcal{T}}(X, Y)$.*

Proof. Assertions (1), (2) and (6) are shown in [1, Proposition 2.2], while (4) and (5) are direct consequences of the definition. Let us prove (3). We may assume $\delta_t(X, Z) < \infty$. Take $Z' \in \mathcal{T}$ and $n_1, \dots n_r \in \mathbb{Z}$ such that $Z \oplus Z' \in X[n_1] * X[n_2] * \cdots * X[n_r]$. We easily see that

$$Z \oplus Z' \oplus Y[n_1 + n_2 + \cdots + n_r] \in (X \oplus Y)[n_1] * (X \oplus Y)[n_2] * \cdots * (X \oplus Y)[n_r]$$

holds. Therefore, the inequality $\delta_t(X \oplus Y, Z) \le \delta_t(X, Z)$ follows. ■

An object G of a triangulated category $\mathcal{T}$ is called a *split generator* if $\mathcal{T} = \mathsf{thick}\, G$. For an excellent scheme X, the derived category $\mathsf{D^b}(\mathsf{coh}\, X)$ of bounded complexes of coherent sheaves on X has a split generator by [2, Theorem 4.15]; see also [8, Theorem 1.1].

Definition 2.5 ([1, Definition 2.4])**.** Let $\mathcal{T}$ be a triangulated category with a split generator G. Let $\Phi : \mathcal{T} \to \mathcal{T}$ be an exact endofunctor. For a real number t, we put

$$\mathbf{h}_t^{\mathcal{T}}(\Phi) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{1}{n} \log \delta_t^{\mathcal{T}}(G, \Phi^n(G)) & \text{if } \delta_t(G, \Phi^e(G)) \neq 0 \text{ for all } e \gg 0, \\ -\infty & \text{otherwise,} \end{cases}$$

and call it the *categorical entropy* of Φ. It follows from [1, Lemma 2.6] that $\mathbf{h}_t^{\mathcal{T}}(\Phi)$ exists in $[-\infty, \infty)$ and is independent of the choice of a split generator G. Omitting the superscript $\mathcal{T}$, we may simply write $\mathbf{h}_t(\Phi)$ if there is no danger of confusion.

Remark 2.6. Let $\mathcal{T}$, G and Φ be as in Definition 2.5.

(1) Let n be an integer such that $\delta_t(G, \Phi^n(G)) = 0$. Then it follows by Lemma 2.4(1)(6) that
$$\delta_t(G, \Phi^{n+1}(G)) \le \delta_t(G, \Phi^n(G))\delta_t(\Phi^n(G), \Phi^{n+1}(G)) \le \delta_t(G, \Phi^n(G))\delta_t(G, \Phi(G)) = 0,$$
and this shows $\delta_t(G, \Phi^{n'}(G)) = 0$ for all $n' \ge n$.
(2) Let G' be another split generator. Then, by Lemma 2.4(1)(6), for each $n \ge 0$ one has
$$\begin{aligned} \delta_t(G, \Phi^n(G)) &\le \delta_t(G, G')\delta_t(G', \Phi^n(G'))\delta_t(\Phi^n(G'), \Phi^n(G)) \\ &\le \delta_t(G, G')\delta_t(G', \Phi^n(G'))\delta_t(G', G). \end{aligned}$$
Therefore, if $\delta_t(G, \Phi^n(G)) \neq 0$ for all $n \ge 0$, then $\delta_t(G', \Phi^n(G')) \neq 0$ for all $n \ge 0$, and moreover, $\delta_t(G, G') \neq 0$ and $\delta_t(G', G) \neq 0$.

Let us recall several asymptotic notations.

Notation 2.7. For two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of real numbers, we write

- $a_n = O(b_n)$ if there is a real number $C > 0$ such that $a_n \le C b_n$ for all $n \gg 1$,
- $a_n = \Omega(b_n)$ if there is a real number $C > 0$ such that $a_n \ge C b_n$ for all $n \gg 1$, and
- $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$.

We present some elementary facts about the asymptotic notations introduced above.

Lemma 2.8. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $a_{m+n} \le a_m a_n$ for all $m,n \ge 1$. Let u be a positive real number. Set $\alpha = \lim_{n\to\infty} \frac{\log a_n}{n} \in [-\infty, \infty)$ (this limit exists by Fekete's lemma). Then the following statements hold.*

(1) (a) *If $a_n = O(u^n)$, then $\alpha \le \log u$.*
(b) *If $a_n = \Omega(u^n)$, then $\alpha \ge \log u$.*
(c) *If $a_n = \Theta(u^n)$, then $\alpha = \log u$.*

(2) *For any real number $\beta > \alpha$, one has $a_n = O(e^{n\beta})$.*

Proof. (1) Let us show assertion (a). By definition, there is a real number $C > 0$ such that $a_n/u^n \le C$ for all $n \gg 1$. It holds that

$$0 = \limsup_{n\to\infty}(\tfrac{1}{n}\log C) \ge \limsup_{n\to\infty}(\tfrac{1}{n}\log \tfrac{a_n}{u^n})$$
$$= \limsup_{n\to\infty}(\tfrac{\log a_n}{n} - \log u) = \lim_{n\to\infty}\tfrac{\log a_n}{n} - \log u,$$

which implies that $\alpha = \lim_{n\to\infty}\frac{\log a_n}{n} \le \log u$. Assertions (b) and (c) can be shown similarly.

(2) Since $\lim_{n\to\infty}\frac{\log a_n}{n} < \beta$, we have $\lim_{n\to\infty}\log\frac{\sqrt[n]{a_n}}{e^\beta} = \lim_{n\to\infty}(\frac{\log a_n}{n} - \beta) < 0$. This means that $\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{e^\beta} < 1$. Hence the inequality $\frac{\sqrt[n]{a_n}}{e^\beta} < 1$ holds for all $n \gg 1$, and we get $a_n < e^{n\beta}$ for all $n \gg 1$. Now the conclusion $a_n = O(e^{n\beta})$ follows. ■

Using the above lemma, we can prove the following proposition, which connect the order of the complexity and the categorical entropy.

Proposition 2.9. *Let $\mathcal{T}$ be a triangulated category. Let G be a split generator of $\mathcal{T}$. Let $\Phi : \mathcal{T} \to \mathcal{T}$ be an exact functor. Then the following statements hold true.*

(1) (a) *If $\delta_t(G, \Phi^n(G)) = O(u^n)$, then $\mathbf{h}_t(\Phi) \le \log u$.*
(b) *If $\delta_t(G, \Phi^n(G)) = \Omega(u^n)$, then $\mathbf{h}_t(\Phi) \ge \log u$.*
(c) *If $\delta_t(G, \Phi^n(G)) = \Theta(u^n)$, then $\mathbf{h}_t(\Phi) = \log u$.*

(2) *For any $\beta > \mathbf{h}_t(\Phi)$, one has the equality $\delta_t(G, \Phi^n(G)) = O(e^{n\beta})$.*

Remark 2.10. Let $\mathcal{T}, G, \Phi$ be as in Proposition 2.9. The *categorical polynomial entropy*

$$\mathbf{h}_t^{\mathrm{pol}}(\Phi) := \limsup_{n\to\infty} \frac{\log \delta_t(G, \Phi^n(G)) - nh_t(\Phi)}{\log n}$$

has been introduced by Fan, Fu and Ouchi [3]. If $\mathbf{h}_t^{\mathrm{pol}}(\Phi)$ is positive, then it is easy to see that $\delta_t(G, \Phi^n(G)) \ne O(e^{n\mathbf{h}_t(\Phi)})$. Therefore, for each $\alpha \in \mathbb{R}$, the equality $\delta_t(G, \Phi^n(G)) = O(e^{n\alpha})$ is stronger than the equality $\mathbf{h}_t(\Phi) = \alpha$ in general.

2.2. **Local entropy.** Next we recall the notion of the local entropy of a local dynamical system introduced in [12]. We start by basic notions from (local) commutative algebra.

Definition 2.11. A local homomorphism $\phi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ of local rings is of *finite length* if the ideal $\phi(\mathfrak{m})S$ of S is $\mathfrak{n}$-primary. If ϕ is finite, then it is of finite length.

Here are examples of local endomorphisms of finite length which we consider in this paper.

Example 2.12. (1) If a local ring R has prime characteristic $p > 0$, then the *Frobenius endomorphism* $F : R \to R$ given by $x \mapsto x^p$ is of finite length.

(2) Let $\Gamma = \sum_{i=1}^r \mathbb{Z}_{\ge 0}\, a_i$ be a finitely generated additive monoid and k a field. Let $k[\![\Gamma]\!]$ denote the completion of the monoid ring $k[\Gamma] = k[t^{a_i} \mid i = 1, \dots, r]$ with respect to its homogeneous maximal ideal $(t^{a_i} \mid i = 1, \dots, r)$. For an integer $m \ge 1$, the morphism of monoids $\Gamma \to \Gamma$ given by $x \mapsto mx$ induces a local ring endomorphism $F_m : k[\![\Gamma]\!] \to k[\![\Gamma]\!]$ which is of finite length.

For a local endomorphism ϕ of a local ring R, an integer $e \geq 1$, and an R-module M, we write ${}^{\phi^e}M$ for $\phi^e_*(M)$. When ϕ is understood from the context, we simply write ${}^eM$.

Definition 2.13. Let ϕ be a local endomorphism of a local ring R. We say that R is *ϕ-finite* provided that ${}^{\phi}R$ is finitely generated as an R-module.

Remark 2.14. (1) If a local ring R is ϕ-finite, then so is the residue field k, i.e., $[{}^{\phi}k : k] < \infty$.
(2) Let R be a local ring of prime characteristic p. If R is F-finite, then R is excellent. The converse holds if the residue field k is F-finite; see [10, Theorem 2.5 and Corollary 2.6].
(3) Let k be a field. Let $\Gamma = \sum_{i=1}^{r} \mathbb{Z}_{\geq 0}\, a_i$ be a finitely generated additive monoid. Then the local ring $k[\![\Gamma]\!]$ is F_m-finite for every positive integer m. Indeed, ${}^{F_m}k[\![\Gamma]\!]$ is generated by the monomials $\{t^{i_1 a_1} \cdots t^{i_r a_r} \mid 0 \leq i_1, \dots, i_r < m\}$ as an $k[\![\Gamma]\!]$-module.

The following easy lemma is frequently used later.

Lemma 2.15. *Let ϕ be a finite local endomorphism of a local ring R. For an R-module M of finite length and $e \geq 0$, the R-module ${}^eM$ has finite length with $\ell_R({}^eM) = [{}^1k : k]^e \cdot \ell_R(M)$.*

Proof. First we prove $\ell_R({}^eM) = [{}^ek : k] \cdot \ell_R(M)$ by induction on $n := \ell_R(M)$. The case $n = 1$ is clear as $M \cong k$. Let $n > 1$. Then there is an exact sequence $0 \to N \to M \to k \to 0$ of R-modules. Applying ${}^e(-)$ to this sequence, we get an exact sequence $0 \to {}^eN \to {}^eM \to {}^ek \to 0$. By the induction hypothesis, $\ell_R({}^eN) = [{}^ek : k] \cdot \ell_R(N)$ and $\ell_R({}^ek) = [{}^ek : k]$, which yield

$$\ell_R({}^eM) = \ell_R({}^eN) + \ell_R({}^ek) = [{}^ek : k] \cdot \ell_R(N) + [{}^ek : k] = [{}^ek : k](\ell_R(N) + 1) = [{}^ek : k] \cdot \ell_R(M).$$

Next we prove $[{}^ek : k] = [{}^1k : k]^e$. Since ${}^1k \cong k^{\oplus[{}^1k:k]}$, one has ${}^ek = {}^{e-1}({}^1k) \cong {}^{e-1}(k^{\oplus[{}^1k:k]}) \cong ({}^{e-1}k)^{\oplus[{}^1k:k]}$. We thus get an isomorphism ${}^ek \cong k^{\oplus[{}^1k:k]^e}$ by induction on e. ■

Definition 2.16 ([12, Definition 5])**.** By a *local algebraic dynamical system* (R, ϕ), we mean a pair of a local ring R and a local endomorphism ϕ of R which is of finite length. A local algebraic dynamical system (R, ϕ) is called *finite* if R is ϕ-finite.

For a local algebraic dynamical system (R, ϕ), the *local entropy* of ϕ is defined by

$$\mathbf{h}_{\mathrm{loc}}(\phi) := \lim_{e \to \infty} \frac{\log \ell_R(R/\phi^e(\mathfrak{m})R)}{e}.$$

This limit exists and is nonnegative by [12, Theorem 1].

For a local ring R we denote by $\mathrm{e}(R)$ and $\operatorname{edim} R$ the (Hilbert–Samuel) multiplicity and the embedding dimension of a local ring R, respectively. Under a certain assumption, we can explicitly compute the local entropy.

Lemma 2.17. *Let (R, ϕ) be a local algebraic dynamical system. Put $d = \dim R$ and $\nu = \operatorname{edim} R$. Assume that there exists an integer $u \geq 1$ such that $\mathfrak{m}^{\nu u^e} \subseteq \phi^e(\mathfrak{m})R \subseteq \mathfrak{m}^{u^e}$ for all $e \gg 1$. Then $\ell_R(R/\phi^e(\mathfrak{m})R) = \Theta(u^{de})$. In particular, there is an equality $\mathbf{h}_{\mathrm{loc}}(\phi) = d \log u$.*

Proof. By assumption, we have inequalities $\ell_R(R/\mathfrak{m}^{u^e}) \leq \ell_R(R/\phi^e(\mathfrak{m})R) \leq \ell_R(R/\mathfrak{m}^{\nu u^e})$ for all $e \gg 1$. Since $\ell_R(R/\mathfrak{m}^{u^e})/u^{de}$ and $\ell_R(R/\mathfrak{m}^{\nu u^e})/u^{de}$ converge to the nonzero real numbers $\mathrm{e}(R)d!$ and $\mathrm{e}(R)\nu d!$ respectively, we obtain the equality $\ell_R(R/\phi^e(\mathfrak{m})R) = \Theta(u^{de})$. The equality $\mathbf{h}_{\mathrm{loc}}(\phi) = d \log u$ follows from $\ell_R(R/\phi^e(\mathfrak{m})R) = \Theta(u^{de})$ and Proposition 2.9(1c). ■

Since the endomorphisms in Example 2.12 satisfy the assumption in this lemma, we get:

Corollary 2.18. (1) *Let R be a d-dimensional F-finite local ring of characteristic p. Then the equalities $\ell_R(R/F^e_*(\mathfrak{m})R) = \Theta(p^{de})$ and $\mathbf{h}_{\mathrm{loc}}(F) = d \log p$ hold.*

(2) *Let k be a field, Γ a finitely generated additive monoid and $R = k[\![\Gamma]\!]$. Let $m \geq 1$ be an integer. Then one has the equalities $\ell_R(R/(F_m)_*^e(\mathfrak{m})R) = \Theta(m^{de})$ and $\mathbf{h}_{\mathrm{loc}}(F_m) = d \log m$.*

In this paper, we are mainly concerned with the Frobenius functor. Let $(R, \mathfrak{m}, k)$ be an F-finite local ring with characteristic p. Then the Frobenius endomorphism $F : R \to R$ induces the *Frobenius pushforward*

$$\begin{array}{cccc} F_* = \mathbb{R}F_* : & \mathsf{D}^{\mathsf{b}}(R) & \longrightarrow & \mathsf{D}^{\mathsf{b}}(R) \\ & \cup\!\mid & & \cup\!\mid \\ & \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R) & \longmapsto & \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R), \end{array}$$

and the *Frobenius pullback* (see [11, Proposition 1.10])

$$\begin{array}{cccc} \mathbb{L}F^* : & \mathsf{D}^{\mathsf{perf}}(R) & \longrightarrow & \mathsf{D}^{\mathsf{perf}}(R) \\ & \cup\!\mid & & \cup\!\mid \\ & \mathsf{D}^{\mathsf{perf}}_{\mathsf{fl}}(R) & \longmapsto & \mathsf{D}^{\mathsf{perf}}_{\mathsf{fl}}(R). \end{array}$$

Here, $\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R), \mathsf{D}^{\mathsf{perf}}_{\mathsf{fl}}(R)$ stand for the subcategories of $\mathsf{D}^{\mathsf{b}}(R), \mathsf{D}^{\mathsf{perf}}(R)$ consisting of complexes with finite length homologies, respectively. The categorical entropy of the Frobenius pullback on $\mathsf{D}^{\mathsf{perf}}_{\mathsf{fl}}(R)$ is computed by Majidi-Zolbanin and Miasnikov:

Theorem 2.19 ([11, Corollary 2.6])**.** *Let R be a d-dimensional complete local ring of prime characteristic $p > 0$. Then the equality $\mathbf{h}_t^{\mathsf{D}^{\mathsf{perf}}_{\mathsf{fl}}(R)}(\mathbb{L}F^*) = d \log p$ holds.*

On the other hand, for the Frobenius pullback on $\mathsf{D}^{\mathsf{perf}}(R)$, the following holds.

Proposition 2.20. *Let R be a local ring of prime characteristic $p > 0$. Then one has the equality $\delta_t^{\mathsf{D}^{\mathsf{perf}}(R)}(R, (\mathbb{L}F^*)^e(R)) = 1$ for all $e \geq 0$. In particular, the equality $\mathbf{h}_t^{\mathsf{D}^{\mathsf{perf}}(R)}(\mathbb{L}F^*) = 0$ holds.*

Proof. Let us prove the first equality. As $(\mathbb{L}F^*)^e(R) \cong R$ for all $e \geq 0$, it suffices to show $\delta_t^{\mathsf{D}^{\mathsf{perf}}(R)}(R, R) = 1$. The inequality $\delta_t^{\mathsf{D}^{\mathsf{perf}}(R)}(R, R) \leq 1$ obviously holds. Assume that there exist $n_1, \dots, n_r \in \mathbb{Z}$ and $X \in \mathsf{D}^{\mathsf{perf}}(R)$ with $R \oplus X \in R[n_1] * \cdots * R[n_r]$. Then at least one of the numbers $n_1, \dots, n_r$ is zero. Indeed, if $n_1, \dots, n_r$ are all nonzero, then the equalities $\mathsf{Hom}_{\mathsf{D}^{\mathsf{perf}}(R)}(R, R[n_i]) = 0$ with $i = 1, \dots, r$ yield $\mathsf{Hom}_{\mathsf{D}^{\mathsf{perf}}(R)}(R, Y) = 0$ for any $Y \in R[n_1] * \cdots * R[n_r]$. In particular, we have $\mathsf{Hom}_{\mathsf{D}^{\mathsf{perf}}(R)}(R, R \oplus X) = 0$, which leads a contradiction. Thus $\sum_{i=1}^r e^{n_i t} \geq 1$, and so $\delta_t^{\mathsf{D}^{\mathsf{perf}}(R)}(R, R) \geq 1$. Now the first equality of the proposition follows. As R is a split generator of $\mathsf{D}^{\mathsf{perf}}(R)$, the second equality follows from the first. ■

We can also compute the categorical entropy of the Frobenius pushforward on $\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$.

Proposition 2.21. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional F-finite local ring. Then for every $e \geq 1$ the equality $\delta_t^{\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)}(k, {}^e k) = [{}^1 k : k]^e$ holds. In particular, one has $\mathbf{h}_t^{\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)}(F_*) = \log[{}^1 k : k]$.*

Proof. As k is a split generator of $\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$, it is enough to show the first equality. The inequality $\delta_t^{\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)}(k, {}^e k) \leq [{}^1 k : k]^e$ is trivial because ${}^e k \cong k^{\oplus [{}^1 k : k]^e}$. Take integers $n_1, \dots, n_r$ and $X \in \mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)$ such that $k^{\oplus [{}^1 k : k]^e} \oplus X \in k[n_1] * \cdots * k[n_r]$. Since $\ell_R(\mathrm{H}^0(-))$ is a subadditive function with respect to exact triangles, we obtain the (in)equalities

$$[{}^1 k : k]^e \leq \ell_R(\mathrm{H}^0(k^{\oplus [{}^1 k : k]^e})) \leq \ell_R(\mathrm{H}^0(k^{\oplus [{}^1 k : k]^e} \oplus X)) \leq \textstyle\sum_{i=1}^r \ell_R(\mathrm{H}^0(k[n_i])) = \#\{i \mid n_i = 0\}.$$

Hence $\sum_{i=1}^r e^{n_i t} \geq \sum_{n_i = 0} 1 \geq [{}^1 k : k]^e$. The inequality $\delta_t^{\mathsf{D}^{\mathsf{b}}_{\mathsf{fl}}(R)}(k, {}^e k) \geq [{}^1 k : k]^e$ follows. ■

From these results, the remaining problem is to compute the categorical entropy of the Frobenius pushforward on $\mathsf{D^b}(R)$, which we shall deal with in the subsequent sections.

We close this section by giving a remark.

Remark 2.22. The categorical entropy has been introduced as a categorical analogue of the topological entropy $\mathbf{h}_{top}(f)$, which is defined for a pair (X, f) of a Hausdorff space X and a continuous self-map $f : X \to X$. The topological entropy satisfies the following properties (cf. [4, Section 1.6]):

(a) If $Y \subseteq X$ is a closed subspace such that $f(Y) \subseteq Y$, then $\mathbf{h}_{top}(f|_Y) \leq \mathbf{h}_{top}(f)$.
(b) If $X = \bigcup_{i=1}^n Y_i$ where each Y_i is a closed subspace such that $f(Y_i) \subseteq Y_i$, then $\mathbf{h}_{top}(f) = \max\{\mathbf{h}_{top}(f|_{Y_i}) \mid i = 1, 2, \dots, n\}$.

It is natural to ask if the following categorical analogues of these properties hold. However, the above Theorem 2.19 and Proposition 2.20 show that they do not hold in general:

Let $(\mathcal{T}, \Phi)$ be a categorical dynamical system.

(a') Let $\mathcal{S} \subseteq \mathcal{T}$ be a thick subcategory such that $\Phi(\mathcal{S}) \subseteq \mathcal{S}$. Then $\mathbf{h}_t^{\mathcal{S}}(\Phi|_{\mathcal{S}}) \leq \mathbf{h}_t^{\mathcal{T}}(\Phi)$.
(b') Let $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n \subseteq \mathcal{T}$ be thick subcategories such that $\mathcal{T} = \mathsf{thick}(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n)$ and $\Phi(\mathcal{S}_i) \subseteq \mathcal{S}_i$. Then $\mathbf{h}_t^{\mathcal{T}}(\Phi) = \max\{\mathbf{h}_t^{\mathcal{S}_i}(\Phi|_{\mathcal{S}_i}) \mid i = 1, 2, \dots, n\}$.

Here, (b') has been addressed and proved to be true in [9] when $\langle \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n \rangle$ is a semi-orthogonal decomposition of $\mathcal{T}$.

3. Lower bounds

In this section, we give in terms of local entropies a lower bound on the categorical entropy $\mathbf{h}_t(\phi)$ of the pushforward $\phi_* : \mathsf{D^b}(R) \to \mathsf{D^b}(R)$ along a finite local endomorphism ϕ of a local ring R. The following lemma plays a key role in showing the main theorem in this section, ideas of whose proof are taken from [11, Lemma 2.1]. Denote by $\mathrm{K}(-)$ the Koszul complex.

Lemma 3.1. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $\underline{x}$ be a sequence of elements of R with $\sqrt{(\underline{x})} = \mathfrak{m}$. Let $G \in \mathsf{D^b}(R)$. Fix an integer N such that $\mathrm{H}^i(G \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x})) = 0$ for all $|i| > N$, and set $B = \max\{\ell_R(\mathrm{H}^i(G \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x}))) \mid -N \leq i \leq N\}$. Then for any $E \in \mathsf{D^b}(R)$ and $m \in \mathbb{Z}$, one has*

$$\ell_R(\mathrm{H}^m(E \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x}))) \leq B e^{mt} e^{N|t|} \delta_t(G, E).$$

Proof. We can assume $E \in \mathsf{thick}\, G$, because otherwise, the right-hand side is positive infinity. We find $E' \in \mathsf{D^b}(R)$ and integers $n_1, \dots, n_r$ such that $E \oplus E' \in G[n_1] * G[n_2] * \cdots * G[n_r]$. Applying $- \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x})$, we get a containment

$$(E \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x})) \oplus (E' \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x})) \in (G \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x}))[n_1] * (G \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x})[n_2] * \cdots * (G \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x})[n_r].$$

Since $\ell_R(\mathrm{H}^i(-))$ is a subadditive function with respect to exact triangles, the inequalities

$$\begin{aligned} \ell_R(\mathrm{H}^m(E \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x}))) &\leq \ell_R(\mathrm{H}^m(E \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x})) \oplus \mathrm{H}^m(E' \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x}))) \\ &\leq \textstyle\sum_{i=1}^r \ell_R(\mathrm{H}^{m+n_i}(G \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x}))) \leq B|S_m| \end{aligned}$$

follow, where $S_m := \{i \mid -N \leq m + n_i \leq N\}$. Using the inequality $e^x \geq e^{-|x|}$, we have

$$e^{-N|t|}|S_m| \leq \textstyle\sum_{i \in S_m} e^{-|(m+n_i)t|} \leq \sum_{i=1}^r e^{-|(m+n_i)t|} \leq \sum_{i=1}^r e^{(m+n_i)t} = e^{mt} \sum_{i=1}^r e^{n_i t}.$$

Thus $\ell_R(\mathrm{H}^m(E \otimes_R^{\mathbb{L}} \mathrm{K}(\underline{x}))) \leq B|S_m| \leq B e^{mt} e^{N|t|} \sum_{i=1}^r e^{n_i t}$, and the assertion follows. ■

The following theorem is the main result of this section.

Theorem 3.2. *Let $(R, \mathfrak{m}, k)$ be a local ring with Krull dimension d and embedding dimension ν. Let (R, ϕ) be a finite local algebraic dynamical system. Suppose that $\mathsf{D}^{\mathsf{b}}(R)$ has a split generator (e.g., if R is excellent). Then there is an inequality*

$$\mathbf{h}_t(\phi_*) \geq \mathbf{h}_{\mathrm{loc}}(\phi) + \log[{}^1k : k].$$

If there exists an integer $u \geq 1$ such that the inclusions $\mathfrak{m}^{\nu u^e} \subseteq \phi^e(\mathfrak{m})R \subseteq \mathfrak{m}^{u^e}$ hold for all $e \gg 1$, then the following stronger equality holds for any split generator G of $\mathsf{D}^{\mathsf{b}}(R)$.

$$\delta_t(G, {}^eG) = \Omega(([{}^1k : k]u^d)^e).$$

Proof. We begin with showing the first assertion. Let G be a split generator of $\mathsf{D}^{\mathsf{b}}(R)$. We may assume $\inf G := \inf\{i \mid \mathrm{H}^i(G) \neq 0\} = 0$ and R is a direct summand of $\mathrm{H}^0(G)$ by Remark 2.6(2). By Lemma 3.1, for any fixed t there exists $D_t > 0$ such that $\ell_R(\mathrm{H}^0({}^eG \otimes \mathrm{K}(\underline{x}))) \leq D_t \cdot \delta_t(G, {}^eG)$, where $\underline{x}$ is a system of generators of $\mathfrak{m}$. As $\inf G = 0$, we get equalities

$$\mathrm{H}^0({}^eG \otimes \mathrm{K}(\underline{x})) = \mathrm{H}^0({}^eG)/\mathfrak{m}\mathrm{H}^0({}^eG) = {}^e\mathrm{H}^0(G)/\mathfrak{m}({}^e\mathrm{H}^0(G)) = {}^e(\mathrm{H}^0(G)/\phi^e(\mathfrak{m})\mathrm{H}^0(G)).$$

Here, the second equality follows since the Frobenius pushforward F_* is exact on the category of R-modules. From these observations, we obtain (in)equalities

$$(*)\quad \begin{aligned} \delta_t(G, {}^eG) &\geq D_t^{-1} \cdot \ell_R({}^e\big[\mathrm{H}^0(G)/\phi^e(\mathfrak{m})\mathrm{H}^0(G)\big]) = D_t^{-1} \cdot [{}^ek : k] \cdot \ell_R(\mathrm{H}^0(G)/\phi^e(\mathfrak{m})\mathrm{H}^0(G)) \\ &= D_t^{-1} \cdot [{}^1k : k]^e \cdot \ell_R(\mathrm{H}^0(G)/\phi^e(\mathfrak{m})\mathrm{H}^0(G)) \geq D_t^{-1} \cdot [{}^1k : k]^e \cdot \ell_R(R/\phi^e(\mathfrak{m})R), \end{aligned}$$

where for the first equality we use Lemma 2.15 and for the last inequality we use the assumption that R is a direct summand of $\mathrm{H}^0(G)$. Take the logarithms of both sides of $(*)$, divide them by e, and take the limits to get the inequality $\mathbf{h}_t(\phi_*) \geq \mathbf{h}_{\mathrm{loc}}(\phi) + \log[{}^1k : k]$.

Finally, we show the second assertion of the theorem. Lemma 2.17 implies $\ell_R(R/\phi^e(\mathfrak{m})R) = \Theta(u^{de})$. It follows from Remark 2.6(2) and $(*)$ that $\delta_t(G, {}^eG) = \Omega(([{}^1k : k]u^d)^e)$. ■

As a direct consequence of Theorem 3.2, we get the following corollary.

Corollary 3.3. (1) *Let $(R, \mathfrak{m}, k)$ be a d-dimensional F-finite local ring with characteristic p. Then the equality $\delta_t(G, {}^eG) = \Omega(([{}^1k : k]p^d)^e)$ holds for every split generator G of $\mathsf{D}^{\mathsf{b}}(R)$. In particular, there is an inequality $\mathbf{h}_t(F_*) \geq d \log p + \log[{}^1k : k]$.*

(2) *Let k be a field, Γ a finitely generated additive monoid and $R = k[\![\Gamma]\!]$. Let m be a positive integer. Then one has the equality $\delta_t(G, (F_m)_*^eG) = \Omega(m^{de})$ for every split generator G of $\mathsf{D}^{\mathsf{b}}(R)$. In particular, there is an inequality $\mathbf{h}_t((F_m)_*) \geq d \log m$.*

Next, we generalize Corollary 3.3(1) to the global case. To this end, we need a couple of lemmas. For a prime ideal $\mathfrak{p}$ of a ring R, set $\alpha_{\mathfrak{p}} = \log_p[{}^1k(\mathfrak{p}) : k(\mathfrak{p})]$.

Lemma 3.4. *Let R be an F-finite ring. Let $\mathfrak{p} \subseteq \mathfrak{q}$ be in $\mathsf{Spec}\, R$. Then $\alpha_{\mathfrak{p}} + \mathsf{ht}\,\mathfrak{p} = \alpha_{\mathfrak{q}} + \mathsf{ht}\,\mathfrak{q}$.*

Proof. It follows from [10, Proposition 2.3] that $[{}^1k(\mathfrak{p}) : k(\mathfrak{p})] = [{}^1k(\mathfrak{q}) : k(\mathfrak{q})] \cdot p^{\dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}}$, where $p = \mathrm{char}\, R$. The ring R is excellent (see Remark 2.14(2)), and in particular, it is catenary. Therefore, we have the equalities $\alpha_{\mathfrak{p}} + \mathsf{ht}\,\mathfrak{p} = \alpha_{\mathfrak{q}} + \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} + \mathsf{ht}\,\mathfrak{p} = \alpha_{\mathfrak{q}} + \mathsf{ht}\,\mathfrak{q}$. ■

Modifying the definition of a Hochster–Huneke graph [6], we introduce the following graph $\mathrm{G}(R)$ associated to a noetherian ring R.

- The set of vertices is $\mathsf{Min}\, R$, the set of minimal prime ideals of R.
- There is an edge between two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ if $\mathfrak{p} + \mathfrak{q} \neq R$.

Lemma 3.5 (cf. [6, Theorem 3.6])**.** *Let R be a ring. If $\mathsf{Spec}\, R$ is connected as a topological space, then $\mathrm{G}(R)$ is connected as a graph.*

Proof. Assume that $\mathrm{G}(R)$ is not connected as a graph. Then there exists a nontrivial partition $\mathsf{Min}\, R = A \sqcup B$ such that $\mathfrak{p} + \mathfrak{q} = R$ for all $\mathfrak{p} \in A$ and $\mathfrak{q} \in B$. Neither the ideal $I = \bigcap_{\mathfrak{p} \in A} \mathfrak{p}$ nor the ideal $J = \bigcap_{\mathfrak{q} \in B} \mathfrak{q}$ is nilpotent, while the ideal IJ is nilpotent. We have

$$\mathrm{V}(I) \cap \mathrm{V}(J) = (\textstyle\bigcup_{\mathfrak{p} \in A} \mathrm{V}(\mathfrak{p})) \cap (\bigcup_{\mathfrak{q} \in B} \mathrm{V}(\mathfrak{q})) = \bigcup_{\mathfrak{p} \in A, \mathfrak{q} \in B} \mathrm{V}(\mathfrak{p} + \mathfrak{q}) = \emptyset.$$

We obtain a nontrivial decomposition $\mathsf{Spec}\, R = \mathrm{V}(IJ) = \mathrm{V}(I) \sqcup \mathrm{V}(J)$ into disjoint closed subsets. Therefore, $\mathsf{Spec}\, R$ is not connected as a topological space. ■

Lemma 3.5 says that for any two prime ideals $\mathfrak{p}, \mathfrak{q}$ of a ring R with $\mathsf{Spec}\, R$ connected, there is a sequence of prime ideals $\mathfrak{p}_1 = \mathfrak{p}, \mathfrak{p}_2, \dots, \mathfrak{p}_n = \mathfrak{q}, \mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_{n-1}$ such that $\mathfrak{p}_i, \mathfrak{p}_{i+1} \subseteq \mathfrak{q}_i$ for $i = 1, 2, \dots, n-1$. Lemma 3.4 implies $\alpha_{\mathfrak{p}} + \mathsf{ht}\, \mathfrak{p} = \alpha_{\mathfrak{q}} + \mathsf{ht}\, \mathfrak{q}$ and hence the number $\alpha_{\mathfrak{p}} + \mathsf{ht}\, \mathfrak{p}$ is constant for $\mathfrak{p} \in \mathsf{Spec}\, R$. As a result, for an F-finite noetherian scheme X, the function

$$X \to \mathbb{Z},\ x \mapsto \dim \mathcal{O}_{X,x} + \log_p [{}^1 k(x) : k(x)]$$

is continuous. If X is connected, then this is a constant function, namely, the number

$$\beta_X := \dim \mathcal{O}_{X,x} + \log_p [{}^1 k(x) : k(x)]$$

is independent of the choice of $x \in X$.

Corollary 3.6. *Let X be a d-dimensional F-finite connected noetherian scheme of characteristic p. Then the equality $\delta_t(G, {}^e G) = \Omega(p^{e \beta_X})$ holds for any split generator G of $\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\, X)$. In particular, the following holds for any $x \in X$.*

$$\mathbf{h}_t^{\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\, X)}(F_*) \geq \beta_X \log p = \dim \mathcal{O}_{X,x} \cdot \log p + \log [{}^1 k(x) : k(x)].$$

Proof. Since X is F-finite, it is excellent by Remark 2.14(2). It follows from [2, Theorem 4.15] that there is a split generator G of $\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\, X)$. Let x be any point of X. Then G_x is a split generator of $\mathsf{D}^{\mathsf{b}}(\mathcal{O}_{X,x})$. Indeed, there is an equivalence of triangulated categories

$$\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\, X)/\mathcal{S}(x) \cong \mathsf{D}^{\mathsf{b}}(\mathcal{O}_{X,x}),\ E \mapsto E_x,$$

where $\mathcal{S}(x) := \{E \in \mathsf{D}^{\mathsf{b}}(\mathsf{coh}\, X) \mid E_x \cong 0\}$; see [14, Lemma 2.2] and [13, Lemma 3.2]. Note that there is a commutative diagram

$$\begin{array}{ccc} \mathsf{D}^{\mathsf{b}}(\mathsf{coh}\, X) & \xrightarrow{F_*} & \mathsf{D}^{\mathsf{b}}(\mathsf{coh}\, X) \\ {\scriptstyle (-)_x}\downarrow & & \downarrow{\scriptstyle (-)_x} \\ \mathsf{D}^{\mathsf{b}}(\mathcal{O}_{X,x}) & \xrightarrow[F_*]{} & \mathsf{D}^{\mathsf{b}}(\mathcal{O}_{X,x}). \end{array}$$

Here, the vertical functors are the stalk functors and the horizontal ones are the Frobenius pushforwards. Using [1, Proposition 2.2(c)], we get $\delta_t^{\mathsf{D}^{\mathsf{b}}(\mathcal{O}_{X,x})}(G_x, {}^e G_x) \leq \delta_t^{\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\, X)}(G, {}^e G)$. Corollary 3.3(1) implies $\delta_t^{\mathsf{D}^{\mathsf{b}}(\mathcal{O}_{X,x})}(G_x, {}^e G_x) = \Omega(([{}^1 k(x) : k(x)] d^{\dim \mathcal{O}_{X,x}})^e) = \Omega(p^{e \beta_X})$. ■

Remark 3.7. If X is a d-dimensional F-finite algebraic variety over an algbraically closed field k, then $\beta_X = d + \log_p [{}^1 k : k]$. Hence the inequality $\mathbf{h}_t(F_*) \geq d \log p + \log [{}^1 k : k]$ holds.

4. Upper bounds

In this section, we give an upper bound of the categorical entropy of the Frobenius push-forward and complete the proof of Theorem 1.1. We begin with providing an easy lemma.

Lemma 4.1. *Let R be a local ring, and let x be an R-regular element. Then, for all positive integers n, one has that $R/x^nR \in (R/xR)^{*n}$.*

Proof. We use induction on n. The case $n = 1$ is clear. Let $n \geq 2$. As x is R-regular, there is an exact sequence $0 \to R/xR \to R/x^nR \to R/x^{n-1}R \to 0$. Using the induction hypothesis, we get $R/x^nR \in R/xR * R/x^{n-1}R \subseteq R/xR * (R/xR)^{*(n-1)} = (R/xR)^{*n}$. ■

Let R be a local ring. Let M be a finitely generated module M. Then we denote by $\mu_R(M)$ the minimal number of generators of M. For an integer $n \geq 0$, we denote by $\Omega_R^n M$ the nth syzygy of M in the minimal free resolution of M and by $\beta_n^R(M)$ the nth Betti number of M. Now we are ready to give a proof of the main result of this section.

Theorem 4.2. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional F-finite local ring of characteristic p. Then:*

(1) *There is an equality $\delta_t(G, {}^eG) = O(([{}^1k : k]p^d)^e)$ for any split generator G of $\mathsf{D^b}(R)$.*
(2) *There is an inequality $\mathbf{h}_t(F_*) \leq d \log p + \log[{}^1k : k]$.*

Proof. It is enough to prove (1) because it implies (2). By Remark 2.6(2), we have only to show that there exists a split generator G such that $\delta_t(G, {}^eG) = O(([{}^1k : k]p^d)^e)$. Let us show it by induction on d. Assume $d = 0$. Then k is a split generator of $\mathsf{D^b}(R)$. Since ${}^ek \cong k^{\oplus[{}^1k:k]^e}$, we have $\delta_t(k, {}^ek) \leq [{}^1k : k]^e = p^{e\alpha}$, and this implies $\delta_t(k, {}^ek) = O([{}^1k : k]^e)$.

Now suppose that $d > 0$ and that the equality $\delta_t(G, {}^eG) = O(([{}^1k : k]p^{d'})^e)$ holds for any F-finite local ring R of dimension $d' < d$ and some split generator G of $\mathsf{D^b}(R)$. Let R be a d-dimensional F-finite local ring. Take a filtration $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_r = R$ of ideals with $I_i/I_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} R$. For a complex $X \in \mathsf{D^b}(R)$, there is a filtration $0 = I_0X \subseteq I_1X \subseteq \cdots \subseteq I_rX = X$ of complexes such that $I_iX/I_{i-1}X$ is a bounded complex of finitely generated $R/\mathfrak{p}_i$-modules. If each $\mathsf{D^b}(R/\mathfrak{p}_i)$ has a split generator G_i, then the above filtration shows that $\bigoplus_{i=1}^r G_i$ is a split generator of $\mathsf{D^b}(R)$. Moreover, one has the inequalities

$$\begin{aligned}\delta_t^{\mathsf{D^b}(R)}\Big(\bigoplus_{i=1}^r G_i, {}^e\Big(\bigoplus_{i=1}^r G_i\Big)\Big) &\leq \sum_{i=1}^r \delta_t^{\mathsf{D^b}(R)}\Big(\bigoplus_{i=1}^r G_i, {}^eG_i\Big) \\ &\leq \sum_{i=1}^r \delta_t^{\mathsf{D^b}(R)}(G_i, {}^eG_i) \leq \sum_{i=1}^r \delta_t^{\mathsf{D^b}(R/\mathfrak{p}_r)}(G_i, {}^eG_i),\end{aligned}$$

where the first, second, and third inequalities follow from Lemma 2.4(5), (3) and (6), respectively. Thus, it suffices to show the statement in the case where R is an integral domain.

By virtue of [7, Theorem 5.3], there exists a nonzero element x of R such that $x\operatorname{Ext}_R^{2d+1}(M, N) = 0$ for all R-modules M and N. Let G be a split generator of $\mathsf{D^b}(R/xR)$ of which R/xR is a direct summand. Then $\widetilde{G} := G \oplus R$ is a split generator of $\mathsf{D^b}(R)$. Indeed, let M be a finitely generated R-module M. As x kills $\operatorname{Ext}_R^1(\Omega_R^{2d}M, -) \cong \operatorname{Ext}_R^{2d+1}(M, -)$, the module M is a direct summand of $\Omega_R(\Omega_R^{2d}M/x\Omega_R^{2d}M)$ by [5, Lemma 2.2]. Then

$$\Omega_R^{2d}M/x\Omega_R^{2d}M \in \mathsf{D^b}(R/xR) = \mathsf{thick}_{\mathsf{D^b}(R/xR)}(G) \subseteq \mathsf{thick}_{\mathsf{D^b}(R)}(\widetilde{G}).$$

There is an exact sequence $0 \to \Omega_R(\Omega_R^{2d}M/x\Omega_R^{2d}M) \to P \to \Omega_R^{2d}M/x\Omega_R^{2d}M \to 0$ with P free, and we see that $M \in \mathsf{thick}_{\mathsf{D^b}(R)}(\widetilde{G})$. It follows that $\widetilde{G}$ is a split generator of $\mathsf{D^b}(R)$.

Using Lemma 2.4(2), we have an inequality $\delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^e\widetilde{G}) \leq \delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^eG) + \delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^eR)$. Also, by Lemma 2.4(3)(6) and the induction hypothesis, we get

$$\delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^eG) \leq \delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(G, {}^eG) \leq \delta_t^{\mathsf{D}^{\mathsf{b}}(R/xR)}(G, {}^eG) = O(([{}^1k : k]p^{d-1})^e).$$

Therefore, it is enough to prove that $\delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^eR) = O(([{}^1k : k]p^d)^e)$.

The minimal free resolution of the R-module ${}^eR$ gives rise to an exact sequence

$$0 \to \Omega_R^{2d}({}^eR) \to R^{\oplus \beta_{2d-1}^R({}^eR)} \to \cdots \to R^{\oplus \beta_0^R({}^eR)} \to {}^eR \to 0.$$

As x is regular on R and ${}^eR$, we obtain an exact sequence

$$0 \to \Omega_R^{2d}({}^eR)/x\Omega_R^{2d}({}^eR) \to (R/xR)^{\oplus \beta_{2d-1}^R({}^eR)} \to \cdots \to (R/xR)^{\oplus \beta_0^R({}^eR)} \to {}^eR/x\,{}^eR \to 0.$$

Also, there is a canonical short exact sequence

$$0 \to \Omega_R(\Omega_R^{2d}({}^eR)/x\Omega_R^{2d}({}^eR)) \to R^{\oplus \beta_{2d}^R({}^eR)} \to \Omega_R^{2d}({}^eR)/x\Omega_R^{2d}({}^eR) \to 0.$$

Here, we use the equalities $\mu_R(\Omega_R^{2d}({}^eR)/x\Omega_R^{2d}({}^eR))) = \mu_R(\Omega_R^{2d}({}^eR)) = \beta_{2d}^R({}^eR)$. As we have already seen, ${}^eR$ is a direct summand of $\Omega_R(\Omega_R^{2d}({}^eR)/x\Omega_R^{2d}({}^eR))$. Hence there exists a finitely generated R-module M such that the following containment holds true.

$${}^eR \oplus M \in ({}^eR/x\,{}^eR)[-2d+1] * (R/xR)^{\oplus \beta_0^R({}^eR)}[-2d] * \cdots * (R/xR)^{\oplus \beta_{2d-1}^R({}^eR)}[-1] * R^{\oplus \beta_{2d}^R({}^eR)}.$$

Since $\widetilde{G}$ contains R and R/xR as direct summands, this yields

$$\begin{aligned}\delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^eR) &\leq \delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^eR/x\,{}^eR)\mathrm{e}^{(-2d+1)t} + \textstyle\sum_{i=0}^{2d} \delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, (R/xR)^{\oplus \beta_i^R({}^eR)})\mathrm{e}^{(-2d+i)t} \\ &\leq \delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^eR/x\,{}^eR)\mathrm{e}^{(-2d+1)t} + \textstyle\sum_{i=0}^{2d} \beta_i^R({}^eR)\mathrm{e}^{(-2d+i)t}\end{aligned}$$

It follows from Lemma 4.1 that ${}^eR/x\,{}^eR = {}^e(R/x^{p^e}R) \in {}^e((R/xR)^{*p^e}) \subseteq ({}^e(R/xR))^{*p^e}$. Hence,

$$\begin{aligned}\delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^eR/x\,{}^eR) &\leq p^e\delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^e(R/xR)) \leq p^e\delta_t^{\mathsf{D}^{\mathsf{b}}(R/xR)}(G, {}^e(R/xR)) \\ &\leq p^e\delta_t^{\mathsf{D}^{\mathsf{b}}(R/xR)}(G, {}^eG) = p^eO(([{}^1k : k]p^{d-1})^e) = O(([{}^1k : k]p^d)^e).\end{aligned}$$

Here, the first inequality follows by Lemma 2.4(5), the second by Lemma 2.4(3)(6), the third by Lemma 2.4(2), and the last by the induction hypothesis. On the other hand, it is shown by [15, Theorem] that $\beta_i^R({}^eR) = O(([{}^1k : k]p^d)^e)$. Consequently, we obtain

$$\delta_t^{\mathsf{D}^{\mathsf{b}}(R)}(\widetilde{G}, {}^e\widetilde{G}) = O(([{}^1k : k]p^{d-1})^e) + O(([{}^1k : k]p^d)^e) = O(([{}^1k : k]p^d)^e).$$

The proof of the theorem is now completed. ■

The combination of Corollary 3.3(1) with Theorem 4.2 yields the following result.

Corollary 4.3. *Let R be a d-dimensional F-finite local ring with characteristic p. Then for any split generator G of $\mathsf{D}^{\mathsf{b}}(R)$, one has $\delta_t(G, {}^eG) = \Theta(([{}^1k : k]p^d)^e)$. In particular, the equality $\mathbf{h}_t(F_*) = d\log p + \log[{}^1k : k]$ holds.*

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