DOMINANT LOCAL RINGS AND SUBCATEGORY CLASSIFICATION

RYO TAKAHASHI

Abstract. We introduce a new notion of commutative noetherian local rings which we call dominant. We explore fundamental properties of dominant local rings, and compare them with other local rings. We also provide several methods to get a new dominant local ring from a given one. Finally, we classify resolving subcategories of the module category $\mathsf{mod}\,R$, and thick subcategories of the derived category $\mathsf{D^b}(R)$ and the singularity category $\mathsf{D^{sg}}(R)$ for a local ring R whose certain localizations are dominant local rings. Our results recover and refine all the known classification theorems described in this context.

Contents

1. Introduction

Given an abelian or triangulated category $\mathcal{C}$, one can discuss *subcategory classification* for $\mathcal{C}$, which means classifying reasonable full subcategories of $\mathcal{C}$ such as Serre subcategories, resolving subcategories, thick subcategories and localizing subcategories. Subcategory classification has been actively studied in a wide range of areas including stable homotopy theory [12, 26, 32, 33, etc.], modular representation theory [13, 14, 15, 27, etc.], algebraic geometry [52, 53, 64, etc.] and ring theory [1, 6, 28, 34, 37, 47, etc.].

Let R be a (commutative, noetherian) local ring. We denote by $\mathsf{mod}\,R$ the category of finitely generated R-modules, and by $\mathsf{D^b}(R)$ the bounded derived category of $\mathsf{mod}\,R$. Let $\mathsf{D^{sg}}(R)$ stand for the singularity category of R, that is, the Verdier quotient of $\mathsf{D^b}(R)$ by perfect complexes. The author and his coauthors [22, 24, 46, 57, 58, 59, 60, etc.] have worked on subcategory classification for the abelian category $\mathsf{mod}\,R$ and the triangulated categories $\mathsf{D^b}(R)$ and $\mathsf{D^{sg}}(R)$. We continue this study in the present paper.

We introduce the full subcategory $\mathsf{C}(R)$ of $\mathsf{mod}\,R$ consisting of modules M with $\operatorname{depth} M_\mathfrak{p} \geqslant \operatorname{depth} R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. This is none other than the full subcategory $\mathsf{CM}(R)$ of $\mathsf{mod}\,R$ consisting of maximal Cohen–Macaulay modules when R is Cohen–Macaulay. We obtain in Theorem 3.8 a result on the structure of modules in $\mathsf{C}(R)$, and some direct consequences in Corollary 4.7 about the Rouquier dimensions and the ultimate dimensions (in the sense of [11, 50]) of certain Verdier quotients of $\mathsf{D^b}(R)$.

Motivated by these results, we shall introduce a new notion of local rings: a *dominant local ring* is a local ring R for which the thick closure of each nonzero object of $\mathsf{D^{sg}}(R)$ contains the residue field (see Corollary 10.8). We prove that dominance is preserved under various fundamental operations in local

2020 *Mathematics Subject Classification.* 13C60, 13D09, 13H10.

Key words and phrases. Burch ring, derived category, hypersurface, minimal multiplicity, module category, resolving subcategory, Rouquier dimension, singularity category, thick subcategory, Tor/Ext-friendly, Tor/Ext-persistent.

The author was partly supported by JSPS Grant-in-Aid for Scientific Research 19K03443.

rings. Moreover, we find out several classes of local rings containing or contained in the class of dominant local rings. The following theorem collects some of our main results in this direction.

Theorem 1.1 (Theorems 5.6, 6.7, Corollary 5.8 and Propositions 5.10, 6.2)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring.*

(1) *Let $x \in \mathfrak{m}$ be an R-regular element. If $R/(x)$ is dominant, then so is R. The converse holds if $x \notin \mathfrak{m}^2$.*
(2) *R is dominant if and only if so is the power series ring $R[\![X]\!]$, if and only if so is the completion $\widehat{R}$.*
(3) *R is dominant if $\mathfrak{m}$ is quasi-decomposable or R is Burch. In particular, R is dominant if it is either a hypersurface, or a Cohen–Macaulay ring with minimal multiplicity and with k infinite.*
(4) *Suppose that R is a complete intersection. Then R is dominant if and only if it is a hypersurface.*
(5) *Suppose that R is dominant. Then R is Tor-friendly, and hence it is Ext-friendly, Tor-persistent and Ext-persistent. In particular, the Auslander–Reiten conjecture holds for R.*

Assertions (3) and (5) of the above Theorem 1.1 are actually complemented in Theorem 9.10, which provides more precise information on the relationships of dominance with other properties of local rings.

After investigating in Section 7 whether and how dominance is inherited from one local ring to another, we discover various dominant local rings, including certain local rings of embedding dimension two (Proposition 8.1 and Corollary 8.4) and certain quotients of regular local rings (Corollary 8.7 and Proposition 8.8). In particular, the quotient of a regular local ring by an ideal generated by at most two elements turns out to be dominant unless it is a complete intersection; see Corollary 8.9.

Finally, we consider subcategory classification for $\mathsf{mod}\, R$, $\mathsf{D^b}(R)$ and $\mathsf{D^{sg}}(R)$. For a subset Φ of $\operatorname{Spec} R$, we denote by $\mathsf{C}_\Phi(R)$ the full subcategory of $\mathsf{C}(R)$ consisting of modules that are locally free outside Φ, by $\mathsf{mod}_\Phi\, R$ the full subcategory of $\mathsf{mod}\, R$ consisting of modules that are locally of finite projective dimension outside Φ, by $\mathsf{D}^{\mathsf{b}}_\Phi(R)$ the full subcategory of $\mathsf{D^b}(R)$ consisting of complexes that are locally perfect outside Φ, and by $\mathsf{D}^{\mathsf{sg}}_\Phi(R)$ the full subcategory of $\mathsf{D^{sg}}(R)$ consisting of objects that are locally zero outside Φ. Using the above-mentioned Theorem 3.8, we obtain the following classification of resolving/thick subcategories.

Theorem 1.2 (Theorem 10.10)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring of depth t. Let Φ be a subset of $\operatorname{Sing} R$.*

(1) *Suppose that $R_\mathfrak{p}$ is dominant for all $\mathfrak{p} \in \Phi \cup \{\mathfrak{m}\}$. Then there exist one-to-one correspondences among:*
    - *the resolving subcategories of $\mathsf{C}_\Phi(R)$,*
    - *the thick subcategories of $\mathsf{C}_\Phi(R)$ containing R,*
    - *the thick subcategories of $\mathsf{mod}_\Phi\, R$ containing R,*
    - *the thick subcategories of $\mathsf{D}^{\mathsf{b}}_\Phi(R)$ containing R,*
    - *the thick subcategories of $\mathsf{D}^{\mathsf{sg}}_\Phi(R)$, and*
    - *the specialization-closed subsets of Φ.*
(2) *Assume that R is a singular local ring. Suppose further that $R_\mathfrak{p}$ is dominant for all $\mathfrak{p} \in \Phi \setminus \{\mathfrak{m}\}$. Let $\Omega^t k$ denote the t-th syzygy of the R-module k. Then there exist one-to-one correspondences among:*
    - *the resolving subcategories of $\mathsf{C}_\Phi(R)$ containing $\Omega^t k$,*
    - *the thick subcategories of $\mathsf{C}_\Phi(R)$ containing R and $\Omega^t k$,*
    - *the thick subcategories of $\mathsf{mod}_\Phi\, R$ containing R and k*
    - *the thick subcategories of $\mathsf{D}^{\mathsf{b}}_\Phi(R)$ containing R and k,*
    - *the thick subcategories of $\mathsf{D}^{\mathsf{sg}}_\Phi(R)$ containing k, and*
    - *the nonempty specialization-closed subsets of Φ.*

We should mention that those one-to-one correspondences which are claimed in this theorem are given explicitly. Here, the resolving subcategories of $\mathsf{C}_\Phi(R)$ mean the resolving subcategories of $\mathsf{mod}\, R$ contained in $\mathsf{C}_\Phi(R)$. The thick subcategories of $\mathsf{C}_\Phi(R)$, $\mathsf{mod}_\Phi\, R$, $\mathsf{D}^{\mathsf{b}}_\Phi(R)$ and $\mathsf{D}^{\mathsf{sg}}_\Phi(R)$ mean the thick subcategories of $\mathsf{C}(R)$, $\mathsf{mod}\, R$, $\mathsf{D^b}(R)$ and $\mathsf{D^{sg}}(R)$ contained in $\mathsf{C}_\Phi(R)$, $\mathsf{mod}_\Phi\, R$, $\mathsf{D}^{\mathsf{b}}_\Phi(R)$ and $\mathsf{D}^{\mathsf{sg}}_\Phi(R)$, respectively. The specialization-closed subsets of Φ mean the specialization-closed subsets of $\operatorname{Spec} R$ contained in Φ.

Theorem 1.2 (and its consequence, Corollary 10.17) recovers all the known classification theorems of the same type, which are the ones proved in [22, 46, 57, 59, 60]. Furthermore, Theorem 1.2 considerably extends/refines them and discovers other classes of local rings R for which such a broad range of resolving subcategories of $\mathsf{mod}\, R$ and thick subcategories of $\mathsf{D^b}(R)$, $\mathsf{D^{sg}}(R)$ are classified completely in terms of specialization-closed subsets of $\operatorname{Spec} R$. In particular, it is worth noting that all the known classification theorems mentioned above assume that the local ring R is Cohen–Macaulay, while Theorem 1.2 does not assume so. As far as the author knows, this is the first time classifying resolving/thick subcategories over a commutative noetherian local ring which is not necessarily Cohen–Macaulay.

We close the section by stating our convention adopted throughout the paper.

Convention. All rings are commutative and noetherian. All modules are finitely generated. All subcategories are nonempty, full and strict. The zero (sub)category is denoted by 0. Each object X of a category $\mathcal{C}$ may be identified with the subcategory $\{X\}$ of $\mathcal{C}$. Let R be a (commutative noetherian) ring. Denote by $\operatorname{mod} R$ the category of (finitely generated) R-modules. We use $\mathbb{N}$ for the set of nonnegative integers.

2. Preliminaries

This section provides a list of basic definitions and fundamental properties used in the later sections.

Definition 2.1. Let $(R, \mathfrak{m})$ be a local ring. We denote by $\widehat{R}$ the completion of R in the $\mathfrak{m}$-adic topology, and by $\operatorname{edim} R$ the embedding dimension of R. We define the *(embedding) codimension* and the *(embedding) codepth* of R by $\operatorname{codim} R = \operatorname{edim} R - \dim R$ and $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$, respectively. For an R-module M we denote by $\mathrm{e}_R(M)$, $\ell_R(M)$, $\mu_R(M)$ and $\mathrm{r}_R(M)$ the (Hilbert–Samuel) multiplicity (with respect to $\mathfrak{m}$), the length, the minimal number of generators and the type of M, respectively. We refer the reader to [16, 44] for the details of these notions about commutative local ring theory.

Definition 2.2. A subset Φ of $\operatorname{Spec} R$ is said to be *specialization-closed* if there is an inclusion $\mathrm{V}(\mathfrak{p}) \subseteq \Phi$ for all $\mathfrak{p} \in \Phi$. Note that a specialization-closed subset is the same thing as a (possibly infinite) union of closed subsets of $\operatorname{Spec} R$ in the Zariski topology. In particular, every closed subset is specialization-closed. The *(Krull) dimension* of Φ is defined by $\dim \Phi = \sup\{n \geqslant 0 \mid \text{there exists a chain } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \Phi\}$. It holds that $\dim \Phi \leqslant \sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \Phi\}$ with equality if Φ is specialization-closed.

Definition 2.3. We denote by $\operatorname{Spec}_0 R$ the set of nonmaximal prime ideals, namely, $\operatorname{Spec}_0 R = \operatorname{Spec} R \setminus \operatorname{Max} R$. If R is a local ring with maximal ideal $\mathfrak{m}$, then $\operatorname{Spec}_0 R$ is none other than the *punctured spectrum* $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ of R. We denote by $\operatorname{Sing} R$ the *singular locus* of R, namely, the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is singular (i.e., nonregular). It is evident that the singular locus is a specialization-closed subset of $\operatorname{Spec} R$. Note also that when $(R, \mathfrak{m})$ is a local ring, R is singular if and only if $\operatorname{Sing} R$ is nonempty, if and only if $\mathfrak{m} \in \operatorname{Sing} R$. For a property $\mathbb{P}$ of local rings (resp. modules over local rings) and a set Φ of prime ideals of R, we say that R (resp. an R-module M) *locally* satisfies $\mathbb{P}$ *on* Φ if the local ring $R_\mathfrak{p}$ (resp. the $R_\mathfrak{p}$-module $M_\mathfrak{p}$) satisfies $\mathbb{P}$ for all $\mathfrak{p} \in \Phi$. A local ring R is said to have an *isolated singularity* if it is locally regular on $\operatorname{Spec}_0 R$, or equivalently, if $\dim \operatorname{Sing} R \leqslant 0$ (i.e., $\dim \operatorname{Sing} R \in \{0, -\infty\}$). Note by definition that a regular local ring has an isolated singularity; see [65, Definition (3.1)] and [41, Definition 7.8 and the paragraph following it]. We denote by $\operatorname{mod}_0 R$ the subcategory of $\operatorname{mod} R$ consisting of modules which are locally free on $\operatorname{Spec}_0 R$.

Definition 2.4. For each positive integer n we denote by $\Omega^n M$ the *nth syzygy* of M, that is to say, the image of the nth differential map in a projective resolution of M in $\operatorname{mod} R$. We set $\Omega^0 M = M$. Note that $\Omega^n M$ is uniquely determined up to projective summands. Whenever the ring R is local, we define $\Omega^n M$ by using a minimal free resolution of M, so that it is uniquely determined up to isomorphism.

Definition 2.5. An R-module M is said to be *maximal Cohen–Macaulay* if $\operatorname{depth} M_\mathfrak{p} \geqslant \dim R_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} R$, namely, if one has either $M = 0$ or $M \neq 0$ and $\operatorname{depth} M_\mathfrak{p} = \dim R_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Supp} M$. (Note by definition that the zero module is the only module having depth ∞.) We denote by $\mathsf{CM}(R)$ the subcategory of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay R-modules. Clearly, R is Cohen–Macaulay if and only if $R \in \mathsf{CM}(R)$. We say that R has *finite* CM*-representation type* if there exist only finitely many isomorphism classes of indecomposable modules in $\mathsf{CM}(R)$. We set $\mathsf{CM}_0(R) = \mathsf{CM}(R) \cap \operatorname{mod}_0 R$. A Cohen–Macaulay local ring R has an isolated singularity if and only if $\mathsf{CM}_0(R) = \mathsf{CM}(R)$.

Definition 2.6. A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called closed under *extensions* (resp. *kernels of epimorphisms*) if for each exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$ with $L, N \in \mathcal{X}$ (resp. $M, N \in \mathcal{X}$) one has $M \in \mathcal{X}$ (resp. $L \in \mathcal{X}$). We say that $\mathcal{X}$ is *resolving* if it contains the projective R-modules and is closed under direct summands, extensions and kernels of epimorphisms. Clearly, the condition that $\mathcal{X}$ contains the projective modules can be replaced with the condition that $\mathcal{X}$ contains R. Also, being closed under kernels of epimorphisms can be replaced with being closed under syzygies; see [66, Lemma 3.2(2)]. It is straightforward to verify that any intersection of resolving subcategories is again resolving.

Definition 2.7. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$. For each $\mathfrak{p} \in \operatorname{Spec} R$ we denote by $\mathcal{X}_\mathfrak{p}$ the subcategory of $\operatorname{mod} R_\mathfrak{p}$ consisting of modules of the form $X_\mathfrak{p}$ with $X \in \mathcal{X}$. The *additive closure* $\operatorname{add}_R \mathcal{X}$ of $\mathcal{X}$ is defined as the subcategory of $\operatorname{mod} R$ consisting of direct summands of finite direct sums of modules in $\mathcal{X}$. The *resolving closure* $\mathsf{res}_R\,\mathcal{X}$ of $\mathcal{X}$ is defined to be the smallest resolving subcategory of $\mathsf{mod}\,R$ containing $\mathcal{X}$. The subcategories $\mathsf{add}\,R$ and $\mathsf{mod}_0\,R$ of $\mathsf{mod}\,R$ are resolving, and so is $\mathsf{CM}(R)$ if (and only if) R is Cohen–Macaulay. For a resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$ and a prime ideal $\mathfrak{p}$ of R, the subcategory $\mathsf{add}_{R_\mathfrak{p}}\,\mathcal{X}_\mathfrak{p}$ of $\mathsf{mod}\,R_\mathfrak{p}$ is resolving; see [24, Lemma 3.2(1)].

Definition 2.8. For an R-module M, we denote by $\mathrm{NF}(M)$ the *nonfree locus* of M, that is, the set of prime ideals $\mathfrak{p}$ of R such that $M_\mathfrak{p}$ is nonfree as a module over $R_\mathfrak{p}$. This is a closed subset of $\operatorname{Spec} R$ in the Zariski topology, since the equality $\mathrm{NF}(M) = \operatorname{Supp}\operatorname{Ext}^1(M, \Omega M)$ holds in general; see [56, Proposition 2.10]. For a subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$, we put $\mathrm{NF}(\mathcal{X}) = \bigcup_{X\in\mathcal{X}} \mathrm{NF}(X)$. This is a specialization-closed subset of $\operatorname{Spec} R$. For a subset Φ of $\operatorname{Spec} R$, we denote by $\mathrm{NF}^{-1}(\Phi)$ the subcategory of $\mathsf{mod}\,R$ consisting of modules M with $\mathrm{NF}(M) \subseteq \Phi$. This is a resolving subcategory of $\mathsf{mod}\,R$; see [57, Proposition 1.15(3)]. Note that there are equalities $\mathsf{mod}_0\,R = \mathrm{NF}^{-1}(\operatorname{Max} R)$ and $\mathsf{add}\,R = \mathrm{NF}^{-1}(\emptyset)$, the latter of which follows from the fact that an R-module M is projective if and only if the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is free for all $\mathfrak{p} \in \operatorname{Spec} R$.

Definition 2.9. Let $\mathcal{T}$ be a triangulated category. A *thick* subcategory of $\mathcal{T}$ is by definition a triangulated subcategory of $\mathcal{T}$ closed under direct summands. For a subcategory $\mathcal{X}$ of $\mathcal{T}$ we denote by $\mathsf{thick}\,\mathcal{X}$ the *thick closure* of $\mathcal{X}$ in $\mathcal{T}$, that is, the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{X}$.

Definition 2.10. Let R be a Gorenstein local ring. We denote by $\underline{\mathsf{CM}}(R)$ the *stable category* of $\mathsf{CM}(R)$, that is, the objects of $\underline{\mathsf{CM}}(R)$ are the maximal Cohen–Macaulay R-modules, and $\operatorname{Hom}_{\underline{\mathsf{CM}}(R)}(M,N)$ is the quotient $\underline{\operatorname{Hom}}_R(M,N)$ of $\operatorname{Hom}_R(M,N)$ by the homomorphisms $M \to N$ factoring through some projective R-modules. The stable category $\underline{\mathsf{CM}}(R)$ is a triangulated category by [31, Theorem 2.6].

Definition 2.11. We denote by $\mathsf{D}^{\mathsf{b}}(R)$ the *bounded derived category* of $\mathsf{mod}\,R$, and by $\mathsf{D}^{\mathsf{perf}}(R)$ the subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ consisting of perfect complexes; recall that a perfect complex means a bounded complex of (finitely generated) projective modules. Note that the inclusion $\mathsf{mod}\,R \subseteq \mathsf{D}^{\mathsf{b}}(R)$ and the equality $\mathsf{D}^{\mathsf{perf}}(R) = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\,R$ hold. The *singularity category* of R is by definition the Verdier quotient $\mathsf{D}^{\mathsf{sg}}(R) = \mathsf{D}^{\mathsf{b}}(R)/\mathsf{D}^{\mathsf{perf}}(R)$. If R is a Gorenstein local ring, then the assignment $M \mapsto M$ with $M \in \underline{\mathsf{CM}}(R)$ gives a triangle equivalence $\underline{\mathsf{CM}}(R) \to \mathsf{D}^{\mathsf{sg}}(R)$; we refer the reader to [18, Theorem 4.4.1] for the proof.

3. Structure of modules in $\mathsf{C}(R)$

First of all, we emphasize that throughout this section R is a (commutative and noetherian) ring. In this section, we introduce the subcategory $\mathsf{C}(R)$ of $\mathsf{mod}\,R$ and investigate modules in it. The main result of this section plays a key role in later sections. We start with recalling restricted flat dimension.

Definition 3.1. For each R-module M, we set $\operatorname{Rfd}_R M = \sup_{\mathfrak{p}\in\operatorname{Spec} R}\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth} M_\mathfrak{p}\}$. This invariant is called the *(large) restricted flat dimension* of M. We always have $\operatorname{Rfd}_R M \in \mathbb{N} \cup \{-\infty\}$, and $\operatorname{Rfd}_R M = -\infty$ if and only if $M = 0$; see [8, Theorem 1.1] and [21, Proposition (2.2) and Theorem (2.4)].

We make the definition of the subcategory $\mathsf{C}(R)$ of $\mathsf{mod}\,R$ and state basic properties.

Definition 3.2. We denote by $\mathsf{C}(R)$ the subcategory of $\mathsf{mod}\,R$ consisting of modules M which satisfy the inequality $\operatorname{Rfd}_R M \leqslant 0$, that is to say, modules M such that $\operatorname{depth} M_\mathfrak{p} \geqslant \operatorname{depth} R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. For each subset Φ of $\operatorname{Spec} R$, we set $\mathsf{C}_\Phi(R) = \mathsf{C}(R) \cap \mathrm{NF}^{-1}(\Phi)$. We put $\mathsf{C}_0(R) = \mathsf{C}(R) \cap \mathsf{mod}_0\,R = \mathsf{C}_{\operatorname{Max} R}(R)$.

Remark 3.3. (1) For every $M \in \mathsf{C}(R)$ and every $\mathfrak{p} \in \operatorname{Spec} R$, one has $M_\mathfrak{p} \in \mathsf{C}(R_\mathfrak{p})$.
(2) By the depth lemma, $\mathsf{C}(R)$ is a resolving subcategory of $\mathsf{mod}\,R$, and so is $\mathsf{C}_\Phi(R)$ for each $\Phi \subseteq \operatorname{Spec} R$.
(3) One has $\mathsf{C}(R) \supseteq \mathsf{CM}(R)$. The equality holds if and only if R is Cohen–Macaulay (as $R \in \mathsf{C}(R)$).
(4) One has $\mathrm{NF}(\mathsf{C}(R)) \subseteq \operatorname{Sing} R$, or in other words, $\mathsf{C}_{\operatorname{Sing} R}(R) = \mathsf{C}(R)$. Indeed, let $\mathfrak{p} \in \mathrm{NF}(\mathsf{C}(R))$. Then $M_\mathfrak{p}$ is not $R_\mathfrak{p}$-free for some $M \in \mathsf{C}(R)$. There is an inequality $\operatorname{depth} M_\mathfrak{p} \geqslant \operatorname{depth} R_\mathfrak{p}$. If $R_\mathfrak{p}$ is regular, then $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ and by the Auslander–Buchsbaum formula $M_\mathfrak{p}$ is $R_\mathfrak{p}$-free, a contradiction.
(5) For an R-module $M \neq 0$ one has $\Omega^r M \in \mathsf{C}(R)$, where $r = \operatorname{Rfd} M$. This is seen by the depth lemma (see [16, Proposition 1.2.9]).
(6) One has $\mathsf{C}(R) = \mathsf{add}\,R$ if and only if R is regular. In fact, the "if" part follows from (3). As for the "only if" part, by (5) one has $\operatorname{pd} M < \infty$ for all $M \in \mathsf{mod}\,R$, which implies that R is regular.
(7) If the ring R is local, then there is an equality $\mathsf{C}_0(R) = \{M \in \mathsf{mod}_0\,R \mid \operatorname{depth} M \geqslant \operatorname{depth} R\}$.
(8) For a local ring $(R, \mathfrak{m}, k)$ of depth t, one has $\mathsf{C}_0(R) = \mathsf{res}\,\Omega^t k$. Indeed, using (7) and the depth lemma shows $\Omega^t k \in \mathsf{C}_0(R)$. Since $\mathsf{C}_0(R)$ is a resolving subcategory of $\mathsf{mod}\,R$ by (2), we have $\mathsf{res}\,\Omega^t k \subseteq \mathsf{C}_0(R)$. Pick $M \in \mathsf{C}_0(R)$. Then $\operatorname{depth} M \geqslant \operatorname{depth} R = t$, and M belongs to $\mathsf{res}\,\Omega^t k$ by [63, Proposition 3.4].

We consider extensions of modules in a given subcategory of $\operatorname{mod} R$.

Definition 3.4. For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\operatorname{mod} R$, we denote by $\mathcal{X} \circ \mathcal{Y}$ the subcategory of $\operatorname{mod} R$ consisting of modules E having an exact sequence $0 \to X \to E \to Y \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Remark 3.5. For subcategories $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ of $\operatorname{mod} R$ it holds that $(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z})$. In fact, let M be an R-module in $(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z}$. There is an exact sequence $0 \to N \xrightarrow{v} M \to Z \to 0$ with $N \in \mathcal{X} \circ \mathcal{Y}$ and $Z \in \mathcal{Z}$. There is an exact sequence $0 \to X \to N \xrightarrow{w} Y \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. The pushout diagram of v, w yields exact sequences $0 \to X \to M \to L \to 0$ and $0 \to Y \to L \to Z \to 0$. Therefore $M \in \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z})$, and thus $(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} \subseteq \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z})$. The opposite inclusion is shown by a dual argument.

Definition 3.6. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$. We set $\mathcal{X}^{\circ 0} = 0$, $\mathcal{X}^{\circ 1} = \mathcal{X}$ and $X^{\circ n} = \mathcal{X}^{\circ (n-1)} \circ \mathcal{X}$ for each $n \geqslant 2$. Taking Remark 3.5 into account, with no ambiguity we may write $\mathcal{X}^{\circ n} = \underbrace{\mathcal{X} \circ \mathcal{X} \circ \cdots \circ \mathcal{X}}_{n}$.

We state a lemma on localization of a subcategory of modules at a prime ideal, which comes from [57].

Lemma 3.7. *Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$ which contains R and is closed under finite direct sums. Let M be an R-module. Then the following two statements hold true.*

(1) *Let $\mathfrak{p}$ be a prime ideal of R. Then the localization $M_\mathfrak{p}$ belongs to $\operatorname{add} \mathcal{X}_\mathfrak{p}$ if and only if there exists an exact sequence $0 \to N \to X \to M \to 0$ of R-modules with $X \in \mathcal{X}$ whose localization at $\mathfrak{p}$ splits.*

(2) *Let Φ be a nonempty finite set of prime ideals of R. Assume that for each $\mathfrak{p} \in \Phi$ the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ belongs to $\operatorname{add} \mathcal{X}_\mathfrak{p}$. Then there exists an exact sequence $0 \to L \to M \oplus N \to X \to 0$ of R-modules such that $X \in \mathcal{X}$, $N \in \operatorname{res}\{M, X\}$, $\operatorname{NF}(L) \subseteq \operatorname{NF}(M)$ and $\operatorname{NF}(L) \cap \Phi = \emptyset$.*

Proof. (1) The proof of [57, Lemma 4.6] actually shows the assertion; [57, Lemma 4.6] concerns a resolving subcategory of $\operatorname{mod} R$ for a local ring R, but the proof does work for a commutative noetherian ring R and a subcategory of $\operatorname{mod} R$ containing R and closed under finite direct sums.

(2) The proof of [57, Proposition 4.7] essentially shows the assertion. We only give an outline. Write $\Phi = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. By (1), for each $1 \leqslant i \leqslant n$ we get an exact sequence $0 \to K_i \to X_i \xrightarrow{\phi_i} M \to 0$ which locally splits at $\mathfrak{p}_i$. There is an exact sequence $\sigma : 0 \to K \xrightarrow{\psi} X \xrightarrow{\phi} M \to 0$, where $X = X_1 \oplus \cdots \oplus X_n \in \mathcal{X}$ and $\phi = (\phi_1, \dots, \phi_n)$. As each $\sigma_{\mathfrak{p}_i}$ splits, we can choose[1] an element $f \in \operatorname{ann}_R \sigma$ such that $f \notin \mathfrak{p}_i$ for all i. Then σ_f splits, and we find a homomorphism $\nu : M \to X$ such that ν_f is a splitting of ϕ_f. Choosing a surjection $\varepsilon : F \to X$ with $F \in \operatorname{add} R$, we get an exact sequence $0 \to L \to M \oplus K \oplus F \xrightarrow{(\nu, \psi, \varepsilon)} X \to 0$. Putting $N = K \oplus F$, we can verify that $N \in \operatorname{res}\{M, X\}$, $\operatorname{NF}(L) \subseteq \operatorname{NF}(M)$ and $\operatorname{NF}(L) \cap \Phi = \emptyset$. ■

Now we state and prove the main result of this section concerning the structure of modules in $\mathsf{C}(R)$. This is used to prove the main results of Sections 4 and 10; it plays a key role in their proofs.

Theorem 3.8. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$ closed under finite direct sums and such that $R \in \mathcal{X} \subseteq \mathsf{C}(R)$. Let Φ be a subset of $\operatorname{Sing} R$ such that $\mathsf{C}_0(R_\mathfrak{p}) \subseteq \operatorname{add}(\mathcal{X}_\mathfrak{p})$ for every $\mathfrak{p} \in \Phi \setminus \{\mathfrak{m}\}$. Then, for each nonfree R-module $M \in \mathsf{C}_\Phi(R)$, there exists an exact sequence $0 \to C \to M \oplus N \to Y \to 0$ of R-modules in $\mathsf{C}(R)$ with $C \in \mathsf{C}_0(R)$, $Y \in \mathcal{X}^{\circ n}$ and $n = \dim \operatorname{NF}(M)$.*

Proof. Since M is nonfree, we have $n \geqslant 0$. We prove the proposition by using induction on n. Let $n = 0$. Then $\mathcal{X}^{\circ n} = 0$ by definition. We have $\operatorname{NF}(M) = \{\mathfrak{m}\}$, whence $M \in \mathsf{C}_0(R)$. Thus the assertion follows by taking the trivial exact sequence $0 \to M \to M \oplus 0 \to 0 \to 0$. Let $n \geqslant 1$. Put $\Psi = \min \operatorname{NF}(M)$. Note that the set Ψ is nonempty and finite. Also, we have that $\Psi \subseteq \Phi$ and $\mathfrak{m} \notin \Psi$. Fix any prime ideal $\mathfrak{p} \in \Psi$. Then $\mathfrak{p} \in \Phi \setminus \{\mathfrak{m}\}$, and $\mathsf{C}_0(R_\mathfrak{p}) \subseteq \operatorname{add} \mathcal{X}_\mathfrak{p}$ by assumption. It is seen that $\operatorname{NF}(M_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$, and we get $M_\mathfrak{p} \in \operatorname{mod}_0 R_\mathfrak{p}$. Hence $M_\mathfrak{p} \in \mathsf{C}(R_\mathfrak{p}) \cap \operatorname{mod}_0 R_\mathfrak{p} = \mathsf{C}_0(R_\mathfrak{p}) \subseteq \operatorname{add} \mathcal{X}_\mathfrak{p}$. Lemma 3.7(2) gives rise to an exact sequence $\alpha : 0 \to L \to M \oplus H \to X \to 0$ of R-modules with $X \in \mathcal{X}$, $H \in \operatorname{res}\{M, X\}$, $\operatorname{NF}(L) \subseteq \operatorname{NF}(M)$ and $\operatorname{NF}(L) \cap \Psi = \emptyset$. As $\mathcal{X}$ is contained in $\mathcal{X}^{\circ n}$, the exact sequence is a desired one if L is free. So, let us assume that L is nonfree. It is observed that $H, L \in \mathsf{C}(R)$, $\operatorname{NF}(L) \subseteq \operatorname{NF}(M) \subseteq \Phi$ and $t := \dim \operatorname{NF}(L) < \dim \operatorname{NF}(M) = n$. Applying the induction hypothesis to L, we obtain an exact sequence $0 \to C \to L \oplus K \xrightarrow{v} Z \to 0$ of R-modules with $C \in \mathsf{C}_0(R)$ and $Z \in \mathcal{X}^{\circ t}$. Taking the direct sum of α and the trivial exact sequence $0 \to K \to K \to 0 \to 0$ and putting $N = H \oplus K \in \mathsf{C}(R)$, we get an exact

[1]There is an obvious error in choosing such an element f in the proof of [57, Proposition 4.7], and the way of choice presented here is what was supposed to be given in [57].

sequence $0 \to L \oplus K \xrightarrow{w} M \oplus N \to X \to 0$. The pushout diagram of v, w gives rise to exact sequences $\beta : 0 \to Z \to Y \to X \to 0$ and $\gamma : 0 \to C \to M \oplus N \to Y \to 0$. Since $Z \in \mathcal{X}^{\circ t}$ and $X \in \mathcal{X}$, the exact sequence β shows $Y \in \mathcal{X}^{\circ(t+1)}$. As $t+1 \leqslant n$, we have $Y \in \mathcal{X}^{\circ n}$. Since $\mathsf{C}(R)$ is a resolving subcategory of $\operatorname{mod} R$, it contains $\mathcal{X}^{\circ n}$ and hence $Y \in \mathsf{C}(R)$. Thus γ is such an exact sequence as we want. ■

Remark 3.9. For $n \geqslant 0$, we denote by $\mathsf{S}_n(R)$ the subcategory of $\operatorname{mod} R$ consisting of modules satisfying *Serre's condition* (S_n), i.e., $\operatorname{depth} M_\mathfrak{p} \geqslant \inf\{n, \operatorname{ht} \mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Put $\mathsf{S}_n^\Phi(R) = \mathsf{S}_n(R) \cap \mathrm{NF}^{-1}(\Phi)$ for each subset Φ of $\operatorname{Spec} R$, and set $\mathsf{S}_n^0(R) = \mathsf{S}_n(R) \cap \operatorname{mod}_0 R = \mathsf{S}_n^{\operatorname{Max} R}(R)$. For every $M \in \mathsf{S}_n(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$, one has $M_\mathfrak{p} \in \mathsf{S}_n(R_\mathfrak{p})$. Whenever R satisfies (S_n), the subcategory $\mathsf{S}_n(R)$ of $\operatorname{mod} R$ is resolving and so is $\mathsf{S}_n^0(R)$. The assertion of Theorem 3.8 with $\mathsf{C}, \mathsf{C}_0, \mathsf{C}_\Phi$ replaced by $\mathsf{S}_n, \mathsf{S}_n^0, \mathsf{S}_n^\Phi$ respectively holds. More precisely, the statement below follows along the same lines as in the proof of Theorem 3.8.

> Let R be a local ring with maximal ideal $\mathfrak{m}$, and let n be a nonnegative integer. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$ closed under finite direct sums and such that $R \in \mathcal{X} \subseteq \mathsf{S}_n(R)$. Let Φ be a subset of $\operatorname{Sing} R$ such that $\mathsf{S}_n^0(R_\mathfrak{p}) \subseteq \operatorname{add}(\mathcal{X}_\mathfrak{p})$ for all $\mathfrak{p} \in \Phi \setminus \{\mathfrak{m}\}$. Then, for each nonfree R-module $M \in \mathsf{S}_n^\Phi(R)$, there exists an exact sequence $0 \to C \to M \oplus N \to Y \to 0$ of R-modules such that $C \in \mathsf{S}_n^0(R)$ and $Y \in \mathcal{X}^{\circ h}$, where $h = \dim \mathrm{NF}(M)$.

Note that in the case where R is Cohen–Macaulay and $n \geqslant \dim R$, one has $\mathsf{S}_n(R) = \mathsf{C}(R) = \mathsf{CM}(R)$, and the above statement is identified with Theorem 3.8 in this case.

Here we recall the definition of a notion.

Definition 3.10. Let Φ be a set of prime ideals of R. A resolving subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is said to be *dominant on* Φ if for each prime ideal $\mathfrak{p} \in \Phi$ there exists an integer $n \geqslant 0$ such that $\Omega^n \kappa(\mathfrak{p}) \in \operatorname{add} \mathcal{X}_\mathfrak{p}$.

The notion of a dominant resolving subcategory is introduced in [24], where the inner structure of each dominant resolving subcategory is described and classification of dominant resolving subcategories is provided. The above theorem yields the following result on dominant resolving subcategories.

Corollary 3.11. *Let $(R, \mathfrak{m})$ be a local ring. Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$ contained in $\mathsf{C}(R)$. Let Φ be a subset of $\operatorname{Sing} R$ such that $\mathcal{X}$ is dominant on $\Phi \setminus \{\mathfrak{m}\}$. Then, for each R-module $M \in \mathsf{C}_\Phi(R)$, there exists an exact sequence $0 \to C \to M \oplus N \to X \to 0$ of R-modules with $C \in \mathsf{C}_0(R)$ and $X \in \mathcal{X}$.*

Proof. The exact sequence $0 \to M \to M \oplus 0 \to 0 \to 0$ is a desired one when M is free, so we may assume that M is nonfree. Let $\mathfrak{p} \in \Phi \setminus \{\mathfrak{m}\}$. By the definition of a dominant resolving subcategory, there exists an integer $n \geqslant 0$ such that $\Omega^n \kappa(\mathfrak{p}) \in \operatorname{add} \mathcal{X}_\mathfrak{p}$. As $\operatorname{add} \mathcal{X}_\mathfrak{p}$ is a resolving subcategory of $\operatorname{mod} R_\mathfrak{p}$, it follows from [63, Corollary 3.3(1)] that $\Omega^{\operatorname{depth} R_\mathfrak{p}} \kappa(\mathfrak{p}) \in \operatorname{add} \mathcal{X}_\mathfrak{p}$. Remark 3.3(8) implies that $\mathsf{C}_0(R_\mathfrak{p})$ is contained in $\operatorname{add} \mathcal{X}_\mathfrak{p}$. As $\mathcal{X}$ is resolving, we have $\mathcal{X}^{\circ i} \subseteq \mathcal{X}$ for all $i \geqslant 0$. The assertion follows from Theorem 3.8. ■

As another corollary of Theorem 3.8, we obtain decompositions of the subcategory $\mathsf{C}(R)$ of $\operatorname{mod} R$.

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a local ring and put $n = \dim \operatorname{Sing} R$. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$ closed under finite direct sums and such that $R \in \mathcal{X} \subseteq \mathsf{C}(R)$. If $\mathsf{C}_0(R_\mathfrak{p}) \subseteq \operatorname{add}(\mathcal{X}_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Sing} R \setminus \{\mathfrak{m}\}$, then $\mathsf{C}(R) = \operatorname{add}(\mathsf{C}_0(R) \circ \mathcal{X}^{\circ n})$. If $\mathsf{C}_0(R_\mathfrak{p}) \subseteq \operatorname{add}(\mathcal{X}_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Sing} R$, then $\mathsf{C}(R) = \operatorname{add}(\mathcal{X}^{\circ(n+1)})$.*

Proof. The first assertion is shown by applying Theorem 3.8 to $\Phi = \operatorname{Sing} R$ and using Remark 3.3(4) and the inequality $\dim \mathrm{NF}(M) \leqslant \dim \operatorname{Sing} R$. In what follows, we show the second assertion. By the first assertion, $\mathsf{C}(R) = \operatorname{add}(\mathsf{C}_0(R) \circ \mathcal{X}^{\circ n})$. If $\mathfrak{m}$ is not in $\operatorname{Sing} R$, then R is regular, and $\mathsf{C}_0(R) \subseteq \mathsf{C}(R) = \operatorname{add} R \subseteq \mathcal{X}$ by Remark 3.3(6). If $\mathfrak{m}$ is in $\operatorname{Sing} R$, then $\mathsf{C}_0(R_\mathfrak{m}) \subseteq \operatorname{add}(\mathcal{X}_\mathfrak{m})$ by assumption, which means $\mathsf{C}_0(R) \subseteq \mathcal{X}$. Thus $\mathcal{X}$ contains $\mathsf{C}_0(R)$ in either case, and we obtain $\mathsf{C}(R) = \operatorname{add}(\mathcal{X}^{\circ(n+1)})$. ■

4. Generation of Verdier quotients of $\mathsf{D}^\mathsf{b}(R)$

In this section, we apply a result obtained in the previous section to explore the structure of some Verdier quotients of the derived category $\mathsf{D}^\mathsf{b}(R)$. To be more precise, applying Corollary 3.12, we investigate the (Rouquier) dimensions and the ultimate dimensions of the Verdier quotients $\mathsf{D}^\mathsf{b}(R)/\operatorname{thick}\{R, k\}$ and $\mathsf{D}^\mathsf{sg}(R)$ of $\mathsf{D}^\mathsf{b}(R)$. These two notions of dimensions for triangulated categories have been introduced by Rouquier [50], and Ballard, Favero and Katzarkov [11], respectively. Let us recall their definitions.

Definition 4.1. Let $\mathcal{T}$ be a triangulated category.

(1) Let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathcal{T}$. We denote by $\langle \mathcal{X} \rangle$ the subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under finite direct sums, direct summands and shifts, by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects $E \in \mathcal{T}$ having an exact triangle $X \to E \to Y \to X[1]$ in $\mathcal{T}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and put $\mathcal{X} \diamond \mathcal{Y} = \langle \mathcal{X} * \mathcal{Y} \rangle$. Set $\langle \mathcal{X} \rangle_0 = 0$, $\langle \mathcal{X} \rangle_1 = \langle \mathcal{X} \rangle$ and $\langle \mathcal{X} \rangle_n = \langle \mathcal{X} \rangle_{n-1} \diamond \langle \mathcal{X} \rangle$ for $n \geqslant 2$. We write $\langle \mathcal{X} \rangle_n^{\mathcal{T}}$ for $\langle \mathcal{X} \rangle_n$ when we specify the ambient triangulated category $\mathcal{T}$.
(2) Let $G \in \mathcal{T}$. The *generation time* $\mathrm{gt}_{\mathcal{T}}(G)$ of G in $\mathcal{T}$ is defined as the infimum of integers $n \geqslant 0$ with $\mathcal{T} = \langle G \rangle_{n+1}$. Note that $\mathrm{gt}_{\mathcal{T}}(G) \in \mathbb{N} \cup \{\infty\}$. We say that G is a *strong generator* of $\mathcal{T}$ if $\mathrm{gt}_{\mathcal{T}}(G) < \infty$.
(3) The *(Rouquier) dimension* $\dim \mathcal{T}$ of $\mathcal{T}$ is defined to be the infimum of the *Orlov spectrum* $\operatorname{OSpec} \mathcal{T} = \{\mathrm{gt}_{\mathcal{T}}(G) \mid G \text{ is a strong generator of } \mathcal{T}\}$ of $\mathcal{T}$. One then has $\dim \mathcal{T} \in \mathbb{N} \cup \{\infty\}$. Note that $\dim \mathcal{T} = \inf\{n \geqslant 0 \mid \mathcal{T} = \langle G \rangle_{n+1} \text{ for some } G \in \mathcal{T}\}$ holds, and that $\dim \mathcal{T} = 0$ if there exist only finitely many isomorphism classes of indecomposable objects in $\mathcal{T}$ and each object of $\mathcal{T}$ is isomorphic to a finite direct sum of indecomposable objects. The *ultimate dimension* $\operatorname{udim} \mathcal{T}$ of $\mathcal{T}$ is defined as follows.
$$\operatorname{udim} \mathcal{T} = \begin{cases} \sup(\operatorname{OSpec} \mathcal{T}) & \text{if } \operatorname{OSpec} \mathcal{T} \neq \emptyset, \\ \infty & \text{if } \operatorname{OSpec} \mathcal{T} = \emptyset. \end{cases}$$
One has $\operatorname{udim} \mathcal{T} \in \mathbb{N} \cup \{\infty\}$.

We need to establish the following general lemma on triangle functors of triangulated categories. The latter assertion of (2) follows from [49, Lemma 1.2]; for the convenience of the reader we give a proof.

Lemma 4.2. *Let $F : \mathcal{T} \to \mathcal{U}$ be a triangle functor of triangulated categories.*
(1) *There is an inclusion relation $F(\mathsf{thick}_{\mathcal{T}} \mathcal{X}) \subseteq \mathsf{thick}_{\mathcal{U}} F(\mathcal{X})$ for each subcategory $\mathcal{X}$ of $\mathcal{T}$.*
(2) *Let $\mathcal{X}$ be a thick subcategory of $\mathcal{U}$. Then $F^{-1}(\mathcal{X}) = \{T \in \mathcal{T} \mid F(T) \in \mathcal{X}\}$ is a thick subcategory of $\mathcal{T}$, and F induces a triangle functor $\overline{F} : \mathcal{T}/F^{-1}(\mathcal{X}) \to \mathcal{U}/\mathcal{X}$. If F is an equivalence, then so is $\overline{F}$.*

Proof. (1) Let $\mathcal{Y}$ be the subcategory of $\mathcal{T}$ consisting of objects Y with $F(Y) \in \mathsf{thick}_{\mathcal{U}} F(\mathcal{X})$. As F is a triangle functor, $\mathcal{Y}$ is seen to be a thick subcategory of $\mathcal{T}$ containing $\mathcal{X}$. Hence $\mathcal{Y}$ contains $\mathsf{thick}_{\mathcal{T}} \mathcal{X}$.

(2) It is easy to deduce the thickness of $F^{-1}(\mathcal{X})$ from the thickness of $\mathcal{X}$ and F being a triangle functor. Let $\alpha : \mathcal{U} \to \mathcal{U}/\mathcal{X}$ be the canonical functor. The composition αF is a triangle functor and $(\alpha F)(F^{-1}(\mathcal{X})) = 0$. By [48, Theorem 2.1.8] there exists a unique triangle functor $\overline{F} : \mathcal{T}/F^{-1}(\mathcal{X}) \to \mathcal{U}/\mathcal{X}$ such that $\overline{F}\beta = \alpha F$, where $\beta : \mathcal{T} \to \mathcal{T}/F^{-1}(\mathcal{X})$ is the canonical functor.

Suppose that F is an equivalence, and let $G : \mathcal{U} \to \mathcal{T}$ be a quasi-inverse to F. Similarly as above, there exists a unique triangle functor $\overline{G} : \mathcal{U}/G^{-1}(F^{-1}(\mathcal{X})) \to \mathcal{T}/F^{-1}(\mathcal{X})$ such that $\overline{G}\gamma = \beta G$, where $\gamma : \mathcal{U} \to \mathcal{U}/G^{-1}(F^{-1}(\mathcal{X}))$. We easily see $G^{-1}(F^{-1}(\mathcal{X})) = \mathcal{X}$ and $\gamma = \alpha$. Using uniqueness, we observe that $\overline{G} \cdot \overline{F} = \overline{GF}$ and $\overline{F} \cdot \overline{G} = \overline{FG}$. As GF and FG are isomorphic to the identity functors, so are $\overline{GF}$ and $\overline{FG}$. Hence $\overline{F}$ is an equivalence to which $\overline{G}$ is a quasi-inverse. ∎

We introduce a certain subcategory of the stable category of maximal Cohen–Macaulay modules.

Definition 4.3. Let R be a Gorenstein local ring. Let $\pi : \mathsf{CM}(R) \to \underline{\mathsf{CM}}(R)$ be the canonical functor, and set $\underline{\mathsf{CM}}_0(R) = \pi(\mathsf{CM}_0(R))$. (Here, note by our convention that if $M \in \underline{\mathsf{CM}}(R)$ and $M \cong \pi(X)$ for some $X \in \mathsf{CM}_0(R)$, then $M \in \pi(\mathsf{CM}_0(R)) = \underline{\mathsf{CM}}_0(R)$.) Hence $\underline{\mathsf{CM}}_0(R)$ is none other than the subcategory of $\underline{\mathsf{CM}}(R)$ consisting of the maximal Cohen–Macaulay R-modules M such that $M_{\mathfrak{p}} \cong 0$ in $\underline{\mathsf{CM}}(R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}_0 R$. Note that $\underline{\mathsf{CM}}_0(R)$ is a thick subcategory of $\underline{\mathsf{CM}}(R)$.

We record a triangle equivalence induced from the one $\underline{\mathsf{CM}}(R) \cong \mathsf{D}^{\mathsf{sg}}(R)$ stated in Definition 2.11.

Proposition 4.4. *Let R be a Gorenstein local ring with residue field k. Then the assignment $M \mapsto M$, where M is a maximal Cohen–Macaulay R-module, gives a triangle equivalence*
$$\underline{\mathsf{CM}}(R)/\,\underline{\mathsf{CM}}_0(R) \cong \mathsf{D}^{\mathsf{b}}(R)/\,\mathsf{thick}\{R, k\}.$$

Proof. As R is a Gorenstein local ring, there is a triangle equivalence $F : \underline{\mathsf{CM}}(R) \to \mathsf{D}^{\mathsf{sg}}(R)$, which is given by $F(M) = M$ for each $M \in \underline{\mathsf{CM}}(R)$. It is seen by [61, Corollary 4.3(3)] that $F^{-1}(\mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}(R)} k) = \underline{\mathsf{CM}}_0(R)$. Lemma 4.2(2) yields a triangle equivalence $\overline{F} : \underline{\mathsf{CM}}(R)/\,\underline{\mathsf{CM}}_0(R) \to \mathsf{D}^{\mathsf{sg}}(R)/\,\mathsf{thick}\, k$ with $\overline{F}(M) = M$ for each maximal Cohen–Macaulay R-module M. Let $\alpha : \mathsf{D}^{\mathsf{b}}(R) \to \mathsf{D}^{\mathsf{sg}}(R)$, $\beta : \mathsf{D}^{\mathsf{sg}}(R) \to \mathsf{D}^{\mathsf{sg}}(R)/\,\mathsf{thick}\, k$ and $\gamma : \mathsf{D}^{\mathsf{b}}(R) \to \mathsf{D}^{\mathsf{b}}(R)/\,\mathsf{thick}\{R, k\}$ be the natural functors. As $\gamma(\mathsf{thick}\, R) = 0$, by [48, Theorem 2.1.8] there exists a triangle functor $\delta : \mathsf{D}^{\mathsf{sg}}(R) = \mathsf{D}^{\mathsf{b}}(R)/\,\mathsf{thick}\, R \to \mathsf{D}^{\mathsf{b}}(R)/\,\mathsf{thick}\{R, k\}$ with $\gamma = \delta\alpha$. Lemma 4.2(1) implies $\beta\alpha(\mathsf{thick}\{R, k\}) \subseteq \mathsf{thick}\, \beta\alpha\{R, k\} = 0$ and $\delta(\mathsf{thick}\, k) \subseteq \mathsf{thick}\, \delta(k) = 0$. By [48, Theorem

2.1.8] again, we get triangle functors $\sigma : \mathsf{D^b}(R)/\operatorname{\mathsf{thick}}\{R,k\} \to \mathsf{D^{sg}}(R)/\operatorname{\mathsf{thick}} k$ and $\tau : \mathsf{D^{sg}}(R)/\operatorname{\mathsf{thick}} k \to \mathsf{D^b}(R)/\operatorname{\mathsf{thick}}\{R,k\}$ such that $\beta\alpha = \sigma\gamma$ and $\delta = \tau\beta$. We then have equalities $\gamma = (\tau\sigma)\gamma$ and $\beta\alpha = (\sigma\tau)\beta\alpha$. We see from [48, Theorem 2.1.8] again that $\tau\sigma$ and $\sigma\tau$ are identity functors. Thus τ is an equivalence. Now the composition $\tau\overline{F} : \underline{\mathsf{CM}}(R)/\underline{\mathsf{CM}}_0(R) \to \mathsf{D^b}(R)/\operatorname{\mathsf{thick}}\{R,k\}$ is a desired triangle equivalence. ■

We establish a lemma which describes maximal Cohen–Macaulay modules over the localizations of R at prime ideals in terms of localizations of maximal Cohen–Macaulay modules over R.

Lemma 4.5. (1) *Let R be a Cohen–Macaulay local ring with a canonical module ω. Let $\mathfrak{p}$ be a prime ideal of R. For any maximal Cohen–Macaulay $R_\mathfrak{p}$-module M there exists an isomorphism $M \oplus \omega_\mathfrak{p}^{\oplus n} \cong X_\mathfrak{p}$ such that n is a nonnegative integer and X is a maximal Cohen–Macaulay R-module. In particular, there is an equality $\mathsf{CM}(R_\mathfrak{p}) = \mathsf{add}(\mathsf{CM}(R)_\mathfrak{p})$ of subcategories of $\operatorname{\mathsf{mod}} R_\mathfrak{p}$.*

(2) *Let R be a Gorenstein local ring. Let $G \in \underline{\mathsf{CM}}(R)$, $\mathfrak{p} \in \operatorname{Spec} R$ and $n \in \mathbb{N}$. Then for every $M \in \langle G_\mathfrak{p}\rangle_n^{\underline{\mathsf{CM}}(R_\mathfrak{p})}$ there exists $N \in \langle G\rangle_n^{\underline{\mathsf{CM}}(R)}$ such that M is a direct summand of $N_\mathfrak{p}$ in $\underline{\mathsf{CM}}(R_\mathfrak{p})$.*

Proof. (1) Write $M = N_\mathfrak{p}$ with some R-module N. As R has a canonical module, N admits a Cohen–Macaulay approximation, that is, an exact sequence $0 \to Y \to X \to N \to 0$ of R-modules such that X is maximal Cohen–Macaulay and Y has finite injective dimension; see [4, Theorem 1.1]. Since $N_\mathfrak{p} = M$ is maximal Cohen–Macaulay and $Y_\mathfrak{p}$ has finite injective dimension, the localized exact sequence $0 \to Y_\mathfrak{p} \to X_\mathfrak{p} \to N_\mathfrak{p} \to 0$ splits (see [16, Exercises 3.1.24]). Hence $M \oplus Y_\mathfrak{p} \cong X_\mathfrak{p}$. This also says that the $R_\mathfrak{p}$-module $Y_\mathfrak{p}$ is maximal Cohen–Macaulay, and we get $Y_\mathfrak{p} \cong \omega_\mathfrak{p}^{\oplus n}$ for some $n \geqslant 0$ by [16, Exercises 3.3.28(a)].

(2) We use induction on n. When $n = 0$, we have $\langle G_\mathfrak{p}\rangle_n = 0$, whence $M = 0$ and we can take $N = 0$. Let $n = 1$. Then in $\underline{\mathsf{CM}}(R_\mathfrak{p})$ the object M is a direct summand of $\bigoplus_{i=1}^a G_\mathfrak{p}^{\oplus b_i}[c_i]$ for some $a, b_i, c_i \geqslant 0$. Set $N = \bigoplus_{i=1}^a G^{\oplus b_i}[c_i] \in \underline{\mathsf{CM}}(R)$. Then N belongs to $\langle G\rangle_1^{\underline{\mathsf{CM}}(R)}$ and M is a direct summand of $N_\mathfrak{p}$ in $\underline{\mathsf{CM}}(R_\mathfrak{p})$. Let $n \geqslant 2$. There exists an exact triangle $A \to B \to C \to A[1]$ in $\underline{\mathsf{CM}}(R_\mathfrak{p})$ such that $A \in \langle G_\mathfrak{p}\rangle_{n-1}$, $C \in \langle G_\mathfrak{p}\rangle_1$ and M is a direct summand of B; see [25, Remark 3.2(1)]. The induction hypothesis gives rise to objects $A' \in \langle G\rangle_{n-1}$ and $C' \in \langle G\rangle_1$ such that in $\underline{\mathsf{CM}}(R_\mathfrak{p})$ the objects A and C are direct summands of $A'_\mathfrak{p}$ and $C'_\mathfrak{p}$, respectively. There are isomorphisms $A \oplus X \cong A'_\mathfrak{p}$ and $C \oplus Y \cong C'_\mathfrak{p}$ in $\underline{\mathsf{CM}}(R_\mathfrak{p})$. Taking the direct sums with the trivial exact triagles $X \to X \to 0 \to X[1]$ and $0 \to Y \to Y \to 0[1]$, we obtain an exact triangle $A'_\mathfrak{p} \to D \to C'_\mathfrak{p} \xrightarrow{f} A'_\mathfrak{p}[1]$ in $\underline{\mathsf{CM}}(R_\mathfrak{p})$ such that M is a direct summand of $D = B \oplus X \oplus Y$. The morphism f belongs to $\underline{\mathrm{Hom}}_{R_\mathfrak{p}}(C'_\mathfrak{p}, A'_\mathfrak{p}[1])$, which is isomorphic to $\underline{\mathrm{Hom}}_R(C', A'[1])_\mathfrak{p}$. There exist elements $g \in \underline{\mathrm{Hom}}_R(C', A'[1])$ and $s \in R \setminus \mathfrak{p}$ such that $f = \frac{1}{s} \cdot g_\mathfrak{p}$. Extend the morphism g to an exact triangle $A' \to N \to C' \xrightarrow{g} A'[1]$ in $\underline{\mathsf{CM}}(R)$, and then we have $N \in \langle G\rangle_n$. We get a commutative diagram

$$\begin{array}{ccccccc}
A'_\mathfrak{p} & \longrightarrow & N_\mathfrak{p} & \longrightarrow & C'_\mathfrak{p} & \xrightarrow{g_\mathfrak{p}} & A'[1] \\
\| & & \downarrow{\scriptstyle h} & & \cong\downarrow{\scriptstyle s} & & \| \\
A'_\mathfrak{p} & \longrightarrow & D & \longrightarrow & C'_\mathfrak{p} & \xrightarrow{f} & A'_\mathfrak{p}[1]
\end{array}$$

of exact triangles in $\underline{\mathsf{CM}}(R_\mathfrak{p})$. Then h is an isomorphism, and M is a direct summand of $N_\mathfrak{p}$ in $\underline{\mathsf{CM}}(R_\mathfrak{p})$. ■

Definition 4.6. Following [36], we say that R has *finite CM_+-representation type* if there exist only a finite number of nonisomorphic indecomposable modules that belong to

$$\mathsf{CM}_+(R) := \mathsf{CM}(R) \setminus \mathsf{CM}_0(R) = \{M \in \mathsf{CM}(R) \mid \dim \mathrm{NF}(M) > 0\}.$$

If R is a Cohen–Macaulay local ring with an isolated singularity, then it has finite CM_+-representation type as $\mathsf{CM}_+(R)$ is empty. If R is a hypersurface of type (A_∞) or (D_∞), then it does not have an isolated singularity but has finite CM_+-representation type; see [2, Theorem 1.1(1)] for the details.

Denote by $\mathsf{D}_0^{\mathsf{sg}}(R)$ the subcategory of $\mathsf{D^{sg}}(R)$ consisting of R-complexes X such that $X_\mathfrak{p} \cong 0$ in $\mathsf{D^{sg}}(R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}_0 R$. Now we can achieve our purpose of stating an application of Corollary 3.12.

Corollary 4.7. *Let R be a Gorenstein local ring with maximal ideal $\mathfrak{m}$ and residue field k.*

(1) *If the singular locus $\operatorname{Sing} R$ has dimension at most one, then there is an inequality*

$$\dim \mathsf{D^b}(R)/\operatorname{\mathsf{thick}}\{R,k\} \leqslant \sup\{\dim \mathsf{D^{sg}}(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Sing} R \setminus \{\mathfrak{m}\}\}.$$

(2) *If R has finite CM_+-representation type, then it holds that $\dim \mathsf{D^b}(R)/\operatorname{\mathsf{thick}}\{R,k\} \leqslant 0$.*

(3) *There is an inequality*

$$\operatorname{udim} \mathsf{D}^{\mathsf{sg}}(R) \leqslant (s+1)(\dim \operatorname{Sing} R + 1) - 1,$$

where s is the supremum of nonnegative integers n such that $\mathsf{D}^{\mathsf{sg}}_0(R_\mathfrak{p}) \subseteq \langle G_\mathfrak{p} \rangle_{n+1}^{\mathsf{D}^{\mathsf{sg}}(R_\mathfrak{p})}$ for every strong generator G of $\mathsf{D}^{\mathsf{sg}}(R)$ and every $\mathfrak{p} \in \operatorname{Sing} R$. In particular, $\operatorname{udim} \mathsf{D}^{\mathsf{sg}}(R) < \infty$ *if* $s < \infty$.

Proof. Since R is Gorenstein, the singularity categories of R and its localization $R_\mathfrak{p}$ at a prime ideal $\mathfrak{p}$ may be replaced with the stable categories of maximal Cohen–Macaulay modules over R and $R_\mathfrak{p}$, respectively.

(1) By [61, Corollary 4.3(3)], we may assume $\dim \operatorname{Sing} R = 1$. We can write $\operatorname{Sing} R = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n, \mathfrak{m}\}$ with $n \geqslant 1$. We may assume $\dim \underline{\mathsf{CM}}(R_{\mathfrak{p}_i}) =: r_i < \infty$ for all $1 \leqslant i \leqslant n$. We find a maximal Cohen–Macaulay $R_{\mathfrak{p}_i}$-module G_i such that $\underline{\mathsf{CM}}(R_{\mathfrak{p}_i}) = \langle G_i \rangle_{r_i+1}$. By Lemma 4.5(1), we may assume $G_i = (C_i)_{\mathfrak{p}_i}$, where C_i is a maximal Cohen–Macaulay R-module. Put $r = \max\{r_1, \dots, r_n\} \in \mathbb{N}$ and $C = C_1 \oplus \cdots \oplus C_n \in \mathsf{CM}(R)$. Let $\mathcal{X}$ be the subcategory of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay R-modules that belong to $\langle C \rangle_{r+1}^{\mathsf{CM}(R)}$. Then $\mathcal{X}$ is closed under finite direct sums and satisfies $R \in \mathcal{X} \subseteq \mathsf{CM}(R)$.

We claim that $\mathsf{CM}(R_{\mathfrak{p}_i})$ is contained in $\operatorname{add} \mathcal{X}_{\mathfrak{p}_i}$ for all $1 \leqslant i \leqslant n$. In fact, pick any object $M \in \mathsf{CM}(R_{\mathfrak{p}_i})$. Then $M \in \underline{\mathsf{CM}}(R_{\mathfrak{p}_i}) = \langle G_i \rangle_{r_i+1} \subseteq \langle C_{\mathfrak{p}_i} \rangle_{r+1}$. By Lemma 4.5(2), there exists an object $N \in \langle C \rangle_{r+1}^{\mathsf{CM}(R)}$ such that M is a direct summand of $N_{\mathfrak{p}_i}$ in $\underline{\mathsf{CM}}(R_{\mathfrak{p}_i})$. As N belongs to $\mathcal{X}$, we see that M belongs to $\operatorname{add} \mathcal{X}_{\mathfrak{p}_i}$.

Applying Corollary 3.12, we obtain $\underline{\mathsf{CM}}(R)/\underline{\mathsf{CM}}_0(R) = \langle \mathcal{X} \rangle_1 = \langle C \rangle_{r+1}$. It follows from Proposition 4.4 that $\mathsf{D}^{\mathsf{b}}(R)/\operatorname{thick}\{R, k\} = \langle C \rangle_{r+1}$, which yields the inequality $\dim \mathsf{D}^{\mathsf{b}}(R)/\operatorname{thick}\{R, k\} \leqslant r$.

(2) According to [36, Theorems 1.2(1) and 1.3], the singular locus $\operatorname{Sing} R$ of R has dimension at most one and $R_\mathfrak{p}$ has finite CM-representation type for every $\mathfrak{p} \in \operatorname{Sing} R \setminus \{\mathfrak{m}\}$. The assertion follows from (1).

(3) We may assume $s < \infty$. Fix a strong generator G of $\underline{\mathsf{CM}}(R)$. Let $\mathcal{X}$ be the subcategory of $\operatorname{mod} R$ consisting of modules X which belong to $\langle G \rangle_{s+1}^{\mathsf{CM}(R)}$. Then $\mathcal{X}$ is closed under finite direct sums and satisfies $R \in \mathcal{X} \subseteq \mathsf{CM}(R)$. Fix $\mathfrak{p} \in \operatorname{Sing} R$. Then $\underline{\mathsf{CM}}_0(R_\mathfrak{p}) \subseteq \langle G_\mathfrak{p} \rangle_{s+1}^{\mathsf{CM}(R_\mathfrak{p})}$. A similar argument as in the claim in the proof of (1) deduces the inclusion $\mathsf{CM}_0(R_\mathfrak{p}) \subseteq \operatorname{add} \mathcal{X}_\mathfrak{p}$. Corollary 3.12 yields $\mathsf{CM}(R) = \operatorname{add}(\mathcal{X}^{\circ r})$, where $r = \dim \operatorname{Sing} R + 1$. We get $\underline{\mathsf{CM}}(R) = \langle \mathcal{X} \rangle_r = \langle G \rangle_{(s+1)r}$, and obtain $\operatorname{udim} \underline{\mathsf{CM}}(R) \leqslant (s+1)r - 1$. ■

5. Basic properties of dominant local rings

Theorem 3.8 and its consequences lead us to get interested in studying local rings R such that the resolving subcategories $\mathcal{X}$ of $\operatorname{mod} R$ with $\operatorname{add} R \neq \mathcal{X} \subseteq \mathsf{C}(R)$ contain $\mathsf{C}_0(R)$. Taking Corollary 3.11 into account, we call such local rings dominant. To be precise, we have the following definition and proposition.

Definition 5.1. Let R be a local ring with residue field k. Put $t = \operatorname{depth} R$. We say that R is *dominant* if for each R-module M of infinite projective dimension, the R-module $\Omega^t k$ belongs to the resolving closure $\operatorname{res} M$ of M. Note that if R is a regular local ring, then it is dominant since $\Omega^t k$ is a free R-module (or since there is no R-module of infinite projective dimension).

Remark 5.2. We will later obtain an equivalent homological characterization of dominant local rings; see Remark 10.9.

Proposition 5.3. *Let R be a local ring. The following three statements are equivalent.*
(1) *The local ring R is dominant.* (2) *For every R-module M with* $\operatorname{pd} M = \infty$ *one has* $\mathsf{C}_0(R) \subseteq \operatorname{res} M$.
(3) *For each resolving subcategory $\mathcal{X}$ of* $\operatorname{mod} R$ *with* $\operatorname{add} R \neq \mathcal{X} \subseteq \mathsf{C}(R)$, *it holds that* $\mathsf{C}_0(R) \subseteq \mathcal{X}$.

Proof. It is a direct consequence of Remark 3.3(8) that (1) and (2) are equivalent.

Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$ such that $\operatorname{add} R \neq \mathcal{X} \subseteq \mathsf{C}(R)$. Then there exists a nonfree R-module $X \in \mathcal{X}$. Since X belongs to $\mathsf{C}(R)$, we observe from the Auslander–Buchsbaum formula that X has infinite projective dimension. If (2) holds, then we have $\mathsf{C}_0(R) \subseteq \operatorname{res} X \subseteq \mathcal{X}$, and hence (3) holds.

Let M be an R-module with $\operatorname{pd} M = \infty$. Put $r = \operatorname{Rfd} M$ and $\mathcal{X} = \operatorname{res} \Omega^r M$. We see from Remark 3.3(5) that $\Omega^r M$ belongs to $\mathsf{C}(R)$, and therefore $\mathcal{X}$ is contained in $\mathsf{C}(R)$. Since $\Omega^r M$ is not R-free, we have $\mathcal{X} \neq \operatorname{add} R$. If (3) holds, then there are inclusions $\mathsf{C}_0(R) \subseteq \mathcal{X} \subseteq \operatorname{res} M$, and thus (2) holds. ■

The corollary below is an immediate consequence of the above proposition.

Corollary 5.4. *Let R be an artinian local ring. Then R is dominant if and only if there exist only trivial resolving subcategories of* $\operatorname{mod} R$, *that is,* $\operatorname{add} R$ *and* $\operatorname{mod} R$ *are the only resolving subcategories of* $\operatorname{mod} R$.

Proof. As R is artinian, we have $\mathsf{C}_0(R) = \mathsf{C}(R) = \operatorname{mod} R$. The assertion follows from Proposition 5.3. ■

In this section, we explore basic properties of dominant local rings. To be more precise, we investigate the relationship of dominance with several fundamental operations of local rings. We first consider how dominance is related to modding out by a regular element, and for this we establish a lemma.

Lemma 5.5. *Let M be an R-module. Let x be an element of R which is regular on R and M. Let N be an R-module with $N \in \operatorname{res}_R M$. Then the element x is regular on N and one has $N/xN \in \operatorname{res}_{R/(x)} M/xM$.*

Proof. Let $\mathcal{X}$ be the subcategory of $\operatorname{mod} R$ consisting of modules X such that x is X-regular and $X/xX \in \operatorname{res}_{R/(x)} M/xM$. By assumption, R and M belong to $\mathcal{X}$. Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules with $C \in \mathcal{X}$. By the snake lemma, the induced sequence $0 \to A/xA \to B/xB \to C/xC \to 0$ is exact. We observe that $A \in \mathcal{X}$ if and only if $B \in \mathcal{X}$. Hence $\mathcal{X}$ is closed under extensions and kernels of epimorphisms. It is easy to check that $\mathcal{X}$ is closed under direct summands. Therefore, $\mathcal{X}$ is a resolving subcategory of $\operatorname{mod} R$ containing M, and thus it contains $\operatorname{res}_R M$. The assertion now follows. ■

Now we prove the following theorem, which tells us that dominance has a good relationship with taking the quotient by a regular element. This theorem plays an important role in the remainder of the paper.

Theorem 5.6. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $x \in \mathfrak{m}$ be an R-regular element. If $R/(x)$ is a dominant local ring, then so is R. The converse is also true if $x \notin \mathfrak{m}^2$.*

Proof. Put $t = \operatorname{depth} R$. Since x is a regular element of R, we have $t \geqslant 1$.

We show the first assertion. Let M be an R-module with $\operatorname{pd}_R M = \infty$. Then x is ΩM-regular, and $\operatorname{pd}_{R/(x)} \Omega M/x\Omega M = \infty$ by [16, Lemma 1.3.5]. Since $R/(x)$ is a dominant local ring of depth $t-1$, the $R/(x)$-module $\Omega^{t-1}_{R/(x)}k$ is in $\operatorname{res}_{R/(x)} \Omega M/x\Omega M$. The R-module $\Omega_R \Omega^{t-1}_{R/(x)}k$ is in $\operatorname{res} \Omega M$ by [57, Lemma 5.8], and hence it is in $\operatorname{res} M$. It follows by [46, Lemma 4.2] that $\Omega^t k$ is in $\operatorname{res} M$. Thus R is dominant.

Now we prove the second assertion. Let M be an $R/(x)$-module of infinite projective dimension. Since x is not in $\mathfrak{m}^2$, it is seen from [7, Theorem 2.2.3] that M has infinite projective dimension as an R-module, and so does ΩM. Since R is dominant, $\Omega^t k$ belongs to $\operatorname{res} \Omega M$. The element x is regular on both R and ΩM. Lemma 5.5 implies that $\Omega^t k/x\Omega^t k$ is in $\operatorname{res}_{R/(x)} \Omega M/x\Omega M$. Thanks to the assumption $x \notin \mathfrak{m}^2$ again, $\Omega^t k/x\Omega^t k$ is isomorphic to $\Omega^t_{R/(x)}k \oplus \Omega^{t-1}_{R/(x)}k$; see [54, Corollary 5.3] for instance. Hence $\Omega^{t-1}_{R/(x)}k$ belongs to $\operatorname{res}_{R/(x)} \Omega M/x\Omega M$. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & R^{\oplus n} & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x\,\mid\,0} & & \\ 0 & \longrightarrow & \Omega M & \longrightarrow & R^{\oplus n} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

with exact rows, where the vertical arrows are multiplication maps by x. The snake lemma gives rise to an exact sequence $0 \to M \to \Omega M/x\Omega M \to (R/(x))^{\oplus n} \to M \to 0$. It is easy to see that $\Omega M/x\Omega M$ is in $\operatorname{res}_{R/(x)} M$. Consequently, $\Omega^{t-1}_{R/(x)}k$ belongs to $\operatorname{res}_{R/(x)} M$, and thus $R/(x)$ is a dominant local ring. ■

To get an application of Theorem 5.6, we establish an elementary lemma.

Lemma 5.7. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $a_1, \dots, a_n \in \mathfrak{m}$ and $e_1, \dots, e_n > 0$. Let $S = R[\![x_1, \dots, x_n]\!]$ be a formal power series ring. For each integer $1 \leqslant i \leqslant n$, let $S_i = S/(x_1^{e_1} - a_1, \dots, x_{i-1}^{e_{i-1}} - a_{i-1})$ be a quotient of S. Then the element $x_i^{e_i} - a_i$ is S_i-regular, and does not belong to the square of the maximal ideal of the local ring S_i if $e_i = 1$. Therefore, the sequence $x_1^{e_1} - a_1, \dots, x_n^{e_n} - a_n$ is S-regular.*

Proof. Fix $1 \leqslant i \leqslant n$, and set $T_i = R[\![x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]\!]/(x_1^{e_1} - a_1, \dots, x_{i-1}^{e_{i-1}} - a_{i-1})$. Then the ring T_i is local, a_i is a nonunit of T_i and $S_i = T_i[\![x_i]\!]$. Denote by $\mathfrak{n}$ the maximal ideal of the local ring T_i.

The element $x_i - a_i$ is not in the square of the maximal ideal of the local ring S_i. In fact, otherwise, the natural surjection $S_i \to S_i/\mathfrak{n}S_i \cong (T_i/\mathfrak{n})[\![x_i]\!] \cong k[\![x_i]\!]$ would give $x_i \in x_i^2 k[\![x_i]\!]$, a contradiction.

We prove that $x_i^{e_i} - a_i$ is S_i-regular. It suffices to show that for a local ring A, a nonunit $a \in A$ and an integer $e > 0$, the element $t^e - a$ is regular on the formal power series ring $A[\![t]\!]$. Assume $(t^e - a)f = 0$ in $A[\![t]\!]$ with $f \in A[\![t]\!]$. Writing $f = c_0 + c_1 t + c_2 t^2 + \cdots$, we see that for each $h \geqslant 0$ the $(e+h)$th coefficient of $(t^e - a)f$ is $c_h - ac_{e+h}$. Hence $c_h = a^j c_{je+h}$ for all $j > 0$, and $c_h \in \bigcap_{j>0}(a^j) = 0$ by Krull's intersection theorem (recall the assumption that $a \in A$ is a nonunit). It follows that $f = 0$, and we are done. ■

The following result is a corollary of Theorem 5.6, which says that dominance also has good relationships with taking a formal power series extension and taking the completion. Thanks to this corollary, when we deal with dominance, we can reduce to the case of a complete local ring.

Corollary 5.8. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Then R is dominant if and only if so is the formal power series ring $R[\![x]\!]$ over R, if and only if so is the completion $\widehat{R}$ of R.*

Proof. We have $R \cong R[\![x]\!]/(x)$, and x is an $R[\![x]\!]$-regular element with $x \in \mathfrak{n} \backslash \mathfrak{n}^2$, where $\mathfrak{n} = \mathfrak{m}R[\![x]\!] + xR[\![x]\!]$ is the maximal ideal of $R[\![x]\!]$. By Theorem 5.6 we see that R is dominant if and only if so is $R[\![x]\!]$.

We write $\mathfrak{m} = (a_1, \dots, a_n)$. We have $\widehat{R} \cong R[\![x_1, \dots, x_n]\!]/(x_1 - a_1, \dots, x_n - a_n)$ by [44, Theorem 8.12]. By what we showed at the beginning of this proof, R is dominant if and only if so is $R[\![x_1, \dots, x_n]\!]$. Using Lemma 5.7 and Theorem 5.6, we inductively see that $R[\![x_1, \dots, x_n]\!]$ is dominant if and only so is $R[\![x_1, \dots, x_n]\!]/(x_1 - a_1, \dots, x_n - a_n)$. As a consequence, R is dominant if and only if so is $\widehat{R}$. ■

Making use of the results which we have obtained in this section so far, we can find out several classes of local rings that are included in that of dominant local rings. In what follows, we first recall the definitions of those local rings, and then show that they are actually dominant.

Definition 5.9. Let R be a d-dimensional local ring with maximal ideal $\mathfrak{m}$ and residue field k.

(1) We say that R is a *hypersurface* if $\operatorname{codepth} R \leqslant 1$. This is equivalent to saying that the completion $\widehat{R}$ of R is isomorphic to the quotient of a regular local ring by a nonzero nonunit element; see [7, §5.1].
(2) Suppose that R is Cohen–Macaulay. Then there is an inequality $\mathrm{e}(R) \geqslant \operatorname{codim} R + 1$. We say that R has *minimal multiplicity* if the equality $\mathrm{e}(R) = \operatorname{codim} R + 1$ holds. If k is infinite, this is equivalent to saying that $\mathfrak{m}^2 = \boldsymbol{x}\mathfrak{m}$ for some system of parameters $\boldsymbol{x} = x_1, \dots, x_d$ of R; see [16, Exercises 4.6.14].
(3) We say that $\mathfrak{m}$ is *quasi-decomposable* if there exists a regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ on R such that $\mathfrak{m}/(\boldsymbol{x})$ is decomposable as a module over $R/(\boldsymbol{x})$ (or equivalently, over R). A local ring with quasi-decomposable maximal ideal is such a local ring that deforms to a fiber product of local rings with common residue field; see [46] for details, where quasi-decomposability has been introduced.
(4) We say that R is *Burch* if there exist a maximal regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ on the completion $\widehat{R}$, a regular local ring $(S, \mathfrak{n})$ and an ideal I of S such that $\mathfrak{n}(I : \mathfrak{n}) \neq \mathfrak{n}I$ and $\widehat{R}/(\boldsymbol{x}) \cong S/I$. For the details of Burch rings, we refer the reader to [22], which gives various observations on Burch rings.

Proposition 5.10. *A local ring $(R, \mathfrak{m}, k)$ is dominant in each of the following four cases.*
(1) *R is a hypersurface.* (2) *R is Cohen–Macaulay and has minimal multiplicity, and k is infinite.*
(3) *$\mathfrak{m}$ is quasi-decomposable.* (4) *R is Burch.*

Proof. It is shown in [22, Propositions 5.1 and 5.2] that if either (1) or (2) holds, then so does (4). We observe from [22, Proposition 7.6] that if R is Burch, then it is dominant.

It remains to show that R is dominant when $\mathfrak{m}$ is quasi-decomposable. Let $\boldsymbol{x} = x_1, \dots, x_n$ be an R-regular sequence such that $\mathfrak{m}/(\boldsymbol{x})$ is decomposable. According to Theorem 5.6, it suffices to show that $R/(\boldsymbol{x})$ is dominant. Replacing R with $R/(\boldsymbol{x})$, we may assume $\mathfrak{m}$ is decomposable. Let M be an R-module with $\operatorname{pd} M = \infty$. By [46, Theorem A], the module $\Omega k = \mathfrak{m}$ is a direct summand of $\Omega^3 M \oplus \Omega^4 M \oplus \Omega^5 M$. Hence Ωk belongs to $\operatorname{res} M$. Using [63, Proposition 3.2], we obtain $\Omega^{\operatorname{depth} R} k \in \operatorname{res}(R \oplus \Omega k) \subseteq \operatorname{res} M$. ■

Remark 5.11. As we saw in the above proposition, the class of dominant local rings include that of Burch rings. An advantage of dominant local rings to Burch rings is the fact that dominant local rings satisfy the property given in Theorem 5.6, while Burch rings do not. More precisely, it is true by the definition of Burch rings that a local ring $(R, \mathfrak{m})$ is Burch if so is the quotient $R/(x)$ by an R-regular element $x \in \mathfrak{m}$, but the converse does not necessarily hold even when $x \notin \mathfrak{m}^2$; see [22, Example 5.8].

6. Tor/Ext-friendliness, Tor/Ext-persistence and dominance

Recently, the notions of Tor/Ext-friendly rings and of Tor/Ext-persistent rings have been introduced and studied in [9]. In this section, we compare dominant local rings with those rings. We begin with recalling the definition of cores.

Definition 6.1. (1) We denote by $\operatorname{fpd} R$ the subcategory of $\operatorname{mod} R$ consisting of those R-modules which have finite projective dimension.

(2) The *(resolving) core* $\mathsf{core}\,R$ of $\mathsf{mod}\,R$ is defined as the intersection of the resolving subcategories of $\mathsf{mod}\,R$ that are not contained in $\mathsf{fpd}\,R$.

We provide a way of interpreting the dominance of a given local ring R in terms of $\mathsf{core}\,R$, and characterize the dominant local rings which are complete intersections.

Proposition 6.2. (1) *If a local ring R is singular, then the inclusion $\mathsf{core}\,R \subseteq \mathsf{C}_0(R)$ holds.*
(2) *Let R be a singular local ring. Then the equality $\mathsf{core}\,R = \mathsf{C}_0(R)$ holds if and only if R is dominant.*
(3) *Let R be a complete intersection local ring. Then R is dominant if and only if R is a hypersurface.*

Proof. (1) Remark 3.3(8) implies that $\mathsf{C}_0(R)$ is not contained in $\mathsf{fpd}\,R$. Hence $\mathsf{C}_0(R)$ contains $\mathsf{core}\,R$.

(2) Let M be an R-module of infinite projective dimension. Then $\mathsf{res}\,M$ is not contained in $\mathsf{fpd}\,R$, so that $\mathsf{res}\,M$ contains $\mathsf{core}\,R$. Hence, if we have $\mathsf{core}\,R = \mathsf{C}_0(R)$, then R is dominant by Proposition 5.3.

Now assume that R is dominant. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$ not contained in $\mathsf{fpd}\,R$. Then $\mathcal{X}$ contains an R-module X of infinite projective dimension. As R is dominant, we have $\mathsf{C}_0(R) \subseteq \mathsf{res}\,X \subseteq \mathcal{X}$ by Proposition 5.3. Hence $\mathsf{C}_0(R) \subseteq \mathsf{core}\,R$, and we get the equality $\mathsf{core}\,R = \mathsf{C}_0(R)$ by (1).

(3) The "if" part follows from Proposition 5.10(1). To show the "only if" part, assume R is singular and dominant. Then $\mathsf{core}\,R = \mathsf{C}_0(R) \ni \Omega^d k$ by (2) and Remark 3.3(8), where $d = \dim R$ and k is the residue field. As $\mathrm{pd}(\Omega^d k) = \infty$, we have $\mathsf{core}\,R \nsubseteq \mathsf{fpd}\,R$. Thus R is a hypersurface by [62, Theorem 1.1]. ■

Now we recall the definitions of Tor/Ext-persistent rings and of Tor/Ext-friendly rings. In [42], Cohen–Macaulay local rings that are Tor-friendly (resp. Ext-friendly) are studied, which are called Cohen–Macaulay local rings that satisfy *trivial Tor-vanishing* (resp. *trivial Ext-vanishing*).

Definition 6.3. (1) We say that R is *Tor-persistent* if every R-module M such that $\mathrm{Tor}^R_{\gg 0}(M, M) = 0$ has finite projective dimension.
(2) We say that R is *Tor-friendly* provided that if M and N are R-modules with $\mathrm{Tor}^R_{\gg 0}(M, N) = 0$, then either M or N has finite projective dimension.
(3) We say that R is *Ext-persistent* if every R-module M such that $\mathrm{Ext}_R^{\gg 0}(M, M) = 0$ has either finite projective dimension or finite injective dimension.
(4) We say that R is *Ext-friendly* provided that if M are N are R-modules with $\mathrm{Ext}_R^{\gg 0}(M, N) = 0$, then either M has finite projective dimension or N has finite injective dimension.

Next we recall the definition of a Golod local ring.

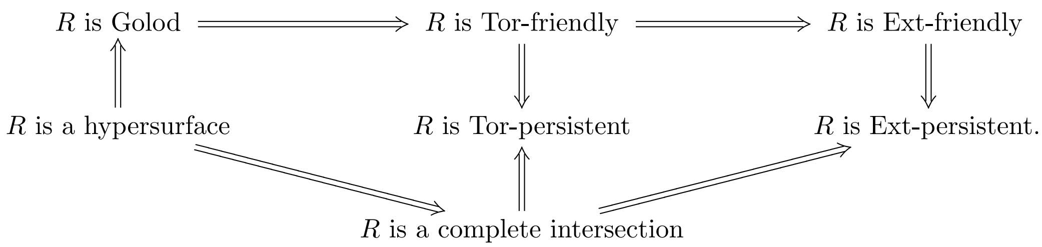
Definition 6.4. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Then R is called *Golod* if the Poincaré series $\mathrm{P}^R_k(t)$ of k over R has the description

$$\mathrm{P}^R_k(t) = \frac{(1+t)^{\mathrm{edim}\,R}}{1 - \sum_{i=1}^{\mathrm{codepth}\,R} \dim_k \mathrm{H}_i(K) t^{i+1}},$$

where K is the Koszul complex on a minimal system of generators of $\mathfrak{m}$. For the details of Golod rings, we refer the reader to [7, Chapter 5].

Here we compare Tor/Ext-friendly rings and Tor/Ext-persistent rings with complete intersections, hypersurfaces and Golod rings.

Remark 6.5. Let R be a local ring. Then the following implications hold, as is shown in [9, Example 1.8 and Corollary 6.4] and (the proof of) [62, Proposition 4.9].



We should also mention that it is asked in [9] whether every commutative noetherian ring is Tor-persistent.

Now we interpret the main theorems of [62] in terms of dominance. Here, $\#\operatorname{\mathsf{ind}}\mathsf{CM}(R)$ stands for the number of isomorphism classes of indecomposable maximal Cohen–Macaulay R-modules.

Proposition 6.6. *Let R be a Cohen–Macaulay local ring with residue field k.*

(1) *Suppose that R is henselian and singular, has finite CM-representation type and admits a canonical module. If R is either Ext-persistent or Tor-persistent with $\mathsf{core}\,R \neq \mathsf{add}\,R$, then R is dominant.*
(2) *Suppose that R is complete and that k is algebraically closed and has characteristic zero. If either $\#\operatorname{\mathsf{ind}}\mathsf{CM}(R) \leqslant 5$ or $\#\operatorname{\mathsf{ind}}\mathsf{CM}(R) \leqslant 7$ and R is Tor-persistent, then R is dominant.*

Proof. In either of the cases (1) and (2), the Cohen–Macaulay local ring R has an isolated singularity by [35, Corollary 2], and hence the equality $\mathsf{CM}(R) = \mathsf{CM}_0(R)$ holds. By [62, Theorem 4.16(2)] for (1) and by [62, Theorem 6.5] for (2) we get $\mathsf{core}\,R = \mathsf{CM}(R)$. Proposition 6.2(2) shows that R is dominant. ■

Here we recall a celebrated conjecture: the *Auslander–Reiten conjecture* asserts that any R-module M with $\operatorname{Ext}_R^{>0}(M, M \oplus R) = 0$ is projective. This conjecture has been presented in the 1970s [5], and is still open. Now we show that each dominant local ring is Tor-friendly, and satisfies that conjecture.

Theorem 6.7. *A dominant local ring is Tor-friendly, and hence it is Ext-friendly, Tor-persistent and Ext-persistent. In particular, the Auslander–Reiten conjecture holds for a dominant local ring.*

Proof. The latter assertion follows from the former and [9, Theorem 7.2]. To show the former assertion, let R be a dominant local ring. In view of Remark 6.5, it suffices to show that R is Tor-friendly. Let M, N be R-modules. Suppose that there is an integer $n \geqslant 0$ with $\operatorname{Tor}_i^R(M, N) = 0$ for all integers $i > n$, and that M has infinite projective dimension. As R is dominant, $\Omega^t k$ is in $\mathsf{res}\,M$, where $t = \operatorname{depth} R$. Let $\mathcal{X}$ be the subcategory of $\mathsf{mod}\,R$ consisting of modules X such that $\operatorname{Tor}_i^R(X, N) = 0$ for all $i > n$. It is easy to verify that $\mathcal{X}$ is a resolving subcategory of $\mathsf{mod}\,R$ containing M. We have $\Omega^t k \in \mathsf{res}\,M \subseteq \mathcal{X}$, and $0 = \operatorname{Tor}_i^R(\Omega^t k, N) = \operatorname{Tor}_{i+t}^R(k, N)$ for all $i > n$. It follows that N has finite projective dimension. ■

For a matrix A with entries in R and an integer r, we denote by $\mathrm{I}_n(A)$ the ideal of R generated by the $n \times n$ minors of A. As an application of the above theorem, we observe that the Auslander–Reiten conjecture holds true for determinantal rings with respect to maximal minors.

Corollary 6.8. *Let $X = (X_{ij})$ be an $m \times n$ generic matrix over a field k such that $m \leqslant n$. Let $R = k[X]/\mathrm{I}_m(X)$ be a determinantal ring. Let $\mathfrak{m}$ be the irrelevant maximal ideal of R. Then the Auslander–Reiten conjecture holds for the localization $R_\mathfrak{m}$ and the completion $\widehat{R_\mathfrak{m}} = k[\![X]\!]/\mathrm{I}_m(X)$.*

Proof. Note in general that the Auslander–Reiten conjecture holds for a local ring L if it holds for some faithfully flat extension of L. Thus, it suffices to prove that the Auslander–Reiten conjecture holds for $S = \widehat{R_\mathfrak{m}} = k[\![X]\!]/\mathrm{I}_m(X)$. Let ℓ be an infinite field containing k (e.g., the algebraic closure of k). Put $T = \ell \otimes_k R = \ell[X]/\mathrm{I}_m(X)$ and $U = \ell[\![X]\!]/\mathrm{I}_m(X)$. Then $U/\mathfrak{m}^i U = T/\mathfrak{m}^i T = T \otimes_R R/\mathfrak{m}^i = \ell \otimes_k R/\mathfrak{m}^i$ is faithfully flat over $S/\mathfrak{m}^i S = R/\mathfrak{m}^i$ for each $i > 0$, as ℓ is faithfully flat over k. Since $\mathfrak{m}S$ is the maximal ideal of the local ring S, we see from [44, Theorem 22.3] that U is faithfully flat over S. Thus, replacing k with ℓ, we may assume that k is infinite. By [22, Proposition 5.6 and Example 5.7] the localization $R_\mathfrak{m}$ is a Burch ring. Proposition 5.10(4) implies that $R_\mathfrak{m}$ is dominant, and so is $\widehat{R_\mathfrak{m}}$ by Corollary 5.8. It follows from Theorem 6.7 that the Auslander–Reiten conjecture holds for $\widehat{R_\mathfrak{m}}$. ■

Remark 6.9. Let R be a Gorenstein ring such that $R_\mathfrak{p}$ is a complete intersection for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht}\mathfrak{p} \leqslant 1$. Let $X = (X_{ij})$ be an $m \times n$ generix matrix with $m \leqslant n$. It is asserted in [39, Theorem 1.2] that the Auslander–Reiten conjecture holds for $R[X]/\mathrm{I}_m(X)$ if $2m \leqslant n + 1$. Compare this with Corollary 6.8.

Here we need to recall a couple of definitions.

Definition 6.10. (1) An R-module C is said to be *semidualizing* if the natural map $R \to \operatorname{Hom}_R(C, C)$ is an isomorphism and $\operatorname{Ext}_R^{>0}(C, C) = 0$. The R-module R is an obvious example of a semidualizing module. If R is a Cohen–Macaulay local ring with a canonical module ω, then ω is a semidualizing R-module. We refer the reader to [20, 29] for the details of semidualizing modules.

(2) An R-module G is said to be *totally reflexive* if G is reflexive and satisfies $\operatorname{Ext}_R^{>0}(G \oplus G^*, R) = 0$, where $(-)^* = \operatorname{Hom}_R(-, R)$. Evidently, every projective module is totally reflexive. If R is Cohen–Macaulay, then any totally reflexive R-module is maximal Cohen–Macaulay. If R is Gorenstein, then total reflexivity is equivalent to maximal Cohen–Macaulayness. For more details of totally reflexive modules, we refer the reader to [3, 19].

(3) Following [55], we say that R is *G-regular* if every totally reflexive R-module is projective.

Applying the theorem stated above, we obtain the following corollary, where we compare dominant local rings with G-regular local rings, and consider semidualizing modules over dominant local rings.

Corollary 6.11. *Let R be a dominant local ring. Then the following two statements hold true.*

(1) *If R is not a Gorenstein ring, then it is G-regular.*
(2) *Let C be a semidualizing R-module, and suppose that C is not isomorphic to R. Then R is a Cohen–Macaulay ring and C is a canonical module of R.*

Proof. First of all, note from Theorem 6.7 that the ring R is Ext-friendly.

(1) Let G be a totally reflexive R-module. Then $\mathrm{Ext}_R^{>0}(G,R)=0$. Ext-friendliness implies $\mathrm{pd}\, G<\infty$ or $\mathrm{id}\, R<\infty$. As R is non-Gorenstein, we get $\mathrm{pd}\, G<\infty$. It is observed from [19, (1.2.10)] that G is free.

(2) Since $\mathrm{Ext}_R^{>0}(C,C)=0$, Ext-friendliness shows that $\mathrm{pd}\, C<\infty$ or $\mathrm{id}\, C<\infty$. In the former case, we have $\mathrm{pd}\, C=\mathrm{depth}\, R-\mathrm{depth}\, C=0$ by [29, Page 68, Item 8], whence $C\cong R$, a contradiction. Therefore $\mathrm{id}\, C<\infty$. Thus C is a dualizing module, so that R is Cohen–Macaulay and C is a canonical module; see [16, Theorem 3.3.10 and Remarks 9.6.4(a)(ii)] and [19, (3.4.1) and (A.8.5)]. ■

Below we provide another application of Theorem 6.7. Compare this result with Theorem 5.6.

Corollary 6.12. *Let $(R,\mathfrak{m})$ be a local ring. Then the following statements hold true.*

(1) *Suppose that there exist a local ring $(S,\mathfrak{n})$ and an S-regular element $f\in\mathfrak{n}^2$ with $\widehat{R}\cong S/(f)$. Then R is dominant if and only if S is regular. In particular, R is dominant if and only if it is a hypersurface.*
(2) *If the local ring R is singular, then $R/(x)$ is not dominant for all R-regular elements $x\in\mathfrak{m}^2$.*

Proof. (1) The latter assertion is deduced by the former and Proposition 5.10(1). If R is dominant, then it is Tor-friendly by Theorem 6.7, and S is regular by [9, Proposition 3.8(2)]. The former assertion follows.

(2) Set $A=R/(x)$. Then the completion $\widehat{A}$ of the local ring A is isomorphic to $\widehat{R}/x\widehat{R}$. The local ring $\widehat{R}$ is singular, and x is $\widehat{R}$-regular and belongs to $(\mathfrak{m}\widehat{R})^2$. It follows from (1) that A is not dominant. ■

Remark 6.13. In the case where the local ring R is not Gorenstein, Corollary 6.12(2) can also be deduced from Corollary 6.11(1) and [10, Example 3.5(3)].

7. Inheritance of dominance

In this section, we study how dominance is inherited by standard operations of local rings. First of all, we consider the relationship of dominance with localization at prime ideals in the following two remarks. It turns out that dominance is not compatible with localization.

Remark 7.1. Let R be a local ring. Then the following implication does not necessarily hold true.

$$R \text{ is dominant} \implies R_{\mathfrak{p}} \text{ is dominant for all prime ideals } \mathfrak{p} \text{ of } R. \tag{7.1.1}$$

In fact, let $R=k[\![x,y,z,w]\!]/(x^2,xyz,y^2,zw)$ with k a field. Then R is a 1-dimensional Cohen–Macaulay complete local ring with a parameter $z-w$. There is an isomorphism $R/(z-w)\cong S/I$, where $S=k[\![x,y,z]\!]$ is a regular local ring and $I=(x^2,xyz,y^2,z^2)$ is an ideal of S. Let $\mathfrak{n}$ be the maximal ideal of S. We have $I:\mathfrak{n}=\mathfrak{n}^2$ and $\mathfrak{n}(I:\mathfrak{n})=\mathfrak{n}^3\ne\mathfrak{n}I$ (as $xyz\in\mathfrak{n}^3\setminus\mathfrak{n}I$). Therefore R is Burch, and dominant by Proposition 5.10(4). However, for the prime ideal $\mathfrak{p}=(x,y,z)$ of R we have $R_{\mathfrak{p}}\cong k[\![x,y,w]\!]_{(x,y)}/(x^2,y^2)$, so that $R_{\mathfrak{p}}$ is a complete intersection which is not a hypersurface. Proposition 6.2(3) shows that $R_{\mathfrak{p}}$ is not dominant.

Another example which does not satisfy the implication (7.1.1) is given in [45, Theorem B]. Indeed, let R be a local ring and $\mathfrak{p}$ a prime ideal of R as in [45, Theorem B]. Then, since the maximal ideal of R is decomposable, R is a dominant local ring by Proposition 5.10(3). Since there exists a semidualizing $R_{\mathfrak{p}}$-module which is neither free nor dualizing, $R_{\mathfrak{p}}$ is not a dominant local ring by Corollary 6.11(2).

The converse of (7.1.1) trivially holds, as one can take $\mathfrak{p}$ to be the maximal ideal of R. The following observation says that the converse is not true in general if we remove the case of the maximal ideal.

Remark 7.2. Let R be a local ring. Then the implication below does not necessarily hold true.

$$R_{\mathfrak{p}} \text{ is dominant for all nonmaximal prime ideals } \mathfrak{p} \text{ of } R \implies R \text{ is dominant.}$$

In fact, let $(R,\mathfrak{m})$ be a local complete intersection that is not a hypersurface but has an isolated singularity (e.g., $k[x,y]/(x^2,y^2)$, $k[\![t^4,t^5,t^6]\!]$ etc., where k is a field). Then R is not dominant by Proposition 6.2(3). However, for each $\mathfrak{p}\in\operatorname{Spec}R\setminus\{\mathfrak{m}\}$ the local ring $R_\mathfrak{p}$ is regular, and hence it is dominant.

Next we investigate how dominance is transferred along a flat local homomorphism.

Remark 7.3. Let $(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a flat local homomorphism of local rings. Then the implications below do not necessarily hold true.

$$R \text{ and } S/\mathfrak{m}S \text{ are dominant} \implies S \text{ is dominant.} \tag{7.3.1}$$

$$R \text{ and } S \text{ are dominant} \implies S/\mathfrak{m}S \text{ is dominant.} \tag{7.3.2}$$

Indeed, let k be a field. As to (7.3.1), we consider the ring extension $R=k[x]/(x^2)\subseteq k[x,t]/(x^2,t^2)=S$ of artinian local complete intersections. Then S is a free R-module with $\{1,t\}$ a basis. The rings R and $S/\mathfrak{m}S\cong k[t]/(t^2)$ are dominant by Proposition 5.10(1), while S is not dominant by Proposition 6.2(3).

One can also construct a non-Gorenstein example. Consider the extension $R=k[x,y]/(x^2,xy,y^2)\subseteq k[x,y,t]/(x^2,xy,y^2,t^2)=S$. Then R,S are non-Gorenstein, and S is a free R-module with $\{1,t\}$ a basis. The rings R and $S/\mathfrak{m}S$ are Burch by [22, Remark 7.15], whence they are dominant by Proposition 5.10(4). By [22, Remark 7.15] again, S is not G-regular. Hence S is not dominant by Corollary 6.11(1).

As for (7.3.2), let $S=k[\![x,y,t]\!]/(x^2,xy,y^2)$, and consider the subring $R=k[\![t^2]\!]$ of S. Then S is a free R-module with $\{1,x,y,t,xt,yt\}$ a basis. The ring R is regular, and the maximal ideal $\mathfrak{n}$ of S is quasi-decomposable; one has $\mathfrak{n}/(t)\cong(x)\oplus(y)$. The local rings R and S are dominant by Proposition 5.10(3). The ring $S/\mathfrak{m}S$ is isomorphic to $k[x,y,t]/(x^2,xy,y^2,t^2)$, which is not dominant as we saw above.

Thus, dominance is not compatible with local flat extensions in full generality. However, the following proposition says that the implication (7.3.1) does hold if we impose the assumption that R is regular. This proposition is regarded as a dominant version of [55, Corollary 4.5] concerning G-regularity.

Proposition 7.4. *Let $\phi:R\to S$ be a flat local homomorphism of local rings. Denote by $\mathfrak{m}$ the maximal ideal of R. If R is regular and $S/\mathfrak{m}S$ is dominant, then S is dominant.*

Proof. Let $\boldsymbol{x}=x_1,\dots,x_d$ be a regular system of parameters of the regular local ring R. Then $\boldsymbol{x}$ is an R-regular sequence. As ϕ is local and flat, $\boldsymbol{x}$ is an S-regular sequence as well. By assumption, $S/\boldsymbol{x}S=S/\mathfrak{m}S$ is dominant. Repeated application of Theorem 5.6 yields that S is dominant. ■

The following question is natural to ask. We have neither a proof nor a counterexample.

Question 7.5. Let $R\to S$ be a flat local homomorphism of local rings. If S is dominant, is R dominant?

To prove our next result, we need to generalize [46, Lemma 4.2] and [57, Lemma 5.8].

Lemma 7.6. *Let $n\geqslant 0$ be an integer. Let I be a proper ideal of R such that $\operatorname{pd}_R R/I\leqslant n$.*

(1) *Let M be an R/I-module. For any $i\geqslant 0$ one has $\Omega_R^n\Omega_{R/I}^iM\cong\Omega_R^{n+i}M$ up to projective R-summands.*

(2) *Let M and N be R/I-modules. If $M\in\operatorname{res}_{R/I}N$, then $\Omega_R^nM\in\operatorname{res}_R(\Omega_R^nN)$.*

Proof. (1) There is an exact sequence $0\to\Omega_{R/I}^iM\to P_{i-1}\to\cdots\to P_0\to M\to 0$ in $\operatorname{mod}R/I$, where each P_j is a projective R/I-module. Applying $\Omega_R^n(-)$ to this exact sequence, we obtain an exact sequence $0\to\Omega_R^n\Omega_{R/I}^iM\to Q_{i-1}\to\cdots\to Q_0\to\Omega_R^nM\to 0$ in $\operatorname{mod}R$ such that each Q_j is a projective R-module since $\operatorname{pd}_R R/I\leqslant n$. This exact sequence says that $\Omega_R^n\Omega_{R/I}^iM$ is an i-th syzygy of the R-module Ω_R^nM, which is isomorphic to $\Omega_R^{n+i}M$ up to projective R-summands.

(2) Put $\mathcal{C}=\operatorname{res}_R(\Omega_R^nN)$. Let $\mathcal{X}$ be the subcategory of $\operatorname{mod}R/I$ consisting of modules X with $\Omega_R^nX\in\mathcal{C}$. Then it is easy to verify that $\mathcal{X}$ is a resolving subcategory of $\operatorname{mod}R/I$. As Ω_R^nN is in $\operatorname{res}_R(\Omega_R^nN)=\mathcal{C}$, the module N is in $\mathcal{X}$. As $\mathcal{X}$ is resolving, it contains $\operatorname{res}_{R/I}N$. Thus $M\in\mathcal{X}$, and $\Omega_R^nM\in\mathcal{C}=\operatorname{res}_R(\Omega_R^nN)$. ■

In the proposition below, we study dominance of local rings by establishing the same kinds of conditions as investigated in [39, Theorem 2.3] for the Auslander–Reiten conjecture.

Proposition 7.7. *Let $(R,\mathfrak{m},k)$ be a local ring. Let $\boldsymbol{x}=x_1,\dots,x_n$ be a regular sequence on R with $n>0$. Let I be the ideal of R generated by $\boldsymbol{x}$. Consider the following four statements.*

(a) *R is dominant.* (b) *R/I^i is dominant for all $i>0$.*

(c) *R/I^i is dominant for all $1\leqslant i\leqslant n$.* (d) *R/I^r is dominant for some $r>0$.*

Then, the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) *always hold true. The implication* (a) $\Rightarrow$ (d) *does not necessarily hold even when* $n = 1$. *The implication* (a) $\Rightarrow$ (b) *does not necessarily hold even when* $i = 1$.

Proof. We prove the ith assertion of the proposition in the paragraph (i) below for each $i \in \{1, 2, 3\}$.

(1) It is evident that the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) hold. To show the implication (d) $\Rightarrow$ (a), suppose that R/I^r is dominant for some $r > 0$. Let M be an R-module of infinite projective dimension, and put $t = \operatorname{depth} R$. By [17, Proposition (2.14)] (or [16, Exercises 1.4.27]) we have $\operatorname{pd}_R R/I^r = n$. The Auslander–Buchsbaum formula implies $\operatorname{depth} R/I^r = t - n$, which says $t \geqslant n$. Set $N = \Omega_R^t M$. Applying [46, Lemma 5.1] to $\Omega_R^{t-n} M$, we see that the sequence $\boldsymbol{x}$ is N-regular. Note that $N/\boldsymbol{x}N$ is an R/I^r-module.

Suppose that $\operatorname{pd}_{R/I^r} N/\boldsymbol{x}N < \infty$. Since $\operatorname{pd}_R R/I^r = n < \infty$, we observe that $\operatorname{pd}_R N/\boldsymbol{x}N < \infty$. Hence $\operatorname{pd}_R N < \infty$ by [16, Exercises 1.3.6]. As $N = \Omega_R^t M$, it follows that $\operatorname{pd}_R M < \infty$, which is a contradiction. Thus $\operatorname{pd}_{R/I^r} N/\boldsymbol{x}N = \infty$. Since R/I^r is dominant, $\Omega_{R/I^r}^{t-n} k$ belongs to $\mathsf{res}_{R/I^r} N/\boldsymbol{x}N$. We obtain

$$\Omega_R^t k \cong \Omega_R^n(\Omega_{R/I^r}^{t-n} k) \in \mathsf{res}_R(\Omega_R^n(N/\boldsymbol{x}N)) \subseteq \mathsf{res}_R N = \mathsf{res}_R(\Omega_R^t M) \subseteq \mathsf{res}_R M,$$

where the isomorphism is up to free summands. The isomorphism and the containment follow from (1) and (2) of Lemma 7.6, respectively, while the first inclusion is obtained by [46, Lemma 4.3]. Thus the local ring R is dominant, that is to say, statement (a) holds.

(2) Suppose that R is not Gorenstein, but dominant and has positive depth. (For example, the quotient $K[\![X, Y, Z]\!](X^2, XY, Y^2)$ with K a field is such a ring by Proposition 5.10(3).) We can choose an R-regular element $x \in \mathfrak{m}^2$. By [10, Examples 3.5(3)], for all $i > 0$ the local ring $R/(x^i)$ is not G-regular. It follows from Corollary 6.11(1) that $R/(x^i)$ is not dominant. Thus (a) $\Rightarrow$ (d) is not necessarily true for $n = 1$.

(3) Suppose that R is regular and has dimension at least 2. Then R is a dominant local ring. We can choose an R-regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{m}^2$ with $n \geqslant 2$. The quotient ring $R/(\boldsymbol{x})$ is a non-hypersurface local complete intersection, and hence it is not dominant by Proposition 6.2(3). This argument shows that the implication (a) $\Rightarrow$ (b) does not necessarily hold for $i = 1$. ∎

8. Construction of dominant local rings

This section provides several ways to construct a dominant local ring from another dominant local ring. By Proposition 5.10(4) a Burch ring is dominant, while a lot of examples of Burch rings are presented in [22]. Thus, applying the methods developed in this section to those Burch rings, we can get a lot of new dominant local rings, which may no longer be Burch.

We start by stating a result on Burch rings, producing a Burch ring which seems to be unknown.

Proposition 8.1. *Let* $(R, \mathfrak{m})$ *be a local ring with* $\mathfrak{m}^3 = 0$ *and* $\operatorname{edim} R = 2$. *Suppose that* R *is not a complete intersection. Then* R *is Burch. Therefore,* R *is a dominant local ring.*

Proof. As R is artinian, it is complete. We can write $R = S/I$, where $(S, \mathfrak{n})$ is a 2-dimensional (complete) regular local ring and I is an $\mathfrak{n}$-primary ideal of S such that $\mathfrak{n}^3 \subseteq I \subseteq \mathfrak{n}^2$. Suppose that the equality $\mathfrak{n}(I : \mathfrak{n}) = \mathfrak{n}I$ holds. Then we have $\mathfrak{n}^3 = \mathfrak{n}\mathfrak{n}^2 \subseteq \mathfrak{n}(I : \mathfrak{n}) = \mathfrak{n}I \subseteq \mathfrak{n}\mathfrak{n}^2 = \mathfrak{n}^3$, and get $\mathfrak{n}^3 = \mathfrak{n}I$. If $I = \mathfrak{n}^2$, then the equality $\mathfrak{n}(I : \mathfrak{n}) = \mathfrak{n}I$ and Nakayama's lemma imply $\mathfrak{n}^2 = 0$, which is a contradiction. Hence $I \neq \mathfrak{n}^2$. Letting x, y be a regular system of parameters of S, we have $\mathfrak{n} = (x, y)$, $\mathfrak{n}^2 = (x^2, xy, y^2)$, and $\ell(\mathfrak{n}^2/\mathfrak{n}^3) = \mu(\mathfrak{n}^2) = 3$. Hence $\mu(I) = \ell(I/\mathfrak{n}I) = \ell(\mathfrak{n}^2/\mathfrak{n}^3) - \ell(\mathfrak{n}^2/I) \leqslant 3 - 1 = 2$. We must have $\mu(I) = 2$ and I is generated by a system of parameters of the regular local ring S. It follows that $R = S/I$ is a complete intersection, which contradicts the assumption of the proposition. Thus we obtain $\mathfrak{n}(I : \mathfrak{n}) \neq \mathfrak{n}I$, and therefore R is Burch. By Proposition 5.10(4) the local ring R is dominant. ∎

To prove our next proposition, we establish an elementary lemma on intersections of ideals.

Lemma 8.2. (1) *For ideals* I, J *of* R *there is an isomorphism* $I \cap J/IJ \cong \operatorname{Tor}_1^R(R/I, R/J)$ *of* R*-modules.*
(2) *Let* I *be an ideal of* R. *Let* $S = R[\![x_1, \dots, x_n]\!]$. *Then* $(x_1, \dots, x_n)S \cap IS = (x_1, \dots, x_n)IS$.

Proof. (1) Applying $- \otimes_R R/J$ to the exact sequence $0 \to I \to R \to R/I \to 0$, we get an exact sequence $0 \to \operatorname{Tor}_1^R(R/I, R/J) \to I/IJ \xrightarrow{\alpha} R/J$, where $\alpha(\overline{x}) = \overline{x}$ for $x \in I$. We have $\operatorname{Ker} \alpha = (I \cap J)/IJ$.

(2) Put $\boldsymbol{x} = x_1, \dots, x_n$. By (1) the quotient $(\boldsymbol{x}S \cap IS)/\boldsymbol{x}IS$ is isomorphic to $\operatorname{Tor}_1^S(S/\boldsymbol{x}S, S/IS)$. This is isomorphic to the first Koszul homology $\mathrm{H}_1(\boldsymbol{x}, S/IS)$, which vanishes since $\boldsymbol{x}$ is a regular sequence on $S/IS \cong (R/I)[\![\boldsymbol{x}]\!]$. Hence $(\boldsymbol{x}S \cap IS)/\boldsymbol{x}IS = 0$, and thus $\boldsymbol{x}S \cap IS = \boldsymbol{x}IS$. ∎

Now we can show the following proposition concerning getting dominant local rings.

Proposition 8.3. *Let R be a 1-dimensional Cohen–Macaulay local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let I be an $\mathfrak{m}$-primary ideal of R. Suppose that the local ring R/aI is dominant for some parameter a of R. Then the local ring R/bI is dominant for every parameter b of R.*

Proof. Let $S = R[\![x]\!]/xIR[\![x]\!]$. Lemma 8.2(2) implies that $xR[\![x]\!] \cap IR[\![x]\!] = xIR[\![x]\!]$. Then it is seen that $xS \cap IS = 0$, and there is an exact sequence $0 \to S \to S/xS \oplus S/IS \to S/xS + IS \to 0$ of S-modules. The local ring $S/xS+IS = R/I$ is artinian, while $S/xS = R$ and $S/IS = (R/I)[\![x]\!]$ are Cohen–Macaulay local rings of dimension 1. We observe that S is a 1-dimensional Cohen–Macaulay local ring.

We claim that $x - c$ is S-regular for any parameter c of R. Indeed, as I, cR are $\mathfrak{m}$-primary, the ring R/cI is artinian and hence complete. We have $S/(x-c) \cong (R/cI)[\![x]\!]/(x-\overline{c}) \cong R/cI$; see [44, Exercise 8.2 and Theorem 8.12]. As S is a Cohen–Macaulay ring of dimension 1, the claim follows.

We also claim that $x - c$ is not in the square of the maximal ideal $\mathfrak{n}$ of S. In fact, if $x - c \in \mathfrak{n}^2$, then by using the surjection $S \to S/\mathfrak{m}S = k[\![x]\!]$ we can deduce that $x \in x^2k[\![x]\!]$, which gives a contradiction.

It follows from the above two claims that $x - a$ and $x - b$ are S-regular elements and $x - b$ is not in $\mathfrak{n}^2$. Since $S/(x-a) \cong R/aI$ is dominant, so is S, and so is $S/(x-b) \cong R/bI$ by Theorem 5.6. ■

Applying the above proposition, we obtain the corollary below on a quotient ring by a monomial ideal. In view of this result, the class of dominant local rings would be much larger than that of Burch rings.

Corollary 8.4. *Let k be a field. Let $a_1 > a_2 > \cdots > a_{n-1} > a_n = 0$ and $0 = b_1 < b_2 < \cdots < b_{n-1} < b_n$ be integers with $n \geqslant 3$. Let $R = k[\![x,y]\!]/(x^{a_1}, x^{a_2}y^{b_2}, \dots, x^{a_{n-1}}y^{b_{n-1}}, y^{b_n})$. Then R is a dominant local ring. The ring R is Burch if and only if $a_r - a_{r+1} = 1$ or $b_{r+1} - b_r = 1$ for some $1 \leqslant r \leqslant n-1$.*

Proof. The latter assertion of the corollary is a consequence of [22, Corollaries 2.7 and 6.5]. To show the former, we consider the 1-dimensional local hypersurface $S = k[\![x,y]\!]/(x^{a_1})$ with maximal ideal $\mathfrak{n} = (x,y)$. The ideal $I = (x^{a_2}, x^{a_3}y^{b_3-b_2}, \dots, x^{a_{n-1}}y^{b_{n-1}-b_2}, y^{b_n-b_2})$ of S is $\mathfrak{n}$-primary. The element y is a parameter of S. The local ring $S/yI = k[\![x,y]\!]/(x^{a_1}, x^{a_2}y, x^{a_3}y^{b_3-b_2+1}, \dots, x^{a_{n-1}}y^{b_{n-1}-b_2+1}, y^{b_n-b_2+1})$ is Burch by [22, Corollary 6.5], and it is dominant by Proposition 5.10(4). As y^{b_2} is a parameter of S as well, Proposition 8.3 shows that $S/y^{b_2}I = k[\![x,y]\!]/(x^{a_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}, \dots, x^{a_{n-1}}y^{b_{n-1}}, y^{b_n}) = R$ is dominant. ■

To state our next proposition, we state an elementary fact on substitution in a formal power series.

Remark 8.5. Let I be an ideal of R. Write $I = (a_1, \dots, a_n)$. Suppose that R is I-adically complete. Let $S = R[\![x_1, \dots, x_n]\!]$ be a formal power series ring. Take $f(x_1, \dots, x_n) \in S$, and write $f(x_1, \dots, x_n) = \sum_{i_1+\cdots+i_n=0}^{\infty} c_{i_1\cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$ with $c_{i_1\cdots i_n} \in R$. Then one can substitute $a_1, \dots, a_n$ for $x_1, \dots, x_n$, that is,

$$f(a_1, \dots, a_n) := \textstyle\sum_{i_1+\cdots+i_n=0}^{\infty} c_{i_1\cdots i_n} a_1^{i_1} \cdots a_n^{i_n} := \lim_{k\to\infty}(\sum_{i_1+\cdots+i_n=0}^{k} c_{i_1\cdots i_n} a_1^{i_1} \cdots a_n^{i_n})$$

is uniquely defined as an element of R. In fact, putting $b_k = \sum_{i_1+\cdots+i_n=0}^{k} c_{i_1\cdots i_n} a_1^{i_1} \cdots a_n^{i_n} \in R$, we get $b_k - b_{k-1} = \sum_{i_1+\cdots+i_n=k} c_{i_1\cdots i_n} a_1^{i_1} \cdots a_n^{i_n} \in I^k$. Thus $\{b_k\}_{k=0}^{\infty}$ is a Cauchy sequence in the I-adic topology, and converges in R since R is I-adically complete (see [44, Page 57]). The assignment $\overline{f(x_1, \dots, x_n)} \mapsto f(a_1, \dots, a_n)$ gives an isomorphism $S/(x_1 - a_1, \dots, x_n - a_n) \cong R$; see [44, Theorem 8.12] and its proof.

The proposition below describes the relationship between the dominance of a quotient of a formal power series ring and the dominance of another ring given by substitution in the defining ideal.

Proposition 8.6. *Let $A = k[\![x_1, \dots, x_n]\!]$ and $B = k[\![y_1, \dots, y_m]\!]$ be formal power series rings over a field k. Let $R = A/(f_1, \dots, f_t)$, where $f_1, \dots, f_t \in (x_1, \dots, x_n)^2$. Let $g_1, \dots, g_n \in B$ be nonunits. Put $\widetilde{f_i} := f_i(g_1, \dots, g_n) \in B$ for each i, and set $S = B/(\widetilde{f_1}, \dots, \widetilde{f_t})$. Suppose that $x_1 - g_1, \dots, x_n - g_n$ is a regular sequence on $R \widehat{\otimes}_k B = R[\![y_1, \dots, y_m]\!]$ (this assumption is satisfied if R is Cohen–Macaulay and the equality $\operatorname{ht}(f_1, \dots, f_t) = \operatorname{ht}(\widetilde{f_1}, \dots, \widetilde{f_t})$ holds). Then R is a dominant local ring if and only if so is S.*

Proof. The ring B is $(y_1, \dots, y_m)$-adically complete, and it is $(g_1, \dots, g_n)$-adically complete as $g_1, \dots, g_n \in (y_1, \dots, y_n)$. Remark 8.5 says $\widetilde{f_1}, \dots, \widetilde{f_t} \in B$ are defined and the assignment $\overline{h(x_1, \dots, x_n)} \mapsto h(g_1, \dots, g_n)$ gives an isomorphism $B[\![x_1, \dots, x_n]\!]/(x_1 - g_1, \dots, x_n - g_n) \to B$, which sends each $\overline{f_i}$ to $\widetilde{f_i}$. Therefore,

$$\begin{aligned} &R[\![y_1, \dots, y_m]\!]/(x_1 - g_1, \dots, x_n - g_n) \\ &\quad = B[\![x_1, \dots, x_n]\!]/(x_1 - g_1, \dots, x_n - g_n, f_1, \dots, f_t) \cong B/(\widetilde{f_1}, \dots, \widetilde{f_t}) = S. \end{aligned}$$

We claim that $x_i - g_i$ is not in the square of the maximal ideal of $R[\![y_1,\dots,y_m]\!]/(x_1-g_1,\dots,x_{i-1}-g_{i-1})$ for each $1 \leqslant i \leqslant n$. In fact, suppose that $x_i - g_i \in (x_1,\dots,x_n,y_1,\dots,y_m)^2 + (f_1,\dots,f_t,x_1-g_1,\dots,x_{i-1}-g_{i-1})$ in $k[\![x_1,\dots,x_n,y_1,\dots,y_m]\!]$. The assumption $f_1,\dots,f_t \in (x_1,\dots,x_n)^2$ implies $x_i - g_i \in (x_1,\dots,x_n,y_1,\dots,y_m)^2 + (x_1-g_1,\dots,x_{i-1}-g_{i-1})$. The surjection $k[\![x_1,\dots,x_n,y_1,\dots,y_m]\!] \to k[\![x_1,\dots,x_n,y_1,\dots,y_m]\!]/(y_1,\dots,y_m) = A$ shows $x_i \in (x_1,\dots,x_n)^2 + (x_1,\dots,x_{i-1})$ in A, a contradiction.

When $x_1-g_1,\dots,x_n-g_n$ is a regular sequence on the ring $R[\![y_1,\dots,y_m]\!]$, by applying Theorem 5.6, the above claim and Corollary 5.8 repeatedly, it is observed that R is dominant if and only if so is S.

Finally, we verify what is stated in the parentheses in the assertion. Assume that R is Cohen–Macaulay and $\mathrm{ht}(f_1,\dots,f_t) = \mathrm{ht}(\widetilde{f_1},\dots,\widetilde{f_t})$. Then $R[\![y_1,\dots,y_m]\!]$ is Cohen–Macaulay as well. The equalities

$$\begin{aligned}\mathrm{grade}(x_1-g_1,\dots,x_n-g_n) &= \dim R[\![y_1,\dots,y_m]\!] - \dim(R[\![y_1,\dots,y_m]\!]/(x_1-g_1,\dots,x_n-g_n))\\ &= (\dim R + m) - \dim S = (n - \mathrm{ht}(f_1,\dots,f_t)) + m - (m - \mathrm{ht}(\widetilde{f_1},\dots,\widetilde{f_t})) = n\end{aligned}$$

hold. By [16, Corollary 1.6.19] the sequence $x_1-g_1,\dots,x_n-g_n$ is regular on $R[\![y_1,\dots,y_m]\!]$. ■

As an application of the above proposition, we obtain the following result regarding a quotient ring by an ideal generated by minors of a matrix.

Corollary 8.7. *Let k be a field. Let $R = k[\![x_1,\dots,x_n]\!]/\mathrm{I}_2\begin{pmatrix} f_{11} & \cdots & f_{1t} \\ f_{21} & \cdots & f_{2t}\end{pmatrix}$, where each f_{ij} is a nonunit of $k[\![x_1,\dots,x_n]\!]$. If $\dim R = n-t+1$, then R is dominant.*

Proof. Let $S = k[y_{11},\dots,y_{1t},y_{21},\dots,y_{2t}]/\mathrm{I}_2\begin{pmatrix} y_{11} & \cdots & y_{1t} \\ y_{21} & \cdots & y_{2t}\end{pmatrix}$ be a determinantal ring, and let $\mathfrak{n}$ be the irrelevant maximal ideal of S. Then S is a Cohen–Macaulay ring of dimension $t+1$ by [16, Theorem 7.3.1(c)]. The local ring $S_\mathfrak{n}$ is Burch by [22, Proposition 5.6 and Example 5.7], and is dominant by Proposition 5.10(4). Let $T = \widehat{S_\mathfrak{n}} = k[\![y_{11},\dots,y_{1t},y_{21},\dots,y_{2t}]\!]/\mathrm{I}_2\begin{pmatrix} y_{11} & \cdots & y_{1t} \\ y_{21} & \cdots & y_{2t}\end{pmatrix}$ be the completion of the local ring of $S_\mathfrak{n}$. Then the local ring T is Cohen–Macaulay, and Corollary 5.8 implies that T is dominant. The computation

$$2t - \dim T = 2t - \dim S = 2t - (t+1) = t-1 = n - (n-t+1) = n - \dim R$$

says that the defining ideals of T, R have the same height. Proposition 8.6 shows that R is dominant. ■

The following result holds, whose context is similar to those of the two propositions stated above.

Proposition 8.8. *Let $(R,\mathfrak{m},k)$ be a regular local ring. Let $\boldsymbol{x} = x_1,\dots,x_n$ and $\boldsymbol{y} = y_1,\dots,y_m$ be sequences of elements of $\mathfrak{m}$ with $n \geqslant 1$ and $m \geqslant 0$. Suppose that $\boldsymbol{x},\boldsymbol{y}$ is an R-regular sequence.*

(1) *If $m \geqslant 1$, then the residue ring $S = R/(\boldsymbol{x}\boldsymbol{y})$ is a dominant local ring.*

(2) *If $z \in \mathfrak{m}$ is an $R/(\boldsymbol{y})$-regular element, then the residue ring $T = R/(\boldsymbol{x}(\boldsymbol{y},z))$ is a dominant local ring.*

Proof. By Corollary 5.8, replacing R,S,T with their completions, we may assume that R is complete.

We begin with proving assertion (2) of the proposition. Let $A = R[\![\boldsymbol{X},\boldsymbol{Y},Z]\!]/\boldsymbol{X}(\boldsymbol{Y},Z)$ be a quotient of a formal power series ring over R, where $\boldsymbol{X} = X_1,\dots,X_n$ and $\boldsymbol{Y} = Y_1,\dots,Y_m$. Then $A/(\boldsymbol{X}-\boldsymbol{x},\boldsymbol{Y}-\boldsymbol{y},Z-z)$ is isomorphic to T (as R is $\mathfrak{m}$-adically complete; see [44, Theorem 8.12]). As $n \geqslant 1$, the local ring $B = A/\mathfrak{m}A = k[\![\boldsymbol{X},\boldsymbol{Y},Z]\!]/\boldsymbol{X}(\boldsymbol{Y},Z)$ has decomposable maximal ideal $(\boldsymbol{X},\boldsymbol{Y},Z)B = \boldsymbol{X}B \oplus (\boldsymbol{Y},Z)B$ by Lemma 8.2(2). Propositions 5.10(3) and 7.4 show that A is dominant. In view of Theorem 5.6, it suffices to prove that $\boldsymbol{X}-\boldsymbol{x},\boldsymbol{Y}-\boldsymbol{y},Z-z$ is a regular sequence on A. The equality $\boldsymbol{X}(\boldsymbol{Y},Z) = (\boldsymbol{X}) \cap (\boldsymbol{Y},Z)$ of ideals of $R[\![\boldsymbol{X},\boldsymbol{Y},Z]\!]$ (following from Lemma 8.2(2)) shows that an exact sequence

$$0 \to A \to A/(\boldsymbol{X}) \oplus A/(\boldsymbol{Y},Z) \to A/(\boldsymbol{X},\boldsymbol{Y},Z) \to 0$$

of A-modules is induced. The sequence $\boldsymbol{X}-\boldsymbol{x}$ is regular on $A/(\boldsymbol{X}) \cong R[\![\boldsymbol{Y},Z]\!]$, $A/(\boldsymbol{Y},Z) \cong R[\![\boldsymbol{X}]\!]$ and $A/(\boldsymbol{X},\boldsymbol{Y},Z) \cong R$, since $-\boldsymbol{x}$ is regular on R and $R[\![\boldsymbol{Y},Z]\!]$, and $\boldsymbol{X}-\boldsymbol{x}$ is regular on $R[\![\boldsymbol{X}]\!]$ by Lemma 5.7. Therefore, $\boldsymbol{X}-\boldsymbol{x}$ is a regular sequence on A, and an exact sequence

$$0 \to A/(\boldsymbol{X}-\boldsymbol{x}) \to A/(\boldsymbol{X},\boldsymbol{x}) \oplus A/(\boldsymbol{Y},Z,\boldsymbol{X}-\boldsymbol{x}) \to A/(\boldsymbol{X},\boldsymbol{Y},Z,\boldsymbol{x}) \to 0$$

is induced. The modules $A/(\boldsymbol{X},\boldsymbol{x})$, $A/(\boldsymbol{Y},Z,\boldsymbol{X}-\boldsymbol{x})$ and $A/(\boldsymbol{X},\boldsymbol{Y},Z,\boldsymbol{x})$ are respectively isomorphic to $(R/(\boldsymbol{x}))[\![\boldsymbol{Y},Z]\!]$, R and $R/(\boldsymbol{x})$. The sequence $\boldsymbol{Y}-\boldsymbol{y}$ is regular on $(R/(\boldsymbol{x}))[\![\boldsymbol{Y},Z]\!]$, while the sequence $-\boldsymbol{y}$ is regular on R and $R/(\boldsymbol{x})$; note that as the sequence $\boldsymbol{x},\boldsymbol{y}$ is R-regular and the ring R is local, the sequence $\boldsymbol{y}$ is R-regular. It is seen that $\boldsymbol{Y}-\boldsymbol{y}$ is a regular sequence on $A/(\boldsymbol{X}-\boldsymbol{x})$, and an exact sequence

$$0 \to A/(\boldsymbol{X}-\boldsymbol{x},\boldsymbol{Y}-\boldsymbol{y}) \to A/(\boldsymbol{X},\boldsymbol{x},\boldsymbol{Y}-\boldsymbol{y}) \oplus A/(\boldsymbol{Y},Z,\boldsymbol{X}-\boldsymbol{x},\boldsymbol{y}) \to A/(\boldsymbol{X},\boldsymbol{Y},Z,\boldsymbol{x},\boldsymbol{y}) \to 0$$

is induced. The modules $A/(\boldsymbol{X},\boldsymbol{x},\boldsymbol{Y}-\boldsymbol{y})$ and $A/(\boldsymbol{Y},Z,\boldsymbol{X}-\boldsymbol{x},\boldsymbol{y})$ are isomorphic to $(R/(\boldsymbol{x}))[\![Z]\!]$ and $R/(\boldsymbol{y})$, respectively. Since $Z-z$ and $-z$ are regular on $(R/(\boldsymbol{x}))[\![Z]\!]$ and $R/(\boldsymbol{y})$ respectively, $Z-z$ is regular on $A/(\boldsymbol{X}-\boldsymbol{x}, \boldsymbol{Y}-\boldsymbol{y})$. We conclude that the sequence $\boldsymbol{X}-\boldsymbol{x}, \boldsymbol{Y}-\boldsymbol{y}, Z-z$ is regular on A, and thus the proof of the second assertion of the proposition is completed.

Assertion (1) is shown by simply ignoring everything on Z or z in the above argument; we only give an outline. Let $A = R[\![\boldsymbol{X}, \boldsymbol{Y}]\!]/(\boldsymbol{X}\boldsymbol{Y})$ with $\boldsymbol{X} = X_1, \dots, X_n$ and $\boldsymbol{Y} = Y_1, \dots, Y_m$. Then $A/(\boldsymbol{X}-\boldsymbol{x}, \boldsymbol{Y}-\boldsymbol{y}) \cong S$. As $n, m \geqslant 1$, the local ring $A/\mathfrak{m}A = k[\![\boldsymbol{X}, \boldsymbol{Y}]\!]/(\boldsymbol{X}\boldsymbol{Y})$ has decomposable maximal ideal, and hence A is dominant. Exact sequences $0 \to A \to A/(\boldsymbol{X}) \oplus A/(\boldsymbol{Y}) \to A/(\boldsymbol{X}, \boldsymbol{Y}) \to 0$ and $0 \to A/(\boldsymbol{X}-\boldsymbol{x}) \to A/(\boldsymbol{X}, \boldsymbol{x}) \oplus A/(\boldsymbol{Y}, \boldsymbol{X}-\boldsymbol{x}) \to A/(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{x}) \to 0$ are induced, and we see that $\boldsymbol{X}-\boldsymbol{x}, \boldsymbol{Y}-\boldsymbol{y}$ is A-regular. ■

Applying the above proposition, we can show the following decisive result on dominant local rings.

Corollary 8.9. *Let $(R, \mathfrak{m})$ be a regular local ring. Let I be a proper ideal of R such that $\mu(I) \leqslant 2$. Then the residue ring R/I is either a complete intersection of codimension two, or a dominant local ring.*

Proof. Proposition 5.10(1) shows that R/I is dominant if $\mu(I) \leqslant 1$. Let $\mu(I) = 2$ and write $I = (x, y)$. Note that R is factorial. Let $g = \gcd(x, y)$. We see that $x = gx', y = gy'$ for some $x', y' \in \mathfrak{m}$ with $\gcd(x', y') = 1$. We have $\operatorname{grade}(x', y') = \operatorname{ht}(x', y') = 2$, which means that x', y' is a regular sequence on R. If $g \notin \mathfrak{m}$, then the local ring R/I is a complete intersection of codimension 2. If $g \in \mathfrak{m}$, then we see by letting $m = 0$ in Proposition 8.8(2) that R/I is a dominant local ring. ■

We should compare the following example with Corollary 8.4. This example would also say that the Burch rings form only a small subclass of the dominant local rings. We should also remark that the local ring R is not Cohen–Macaulay.

Example 8.10. Let $R = k[\![x, y]\!]/x^a(x^b, y^c) = k[\![x, y]\!]/(x^{a+b}, x^a y^c)$ be a quotient of a formal power series ring over a field k, where $a, b, c > 0$ are integers. Then, by [22, Corollaries 2.7 and 6.5], the local ring R is Burch if and only if either $b = 1$ or $c = 1$ holds. The local ring R is always dominant by Corollary 8.9.

Remark 8.11. Let R be a regular local ring and I a proper ideal of R with $\mu(I) \leqslant 2$ as in Corollary 8.9. Then $\operatorname{edim} R/I \leqslant \dim R$ and $\dim R/I = \dim R - \operatorname{ht} I \geqslant \dim R - 2$, whence $\operatorname{codim} R/I \leqslant 2$. Therefore, if R/I is Cohen–Macaulay, then it is either a complete intersection or a Golod ring by [7, Proposition 5.3.4]. This may say that there exists some connection between dominance and Golodness.

9. Comparison of dominance with other properties of local rings

In this section, we seek for implications between dominance and several other properties of local rings. In fact, unfortunately, it turns out that for many of those properties we have no idea whether an implication exists to/from dominance, and we present questions together with some related information.

We start by considering dominance in relation to finite CM-representation type. In [62, Conjecture 4.2], it is conjectured that a Cohen–Macaulay singular local ring R of finite CM-representation type satisfies $\mathsf{core}\, R = \mathsf{CM}(R)$. By Proposition 6.2(2) and [35, Corollary 2], one can interpret it in terms of dominance as follows. Proposition 6.6(2) supports this conjecture, and so does Proposition 10.15(4) appearing later.

Conjecture 9.1. A Cohen–Macaulay local ring of finite CM-representation type is dominant.

Next, we pose the following question about the relationship of dominance with Tor/Ext-friendliness. It looks a bit bold but seems to be natural as well. Proposition 6.6(1) gives a partial affirmative answer to this question, but we do not have a complete answer. It would be very interesting if we could find out relatively general cases where the question is affirmative.

Question 9.2. Let R be a local ring. Suppose that R is Tor-friendly or Ext-friendly. Is then R dominant?

Now we deal with the Golod property of local rings. For this, we recall the following two facts.

Remark 9.3. (1) A local ring of codimension at most one is Golod by [7, Proposition 5.2.5].
(2) A local ring is a hypersurface if and only if it is Gorenstein and Golod. This fact is stated in [7, Page 49, Remark]; see also [10, Examples 3.5(2)].

We know by Proposition 5.10(1) that a local hypersurface is dominant. Combining this with Remarks 9.3 and 8.11 naturally leads us to the following question.

Question 9.4. Is a Golod local ring necessarily dominant? In particular, is every local ring of codimension at most one dominant?

The following example supports Question 9.4 in the affirmative.

Example 9.5. Let $R = k[\![x,y]\!]/(x^{a+b}, x^a y^c)$ be a quotient of a formal power series ring over a field k a field, where $a \geqslant 1$ and $b, c \geqslant 2$. Then $\dim R = \operatorname{codim} R = 1$. Remark 9.3(1) shows that R is Golod. As $\operatorname{depth} R = 0$, the ring R is not Cohen–Macaulay. Example 8.10 says that R is not Burch but dominant.

By Propositions 6.2(3), a local ring is a hypersurface if and only if it is a dominant complete intersection. It may be natural to ask whether the "if" part can be strengthened as follows.

Question 9.6. Is a Gorenstein dominant local ring a hypersurface?

However, the following statement may lead us to think that the above question has a negative answer.

Remark 9.7. If Question 9.6 has an affirmative answer, then, for example, the artinian Gorenstein local ring $R = k[x,y,z]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$ with k a field turns out to be non-dominant, and hence there exists a resolving subcategory $\mathcal{X}$ of $\operatorname{\mathsf{mod}} R$ such that $\operatorname{\mathsf{add}} R \neq \mathcal{X} \neq \operatorname{\mathsf{mod}} R$; see Corollary 5.4.

In the following two remarks, we compare dominance with other two properties of local rings.

Remark 9.8. Recall from Corollary 6.11(1) that a non-Gorenstein dominant local ring is G-regular. We may thus wonder if for a local ring R the implication

$$R \text{ is not Gorenstein but G-regular} \implies R \text{ is dominant}$$

holds, but it does not always. More strongly (see Theorem 6.7), the combination of non-Gorensteinness and G-regularity does not imply Tor-friendliness. Let $R = k[\![x,y,z,w]\!]/(x^2, xy, y^2, z^2, zw, w^2)$ with k a field. Then R is not Gorenstein. Since $\operatorname{Tor}^R_{>0}(R/(x,y), R/(z,w)) = 0$, the local ring R is not Tor-friendly. Suppose that R is not G-regular. Then, since the maximal ideal $\mathfrak{m}$ of R is such that $\mathfrak{m}^3 = 0$, it follows from [65, Theorem 3.1] that the Hilbert series $\mathrm{H}_R(t)$ of R is $1 + (r+1)t + rt^2$, where $r = \mathrm{r}(R)$. However, we have $\mathrm{H}_R(t) = 1 + 4t + 4t^2$, which is a contradiction. We conclude that R is a G-regular local ring.

Remark 9.9. An artinian Gorenstein local ring $(R, \mathfrak{m})$ is called *stretched* if $\mathfrak{m}^{\ell(R) - \operatorname{edim} R} \neq 0$; we refer to [51] for details. For an artinian Gorenstein local ring R, the implication below does not always hold.

$$R \text{ is stretched} \implies R \text{ is dominant}$$

Indeed, consider the artinian complete intersection local ring $R = k[x,y]/(x^2, y^2)$ with k a field. Then $\ell(R) = 4$, $\operatorname{edim} R = 2$ and $\mathfrak{m}^2 \neq 0$, so R is stretched. However, R is not dominant by Proposition 6.2(3).

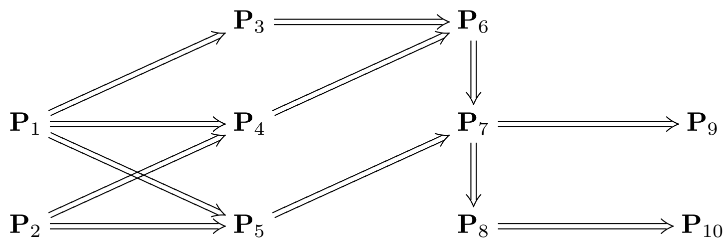
Incidentally, the converse of the above implication holds unless R is a field, provided that Question 9.6 has an affirmative answer. This is because an artinian local hypersurface that is not a field is stretched.

The theorem below discusses various properties of local rings including dominance, and makes a table showing whether the implication from one of them to another holds. This theorem also plays the role of a summary of what we have got so far; some of the statements of the theorem have already appeared.

Theorem 9.10. *Let $(R, \mathfrak{m}, k)$ be a local ring. The table below describes the relationships among those ten properties listed left. Here, the symbol "$\bigcirc$" (resp. "$\times$") in the (i,j) entry means that the implication $\mathbf{P}_i \Rightarrow \mathbf{P}_j$ always holds (resp. does not always hold). The symbol "?" means that we do not know if the corresponding implication always holds or not. The bottom diagram of implications follows by the table.*

$\mathbf{P}_1 =$ *R is Cohen–Macaulay, non-hypersurface, with minimal multiplicity and $|k| = \infty$,*
$\mathbf{P}_2 =$ *R is a hypersurface,*
$\mathbf{P}_3 =$ *$\mathfrak{m}$ is quasi-decomposable,*
$\mathbf{P}_4 =$ *R is Burch,*
$\mathbf{P}_5 =$ *R is Golod,*
$\mathbf{P}_6 =$ *R is dominant,*
$\mathbf{P}_7 =$ *R is Tor-friendly,*
$\mathbf{P}_8 =$ *R is Ext-friendly,*
$\mathbf{P}_9 =$ *R is Tor-persistent,*
$\mathbf{P}_{10} =$ *R is Ext-persistent.*

	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_7$	$\mathbf{P}_8$	$\mathbf{P}_9$	$\mathbf{P}_{10}$
$\mathbf{P}_1$	$\bigcirc$	$\times$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$\mathbf{P}_2$	$\times$	$\bigcirc$	$\times$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$\mathbf{P}_3$	$\times$	$\times$	$\bigcirc$	$\times$	$\times$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$\mathbf{P}_4$	$\times$	$\times$	$\times$	$\bigcirc$	$\times$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$\mathbf{P}_5$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	?	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$\mathbf{P}_6$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$\mathbf{P}_7$	$\times$	$\times$	$\times$	$\times$	$\times$	?	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$
$\mathbf{P}_8$	$\times$	$\times$	$\times$	$\times$	$\times$	?	?	$\bigcirc$	?	$\bigcirc$
$\mathbf{P}_9$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	$\times$
$\mathbf{P}_{10}$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	?	$\bigcirc$



Proof. The implication $\mathbf{P}_1 \Rightarrow \mathbf{P}_3$ holds by [46, Example 4.7], $\mathbf{P}_1 \Rightarrow \mathbf{P}_4 \Leftarrow \mathbf{P}_2$ by [22, Propositions 5.1 and 5.2], $\mathbf{P}_1 \Rightarrow \mathbf{P}_5 \Leftarrow \mathbf{P}_2$ by [7, Example 5.2.8] and Remark 9.3(2), $\mathbf{P}_3 \Rightarrow \mathbf{P}_6 \Leftarrow \mathbf{P}_4$ by Proposition 5.10(3)(4), $\mathbf{P}_5 \Rightarrow \mathbf{P}_7 \Rightarrow \mathbf{P}_8 \Rightarrow \mathbf{P}_{10}$ and $\mathbf{P}_7 \Rightarrow \mathbf{P}_9$ by Remark 6.5, and $\mathbf{P}_6 \Rightarrow \mathbf{P}_7$ by Theorem 6.7, respectively.

We have done with the symbols "◯" and the symbols "×" remain. In view of the implications we have got, it suffices to prove "×" in the entries (1,2), (2,3), (3,4), (3,5), (4,5), (5,4), (9,10) and (10,8). In what follows, we let k be a field, and denote by $\mathfrak{m}$ the maximal ideal of the local ring R.

(1,2): Assume $|k| = \infty$, and let $R = k[x,y]/(x^2, xy, y^2)$. Then R does not satisfy $\mathbf{P}_2$ but satisfies $\mathbf{P}_1$.

(2,3): Consider the local ring $R = k[\![x,y]\!]/(x^3)$. Evidently, R satisfies $\mathbf{P}_2$. Suppose that R satisfies $\mathbf{P}_3$. Then there exists an ideal I generated by a regular sequence such that the module $\mathfrak{m}/I$ is decomposable. As R/I is Gorenstein, it is a 1-dimensional hypersurface and isomorphic to the fiber product $S \times_k T$ of discrete valuation rings S, T with residue field k; see [45, Corollary 2.7]. We must have $I = 0$, and get $3 = \mathrm{e}(R) = \mathrm{e}(S) + \mathrm{e}(T) = 2$ by [45, Fact 2.9], a contradiction. Hence the ring R does not satisfy $\mathbf{P}_3$.

(3,4) and (3,5): Let $R = k[x,y]/(x^2,y^2) \times_k k[z,w]/(z^2,w^2) = k[x,y,z,w]/(x^2,y^2,z^2,w^2,xz,xw,yz,yw)$. Then $\mathfrak{m} = (x,y) \oplus (z,w)$, and $\mathbf{P}_3$ holds for R. The ring $k[x,y]/(x^2,y^2) \cong k[z,w]/(z^2,w^2)$ is neither Burch nor Golod by [22, Proposition 5.1] and Remark 9.3(2). By [22, Proposition 6.15] and [40, Théorème 4.1] the ring R is neither Burch nor Golod. Thus neither $\mathbf{P}_4$ nor $\mathbf{P}_5$ holds for R.

(4,5): Not $\mathbf{P}_5$ but $\mathbf{P}_4$ holds for $R = k[x,y,z,w]/(x,y,z,w)(x^2,y^2,z^2,w^2)$ by [22, Remark 5.3].

(5,4): Let $R = k[x,y]/(x^4, x^2y^2, y^4)$. Then R does not satisfy $\mathbf{P}_4$ but satisfies $\mathbf{P}_5$ by [22, Remark 5.3].

(9,10): Let $A = k[x,y]/(x^2,xy,y^2)$ and $R = A[z,w]/(z^2,zw,w^2) = k[x,y,z,w]/(x^2,xy,y^2,z^2,zw,w^2)$. Then $\mathfrak{m}^3 = 0$, and R satisfies $\mathbf{P}_9$ by [43, Theorem A]. As R is A-free, $C = \mathrm{Hom}_A(R,A)$ is a semidualizing R-module with $\mu_R(C) = \mathrm{r}_R(C) = 2$, whence it is not isomorphic to R or the injective hull $\mathrm{E}_R(k)$ of k; see [20, (7.8)]. Arguments as in the proof of Corollary 6.11(2) show that the ring R does not satisfy $\mathbf{P}_{10}$.

(10,8): Consider the artinian local ring $R = k[x,y]/(x^2,y^2)$. Then R is a complete intersection, whence it satisfies $\mathbf{P}_{10}$ by Remark 6.5. Since $\mathrm{Ext}_R^{\gg 0}(R/(x), R/(y)) = 0$, the local ring R does not satisfy $\mathbf{P}_8$. ■

Regrettably, the table displayed in the above theorem contains many question marks. In particular, those three "?" that appear in the sixth column of the table concern dominance; it would be nice if one could change some of them into "◯" or "×"; as to the one in (5,6), some help may be given by Remark 8.11. Below are comments on the other question marks.

Remark 9.11. Each of the symbols "?" in Theorem 9.10 may become "◯" if we add some assumptions. For instance, under the assumption that R is Cohen–Macaulay and admits a canonical module, the symbol "?" in the entry $(8,7)$, and hence the one in $(8,9)$, become "◯" by [42, Theorem 3.2(2)]. On the other hand, as we mentioned in Remark 6.5, no example is known of a local ring that is not Tor-persistent. Taking this into account, the symbols "?" in the entries $(8,9)$ and $(10,9)$ may be close to "◯".

10. Classification of subcategories and dominant local rings

In this section, we consider classifying resolving subcategories of $\mathsf{mod}\,R$, and thick subcategories of $\mathsf{mod}\,R$, $\mathsf{D}^{\mathsf{b}}(R)$ and $\mathsf{D}^{\mathsf{sg}}(R)$ when certain localizations of R are dominant local rings. We first recall some notions and their basic properties, which are necessary to state and prove the main results of this section.

Definition 10.1. (1) A subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$ (resp. $\mathsf{C}(R)$) is called *thick* provided that $\mathcal{X}$ is closed under direct summands and that for an exact sequence $0 \to L \to M \to N \to 0$ of modules in $\mathsf{mod}\,R$ (resp. $\mathsf{C}(R)$), if two of L, M, N are in $\mathcal{X}$, then so is the third. For each subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$ (resp. $\mathsf{C}(R)$), we denote by $\mathsf{thick}_{\mathsf{mod}\,R}\,\mathcal{X}$ (resp. $\mathsf{thick}_{\mathsf{C}(R)}\,\mathcal{X}$) the *thick closure* of $\mathcal{X}$ in $\mathsf{mod}\,R$ (resp. $\mathsf{C}(R)$), that is to say, the smallest thick subcategory of $\mathsf{mod}\,R$ (resp. $\mathsf{C}(R)$) which contains $\mathcal{X}$.

(2) For each object $C \in \mathsf{D}^{\mathsf{b}}(R)$ we denote by $\mathrm{IPD}(C)$ the *infinite projective dimension locus* of C, that is, the set of prime ideals $\mathfrak{p}$ of R such that the localization $C_{\mathfrak{p}}$ has infinite projective dimension as a

complex over $R_\mathfrak{p}$; see [19, (A.3.9)] for the definition of the projective dimension of a complex, and note that a bounded complex of R-modules has finite projective dimension if and only if it is isomorphic to a perfect complex in $\mathsf{D^b}(R)$. For a subcategory $\mathcal{X}$ of $\mathsf{D^b}(R)$ we put $\mathrm{IPD}(\mathcal{X}) = \bigcup_{X\in\mathcal{X}} \mathrm{IPD}(X)$. For each object $D \in \mathsf{D^{sg}}(R)$ we denote by $\mathrm{Supp}^{\mathrm{sg}} D$ the *singular support* of D, i.e., the set of prime ideals $\mathfrak{p}$ of R such that $D_\mathfrak{p} \not\cong 0$ in $\mathsf{D^{sg}}(R_\mathfrak{p})$. For a subcategory $\mathcal{Y}$ of $\mathsf{D^{sg}}(R)$ we put $\mathrm{Supp}^{\mathrm{sg}} \mathcal{Y} = \bigcup_{Y\in\mathcal{Y}} \mathrm{Supp}^{\mathrm{sg}} Y$. For each bounded complex E of R-modules, the infinite projective dimension locus of E as an object of $\mathsf{D^b}(R)$ is the same as the singular support of E as an object of $\mathsf{D^{sg}}(R)$; see Remark 10.2(9) below.

(3) Let Φ be a set of prime ideals of R. We denote by $\mathsf{mod}_\Phi R$ and $\mathsf{D}^\mathsf{b}_\Phi(R)$ the subcategories of $\mathsf{mod}\, R$ and $\mathsf{D^b}(R)$ consisting of objects C such that $\mathrm{IPD}(C) \subseteq \Phi$, respectively. We denote by $\mathsf{D}^\mathsf{sg}_\Phi(R)$ the subcategories of $\mathsf{D^{sg}}(R)$ consisting of objects C such that $\mathrm{Supp}^{\mathrm{sg}} C \subseteq \Phi$.

Remark 10.2. (1) A thick subcategory of $\mathsf{mod}\, R$ containing R is always a resolving subcategory of $\mathsf{mod}\, R$. The converse is not true in general. For example, $\mathsf{add}\, R$ is resolving, but not thick if $(R, \mathfrak{m})$ is local and $\operatorname{depth} R > 0$ as there exists $x \in \mathfrak{m}$ such that the sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ is exact and $R/(x) \notin \mathsf{add}\, R$. A thick subcategory of $\mathsf{C}(R)$ containing R is always a resolving subcategory of $\mathsf{mod}\, R$ contained in $\mathsf{C}(R)$, since $\mathsf{C}(R)$ is a resolving subcategory of $\mathsf{mod}\, R$. Again, the converse is not true in general; see [23, Proposition 6.1 and its preceding part]. Note that $\mathsf{thick}_{\mathsf{mod}\, R} R$ coincides with $\mathsf{fpd}\, R$, the subcategory of $\mathsf{mod}\, R$ consisting of modules of finite projective dimension.

(2) Let Φ be a set of prime ideals of R. Then $\mathsf{mod}_\Phi R$ and $\mathsf{D}^\mathsf{b}_\Phi(R)$ are thick subcategories of $\mathsf{mod}\, R$ and $\mathsf{D^b}(R)$ containing R, respectively. The subcategory $\mathsf{D}^\mathsf{sg}_\Phi(R)$ of $\mathsf{D^{sg}}(R)$ is thick as well.

(3) For an R-module M one has $\mathrm{IPD}(M) \subseteq \mathrm{NF}(M)$, and the equality holds if $M \in \mathsf{C}(R)$. In particular, there is an equality $\mathsf{C}_\Phi(R) = \mathsf{C}(R) \cap \mathsf{mod}_\Phi R$ for each set Φ of prime ideals of R.

(4) By (3) and Remark 3.3(5) one has $\mathrm{IPD}(M) = \mathrm{NF}(\Omega^r M)$ for an R-module $M \neq 0$, where $r = \mathrm{Rfd}\, M$.

(5) For a bounded complex X of R-modules the subset $\mathrm{IPD}(X)$ of $\operatorname{Spec} R$ is closed in the Zariski topology. Indeed, there is an exact triangle $P \to X \to M[n] \to P[1]$ in $\mathsf{D^b}(R)$ such that $P \in \mathsf{thick}\, R$, $M \in \mathsf{mod}\, R$ and $n \in \mathbb{Z}$. This implies $\mathrm{IPD}(X) = \mathrm{IPD}(M)$. By (4), we see that $\mathrm{IPD}(M)$ is a closed subset of $\operatorname{Spec} R$.

(6) By (5), the subset $\mathrm{IPD}(\mathcal{X})$ of $\operatorname{Spec} R$ is specialization-closed for each subcategory $\mathcal{X}$ of $\mathsf{D^b}(R)$.

(7) For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ one has $\mathrm{IPD}(\mathcal{X}) \supseteq \mathrm{NF}(\mathcal{X} \cap \mathsf{C}(R))$, and the equality holds if $\mathcal{X}$ is closed under syzygies. In fact, for every module X in $\mathcal{X} \cap \mathsf{C}(R)$ one has $\mathrm{NF}(X) = \mathrm{IPD}(X) \subseteq \mathrm{IPD}(\mathcal{X})$ by (3). For any nonzero module $Y \in \mathcal{X}$ with $r = \mathrm{Rfd}\, Y$, it follows from (4) that $\mathrm{IPD}(Y) = \mathrm{NF}(\Omega^r Y)$, and Remark 3.3(5) shows that $\Omega^r Y \in \mathcal{X} \cap \mathsf{C}(R)$ if $\mathcal{X}$ is closed under syzygies.

(8) For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ one has $\mathrm{IPD}(\mathsf{thick}_{\mathsf{mod}\, R}\, \mathcal{X}) = \mathrm{IPD}(\mathcal{X})$. Indeed, the inclusion ($\supseteq$) holds since $\mathsf{thick}_{\mathsf{mod}\, R}\, \mathcal{X}$ contains $\mathcal{X}$. The inclusion ($\subseteq$) is a consequence of the fact that for $\Phi = \mathrm{IPD}(\mathcal{X})$ the subcategory $\mathsf{mod}_\Phi R$ of $\mathsf{mod}\, R$ is thick and contains $\mathcal{X}$, which follows from (2). In a similar way, one also observes that the equality $\mathrm{IPD}(\mathsf{thick}_{\mathsf{D^b}(R)}\, \mathcal{Y}) = \mathrm{IPD}(\mathcal{Y})$ holds for a subcategory $\mathcal{Y}$ of $\mathsf{D^b}(R)$.

(9) Let $\pi : \mathsf{D^b}(R) \to \mathsf{D^{sg}}(R)$ be the canonical functor. Then $\mathrm{Supp}^{\mathrm{sg}}(\pi(C)) = \mathrm{IPD}(C)$ for each $C \in \mathsf{D^b}(R)$, and $\mathsf{D}^\mathsf{sg}_\Phi(R) = \pi(\mathsf{D}^\mathsf{b}_\Phi(R))$. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathsf{D^b}(R)$ and $\mathsf{D^{sg}}(R)$, respectively. Then $\mathrm{IPD}(\mathcal{X}) = \mathrm{Supp}^{\mathrm{sg}}(\pi(\mathcal{X}))$ and $\mathrm{IPD}(\pi^{-1}(\mathcal{Y})) = \mathrm{Supp}^{\mathrm{sg}}(\mathcal{Y})$. If $\mathcal{X}$ contains R, then the equality $\pi(\mathsf{thick}_{\mathsf{D^b}(R)}\, \mathcal{X}) = \mathsf{thick}_{\mathsf{D^{sg}}(R)}\, \pi(\mathcal{X})$ holds. (Hence, if $\mathcal{X}$ is a thick subcategory of $\mathsf{D^b}(R)$ containing R, then $\pi(\mathcal{X})$ is a thick subcategory of $\mathsf{D^{sg}}(R)$.) Indeed, ($\subseteq$) follows from Lemma 4.2(1). For any $M, N \in \mathsf{D^b}(R)$ one has $\pi(M) \cong \pi(N)$ in $\mathsf{D^{sg}}(R)$ if and only if there are exact triangles $E \to M \to A \to E[1]$ and $E \to N \to B \to E[1]$ in $\mathsf{D^b}(R)$ with $A, B \in \mathsf{thick}\, R$; see [48, Proposition 2.1.35]. Using this fact, we see that $\pi(\mathsf{thick}_{\mathsf{D^b}(R)}\, \mathcal{X})$ is a thick subcategory of $\mathsf{D^{sg}}(R)$. The inclusion ($\supseteq$) now follows.

(10) Let Φ be a subset of $\operatorname{Spec} R$. The categories $\mathsf{C}_\Phi(R)$, $\mathsf{mod}_\Phi R$, $\mathsf{D}^\mathsf{b}_\Phi(R)$ and $\mathsf{D}^\mathsf{sg}_\Phi(R)$ depend only on $\Phi \cap \operatorname{Sing} R$. Therefore, whenever we are concerned with some of these categories, Φ can be taken as a subset of $\operatorname{Sing} R$ rather than being just an arbitrary subset of $\operatorname{Spec} R$.

Using an infinite projective dimension locus, we can relate dominance of localizations of R to dominance of resolving subcategories of $\mathsf{mod}\, R$, whose definition is given in Definition 3.10. Actually, the name "dominant local ring" comes from this proposition.

Proposition 10.3. *Let Φ be a set of prime ideals of R. Then the local ring $R_\mathfrak{p}$ is dominant for all $\mathfrak{p} \in \Phi$ if and only if every resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ is dominant on $\Phi \cap \mathrm{IPD}(\mathcal{X})$.*

Proof. The "only if" part: Take any $\mathfrak{p} \in \Phi \cap \mathrm{IPD}(\mathcal{X})$. Then $\mathrm{pd}_{R_\mathfrak{p}} X_\mathfrak{p} = \infty$ for some $X \in \mathcal{X}$, and $R_\mathfrak{p}$ is a dominant local ring by assumption. Hence $\Omega^{\operatorname{depth} R_\mathfrak{p}} \kappa(\mathfrak{p})$ is in $\mathsf{res}\, X_\mathfrak{p}$, and hence it belongs to $\mathsf{add}\, \mathcal{X}_\mathfrak{p}$.

The "if" part: Fix $\mathfrak{p} \in \Phi$. Let M be an $R_\mathfrak{p}$-module of infinite projective dimension. We can choose an R-module N such that $M = N_\mathfrak{p}$, and then $\mathfrak{p} \in \operatorname{IPD}(N)$. Let $\mathcal{X}$ be the resolving closure of N. We see that $\mathfrak{p}$ belongs to $\Phi \cap \operatorname{IPD}(\mathcal{X})$, and by assumption we find an integer $n \geqslant 0$ such that $\Omega^n\kappa(\mathfrak{p}) \in \operatorname{\mathsf{add}} \mathcal{X}_\mathfrak{p}$. Applying [63, Corollary 3.3(1)], we obtain $\Omega^{\operatorname{depth} R_\mathfrak{p}}\kappa(\mathfrak{p}) \in \operatorname{\mathsf{add}} \mathcal{X}_\mathfrak{p} \subseteq \operatorname{\mathsf{res}} M$. ■

Next we recall some notation, which is also necessary to state and prove our main results.

Definition 10.4. We put $\operatorname{NF}^{-1}_{\mathsf{C}}(W) = \operatorname{NF}^{-1}(W) \cap \mathsf{C}(R)$ and $\operatorname{IPD}^{-1}(W) = \operatorname{\mathsf{mod}}_W R$ for each $W \subseteq \operatorname{Spec} R$. For subcategories $\mathcal{V} \subseteq \mathsf{C}(R)$, $\mathcal{Z} \subseteq \operatorname{\mathsf{mod}} R$, $\mathcal{X} \subseteq \mathsf{D^b}(R)$ and $\mathcal{Y} \subseteq \mathsf{D^{sg}}(R)$ we set

$$(\mathsf{thick}_{\mathsf{mod}})(\mathcal{V}) = \mathsf{thick}_{\mathsf{mod}\,R}\,\mathcal{V}, \quad (\mathsf{thick}_{\mathsf{D^b}})(\mathcal{Z}) = \mathsf{thick}_{\mathsf{D^b}(R)}\,\mathcal{Z}, \quad (\mathsf{thick}_{\mathsf{D^{sg}}})(\mathcal{V}) = \mathsf{thick}_{\mathsf{D^{sg}}(R)}\,\pi(\mathcal{V}),$$
$$(\mathsf{rest}_{\mathsf{mod}})(\mathcal{X}) = \mathcal{X} \cap \operatorname{\mathsf{mod}} R, \quad (\mathsf{rest}_{\mathsf{C}})(\mathcal{Z}) = \mathcal{Z} \cap \mathsf{C}(R), \quad (\mathsf{rest}_{\mathsf{C}})(\mathcal{Y}) = \pi^{-1}(\mathcal{Y}) \cap \mathsf{C}(R),$$

where $\pi : \mathsf{D^b}(R) \to \mathsf{D^{sg}}(R)$ stands for the canonical functor.

We establish a lemma on bijective correspondences among thick subcategories of $\operatorname{\mathsf{mod}} R$, $\mathsf{D^b}(R)$ and $\mathsf{D^{sg}}(R)$. This is a consequence of a general theorem about exact categories given in [38].

Lemma 10.5. *Let $\pi : \mathsf{D^b}(R) \to \mathsf{D^{sg}}(R)$ be the canonical functor. There are mutually inverse bijections:*

$$\left\{\begin{matrix}\text{Thick subcategories}\\ \text{of } \mathsf{mod}\,R \text{ containing } R\end{matrix}\right\} \underset{\mathsf{rest}_{\mathsf{mod}}}{\overset{\mathsf{thick}_{\mathsf{D^b}}}{\rightleftarrows}} \left\{\begin{matrix}\text{Thick subcategories}\\ \text{of } \mathsf{D^b}(R) \text{ containing } R\end{matrix}\right\} \underset{\pi^{-1}}{\overset{\pi}{\rightleftarrows}} \left\{\begin{matrix}\text{Thick subcategories}\\ \text{of } \mathsf{D^{sg}}(R)\end{matrix}\right\}.$$

Proof. Fix a thick subcategory $\mathcal{X}$ of $\mathsf{D^b}(R)$ containing R, a thick subcategory $\mathcal{Y}$ of $\mathsf{D^{sg}}(R)$, and a thick subcategory $\mathcal{Z}$ of $\operatorname{\mathsf{mod}} R$ containing R.

Remark 10.2(9) and Lemma 4.2(2) show that the maps π, π^{-1} are well-defined. It is seen by the fact given around the end of Remark 10.2(9) that $\pi^{-1}\pi(\mathcal{X})$ is contained in $\mathcal{X}$. We easily deduce the equalities $\pi^{-1}\pi(\mathcal{X}) = \mathcal{X}$ and $\pi\pi^{-1}(\mathcal{Y}) = \mathcal{Y}$. Thus we obtain the pair (π, π^{-1}) of mutually inverse bijections.

The latter half of the proof of [38, Theorem 1] shows $(\mathsf{thick}_{\mathsf{D^b}} \cdot \mathsf{rest}_{\mathsf{mod}})(\mathcal{X}) = \mathsf{thick}_{\mathsf{D^b}(R)}(\mathcal{X} \cap \operatorname{\mathsf{mod}} R) = \mathcal{X}$. We see from the former half of the proof of [38, Theorem 1] that $\mathsf{thick}_{\mathsf{D^b}(R)}\,\mathcal{Z}$ coincides with the subcategory of $\mathsf{D^b}(R)$ consisting of bounded complexes of modules in $\mathcal{Z}$, and that $(\mathsf{rest}_{\mathsf{mod}} \cdot \mathsf{thick}_{\mathsf{D^b}})(\mathcal{Z}) = (\mathsf{thick}_{\mathsf{D^b}(R)}\,\mathcal{Z}) \cap \operatorname{\mathsf{mod}} R = \mathcal{Z}$. Thus we get the pair $(\mathsf{thick}_{\mathsf{D^b}}, \mathsf{rest}_{\mathsf{mod}})$ of mutually inverse bijections. ■

Applying the above lemma together with a result stated in [63], we obtain one more lemma and a proposition regarding the residue field and resolving/thick closures.

Lemma 10.6. *Let $(R, \mathfrak{m}, k)$ be a local ring of depth t. The following are equivalent for each $C \in \mathsf{C}(R)$.*
(1) *The module $\Omega^t k$ belongs to $\operatorname{\mathsf{res}} C$.*
(2) *The module $\Omega^t k$ belongs to $\mathsf{thick}_{\mathsf{C}(R)}\{R, C\}$.*
(3) *The module k belongs to $\mathsf{thick}_{\mathsf{mod}\,R}\{R, C\}$.*
(4) *The module k belongs to $\mathsf{thick}_{\mathsf{D^b}(R)}\{R, C\}$.*
(5) *The module k belongs to $\mathsf{thick}_{\mathsf{D^{sg}}(R)}\,C$.*

Proof. The inclusions $\operatorname{\mathsf{res}} C \subseteq \mathsf{thick}_{\mathsf{C}(R)}\{R, C\} \subseteq \mathsf{thick}_{\mathsf{mod}\,R}\{R, C\} \subseteq \mathsf{thick}_{\mathsf{D^b}(R)}\{R, C\}$ hold. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) follow from them. Using the canonical functor $\mathsf{D^b}(R) \to \mathsf{D^{sg}}(R)$, we get the implication (4) ⇒ (5). Applying Lemma 10.5, we obtain the equalities below, which show (5) ⇒ (3).

$$\begin{aligned}\mathsf{thick}_{\mathsf{mod}\,R}\{R, C\} &= (\mathsf{rest}_{\mathsf{mod}} \cdot \mathsf{thick}_{\mathsf{D^b}})(\mathsf{thick}_{\mathsf{mod}\,R}\{R, C\}) = \mathsf{thick}_{\mathsf{D^b}(R)}\{R, C\} \cap \operatorname{\mathsf{mod}} R\\ &= \pi^{-1}\pi(\mathsf{thick}_{\mathsf{D^b}(R)}\{R, C\}) \cap \operatorname{\mathsf{mod}} R = \pi^{-1}(\mathsf{thick}_{\mathsf{D^{sg}}(R)}\,C) \cap \operatorname{\mathsf{mod}} R.\end{aligned}$$

Let $\mathcal{X}$ be the subcategory of $\operatorname{\mathsf{mod}} R$ consisting of modules X such that $\Omega^n X \in \operatorname{\mathsf{res}} C$ for some $n \geqslant 0$. Then $\mathcal{X}$ is a thick subcategory of $\operatorname{\mathsf{mod}} R$ containing R and C, whence $\mathcal{X}$ contains $\mathsf{thick}_{\mathsf{mod}\,R}\{R, C\}$. In fact, for example, let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules with $\Omega^n L$ and $\Omega^n M$ being in $\operatorname{\mathsf{res}} C$ for some $n \geqslant 0$. Then there are exact sequences $0 \to \Omega^n L \to \Omega^n M \oplus R^{\oplus a} \xrightarrow{\alpha} \Omega^n N \to 0$ and $0 \to \Omega^{n+1} N \to R^{\oplus b} \xrightarrow{\beta} \Omega^n N \to 0$ with $a, b \geqslant 0$. The pullback diagram of α, β yields an exact sequence $0 \to \Omega^{n+1} N \to \Omega^n L \oplus R^{\oplus b} \to \Omega^n M \oplus R^{\oplus a} \to 0$, which shows $\Omega^{n+1} N \in \operatorname{\mathsf{res}} C$. If k is in $\mathsf{thick}_{\mathsf{mod}\,R}\{R, C\}$, then $\Omega^h k \in \operatorname{\mathsf{res}} C$ for some $h \geqslant 0$, and $\Omega^t k \in \operatorname{\mathsf{res}} C$ by [63, Corollary 3.3(1)]. Thus we get (3) ⇒ (1). ■

Proposition 10.7. *Let $(R, \mathfrak{m}, k)$ be a local ring of depth t. Let $E \in \mathsf{C}(R)$. The following are equivalent.*
(1) *For every $C \in \mathsf{C}(R)$ that is a nonfree R-module, one has $\Omega^t k \in \operatorname{\mathsf{res}}\{E, C\}$.*
(2) *For every $C \in \mathsf{C}(R)$ that is a nonfree R-module, one has $\Omega^t k \in \mathsf{thick}_{\mathsf{C}(R)}\{E, R, C\}$.*

(3) *For every* $M \in \mathsf{mod}\, R$ *with* $\mathrm{pd}_R M = \infty$*, one has* $\Omega^t k \in \mathsf{res}\{E, M\}$*.*
(4) *For every* $M \in \mathsf{mod}\, R$ *with* $\mathrm{pd}_R M = \infty$*, one has* $k \in \mathsf{thick}_{\mathsf{mod}\, R}\{E, R, M\}$*.*
(5) *For every* $X \in \mathsf{D}^{\mathsf{b}}(R)$ *with* $\mathrm{pd}_R X = \infty$*, one has* $k \in \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{E, R, X\}$*.*
(6) *For every* $Y \in \mathsf{D}^{\mathsf{sg}}(R)$ *that is a nonzero object, one has* $k \in \mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}(R)}\{E, Y\}$*.*

Proof. (1) ⇒ (3): Put $r = \mathrm{Rfd}_R M$. Then the syzygy $\Omega^r M$ is a nonfree R-module and belongs to $\mathsf{C}(R)$; see Remark 3.3(5). It follows that $\Omega^t k \in \mathsf{res}\{E, \Omega^r M\} \subseteq \mathsf{res}\{E, M\}$.

(3) ⇒ (4): We have $\Omega^t k \in \mathsf{res}\{E, M\} \subseteq \mathsf{thick}_{\mathsf{mod}\, R}\{E, R, M\}$, and hence $k \in \mathsf{thick}_{\mathsf{mod}\, R}\{E, R, M\}$.

(4) ⇒ (5): There is an exact triangle $P \to X \to M[n] \to P[1]$ with $P \in \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)} R$, $M \in \mathsf{mod}\, R$ and $n \in \mathbb{Z}$. Then $\mathrm{pd}_R M = \infty$, and $k \in \mathsf{thick}_{\mathsf{mod}\, R}\{E, R, M\} \subseteq \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{E, R, M\} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{E, R, X\}$.

(5) ⇒ (6): As an object of $\mathsf{D}^{\mathsf{b}}(R)$ the complex Y has infinite projective dimension. Therefore we have $k \in \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{E, R, Y\}$. Using the canonical functor $\mathsf{D}^{\mathsf{b}}(R) \to \mathsf{D}^{\mathsf{sg}}(R)$, we get $k \in \mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}(R)}\{E, Y\}$.

(6) ⇒ (1): As an object of $\mathsf{D}^{\mathsf{sg}}(R)$ the module C is nonzero. It follows that $k \in \mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}(R)}\{E, C\} = \mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}(R)}(E \oplus C)$. By virtue of Lemma 10.6 we get $\Omega^t k \in \mathsf{res}(E \oplus C) = \mathsf{res}\{E, C\}$.

(1) ⇔ (2): Applying Lemma 10.6 to $E \oplus C$, we observe that the equivalence holds. ■

The following corollary is a direct consequence of the above proposition; in fact, letting E be 0 or R in the proposition immediately yields the corollary.

Corollary 10.8. *Let* R *be a local ring of depth* t *and with residue field* k*. The following are equivalent.*

(1) *For every* $C \in \mathsf{C}(R)$ *that is a nonfree* R*-module, one has* $\Omega^t k \in \mathsf{res}\, C$*.*
(2) *For every* $C \in \mathsf{C}(R)$ *that is a nonfree* R*-module, one has* $\Omega^t k \in \mathsf{thick}_{\mathsf{C}(R)}\{R, C\}$*.*
(3) *For every* $M \in \mathsf{mod}\, R$ *with* $\mathrm{pd}_R M = \infty$*, one has* $\Omega^t k \in \mathsf{res}\, M$*. (Namely,* R *is dominant.)*
(4) *For every* $M \in \mathsf{mod}\, R$ *with* $\mathrm{pd}_R M = \infty$*, one has* $k \in \mathsf{thick}_{\mathsf{mod}\, R}\{R, M\}$*.*
(5) *For every* $X \in \mathsf{D}^{\mathsf{b}}(R)$ *with* $\mathrm{pd}_R X = \infty$*, one has* $k \in \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{R, X\}$*.*
(6) *For every* $Y \in \mathsf{D}^{\mathsf{sg}}(R)$ *that is a nonzero object, one has* $k \in \mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}(R)} Y$*.*

Remark 10.9. Thanks to Corollary 10.8, we may say that the dominant local ring R is the local ring R satisfying one of the six equivalent conditions presented in the corollary.

Now is the time when we state and prove the main result of this section. For the fact that Φ is taken as a subset of $\mathrm{Sing}\, R$, we should recall Remark 10.2(10).

Theorem 10.10. *Let* R *be a local ring with maximal ideal* $\mathfrak{m}$ *and residue field* $k = R/\mathfrak{m}$*. Put* $t = \mathrm{depth}\, R$*. Let* Φ *be a subset of the singular locus* $\mathrm{Sing}\, R$*. Denote by* π *the canonical functor* $\mathsf{D}^{\mathsf{b}}(R) \to \mathsf{D}^{\mathsf{sg}}(R)$*.*

(1) *Assume that the local ring* R *is singular. Suppose that the localization* $R_{\mathfrak{p}}$ *is a dominant local ring for every* $\mathfrak{p} \in \Phi \setminus \{\mathfrak{m}\}$*. Then there is a commutative diagram of mutually inverse bijections:*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\textit{Resolving subcategories of } \mathsf{mod}\, R\\ \textit{contained in } \mathsf{C}_\Phi(R) \textit{ and containing } \Omega^t k\end{array}\right\} & \underset{\mathrm{NF}_{\mathsf{C}}^{-1}}{\overset{\mathrm{NF}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Nonempty specialization-closed}\\ \textit{subsets of } \mathrm{Spec}\, R \textit{ contained in } \Phi\end{array}\right\} \\
\Big\| & & \mathrm{IPD}\ \big\uparrow\big\downarrow\ \mathrm{IPD}^{-1} \\
\left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{C}(R) \textit{ contained}\\ \textit{in } \mathsf{C}_\Phi(R) \textit{ and containing } R, \Omega^t k\end{array}\right\} & \underset{\mathsf{rest}_{\mathsf{C}}}{\overset{\mathsf{thick}_{\mathsf{mod}}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{mod}\, R \textit{ contained}\\ \textit{in } \mathsf{mod}_\Phi R \textit{ and containing } R, k\end{array}\right\} \\
\mathsf{rest}_{\mathsf{C}}\ \big\uparrow\big\downarrow\ \mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}} & & \mathsf{rest}_{\mathsf{mod}}\ \big\uparrow\big\downarrow\ \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}} \\
\left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{D}^{\mathsf{sg}}(R)\\ \textit{contained in } \mathsf{D}^{\mathsf{sg}}_\Phi(R) \textit{ and containing } k\end{array}\right\} & \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{D}^{\mathsf{b}}(R) \textit{ contained}\\ \textit{in } \mathsf{D}^{\mathsf{b}}_\Phi(R) \textit{ and containing } R, k\end{array}\right\}.
\end{array}$$

(2) *Suppose that the localization $R_\mathfrak{p}$ is a dominant local ring for every prime ideal $\mathfrak{p} \in \Phi \cup \{\mathfrak{m}\}$. Then one has the following commutative diagram of mutually inverse bijections.*

$$\begin{array}{ccc}
\left\{\begin{matrix}\textit{Resolving subcategories of } \mathsf{mod}\, R \\ \textit{contained in } \mathsf{C}_\Phi(R)\end{matrix}\right\} & \underset{\mathrm{NF}_{\mathsf{C}}^{-1}}{\overset{\mathrm{NF}}{\rightleftarrows}} & \left\{\begin{matrix}\textit{Specialization-closed subsets of } \operatorname{Spec} R \\ \textit{contained in } \Phi\end{matrix}\right\} \\
\| & & \mathrm{IPD} \uparrow\downarrow \mathrm{IPD}^{-1} \\
\left\{\begin{matrix}\textit{Thick subcategories of } \mathsf{C}(R) \\ \textit{contained in } \mathsf{C}_\Phi(R) \textit{ and containing } R\end{matrix}\right\} & \underset{\mathsf{rest}_{\mathsf{C}}}{\overset{\mathsf{thick}_{\mathsf{mod}}}{\rightleftarrows}} & \left\{\begin{matrix}\textit{Thick subcategories of } \mathsf{mod}\, R \\ \textit{contained in } \mathsf{mod}_\Phi R \textit{ and containing } R\end{matrix}\right\} \\
\mathsf{rest}_{\mathsf{C}} \uparrow\downarrow \mathsf{thick}_{\mathsf{D^{sg}}} & & \mathsf{rest}_{\mathsf{mod}} \uparrow\downarrow \mathsf{thick}_{\mathsf{D^b}} \\
\left\{\begin{matrix}\textit{Thick subcategories of } \mathsf{D^{sg}}(R) \\ \textit{contained in } \mathsf{D}^{\mathsf{sg}}_\Phi(R)\end{matrix}\right\} & \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} & \left\{\begin{matrix}\textit{Thick subcategories of } \mathsf{D^b}(R) \\ \textit{contained in } \mathsf{D}^{\mathsf{b}}_\Phi(R) \textit{ and containing } R\end{matrix}\right\}.
\end{array}$$

Proof. (1) We fix a resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ with $\Omega^t k \in \mathcal{X} \subseteq \mathsf{C}_\Phi(R)$, a thick subcategory $\mathcal{Y}$ of $\mathsf{mod}\, R$ with $R, k \in \mathcal{Y} \subseteq \mathsf{mod}_\Phi R$, a thick subcategory $\mathcal{Z}$ of $\mathsf{D^b}(R)$ with $R, k \in \mathcal{Z} \subseteq \mathsf{D}^{\mathsf{b}}_\Phi(R)$, a thick subcategory $\mathcal{V}$ of $\mathsf{D^{sg}}(R)$ with $k \in \mathcal{V} \subseteq \mathsf{D}^{\mathsf{sg}}_\Phi(R)$, a thick subcategory $\mathcal{U}$ of $\mathsf{C}(R)$ with $R, \Omega^t k \in \mathcal{U} \subseteq \mathsf{C}_\Phi(R)$, and a specialization-closed subset W of $\operatorname{Spec} R$ with $\emptyset \neq W \subseteq \Phi$. We prove the assertion step by step.

(a) The nonfree locus $\mathrm{NF}(\mathcal{X})$ of $\mathcal{X}$ is a specialization-closed subset of $\operatorname{Spec} R$ contained in Φ, while $\mathrm{NF}_{\mathsf{C}}^{-1}(W) = \mathrm{NF}^{-1}(W) \cap \mathsf{C}(R)$ is a resolving subcategory of $\mathsf{mod}\, R$ contained in $\mathrm{NF}_{\mathsf{C}}^{-1}(\Phi) = \mathsf{C}_\Phi(R)$. Since $\Omega^t k$ belongs to $\mathcal{X}$ and R is a singular local ring, we have $\mathfrak{m} \in \mathrm{NF}(\Omega^t k) \subseteq \mathrm{NF}(\mathcal{X})$, which implies that $\mathrm{NF}(\mathcal{X})$ is nonempty. As W is nonempty and specialization-closed, $\mathfrak{m}$ is in W and we get $\Omega^t k \in \mathrm{NF}_{\mathsf{C}}^{-1}(\{\mathfrak{m}\}) \subseteq \mathrm{NF}_{\mathsf{C}}^{-1}(W)$; see Remark 3.3(8). Thus the maps $\mathrm{NF}, \mathrm{NF}_{\mathsf{C}}^{-1}$ in the diagram are well-defined.

(b) We claim that $\mathrm{NF}(\mathrm{NF}_{\mathsf{C}}^{-1}(W)) = W$. Indeed, the inclusion ($\subseteq$) is clear. To show ($\supseteq$), we pick any $\mathfrak{p} \in W$. Put $r = \operatorname{Rfd}_R R/\mathfrak{p}$. Then we have $\mathrm{NF}(\Omega^r(R/\mathfrak{p})) \subseteq \mathrm{V}(\mathfrak{p}) \subseteq W$ as W is specialization-closed, which implies $\Omega^r(R/\mathfrak{p}) \in \mathrm{NF}^{-1}(W)$. Remark 3.3(5) shows that $\Omega^r(R/\mathfrak{p})$ is in $\mathsf{C}(R)$, and hence it belongs to $\mathrm{NF}_{\mathsf{C}}^{-1}(W)$. Since $\mathfrak{p} \in W \subseteq \Phi \subseteq \operatorname{Sing} R$, we get $\mathfrak{p} \in \mathrm{NF}(\Omega^r(R/\mathfrak{p}))$. Thus W is contained in $\mathrm{NF}(\mathrm{NF}_{\mathsf{C}}^{-1}(W))$.

(c) Clearly, $\mathcal{X}$ is contained in $\mathrm{NF}_{\mathsf{C}}^{-1}(\mathrm{NF}(\mathcal{X}))$. Let M be an R-module in $\mathrm{NF}_{\mathsf{C}}^{-1}(\mathrm{NF}(\mathcal{X}))$. To show that M belongs to $\mathcal{X}$, we may assume that M is nonfree. Fix $\mathfrak{p} \in \mathrm{NF}(\mathcal{X}) \setminus \{\mathfrak{m}\}$. Then $\mathfrak{p} \in \Phi \setminus \{\mathfrak{m}\}$ and $R_\mathfrak{p}$ is dominant. Since $\mathsf{add}\, \mathcal{X}_\mathfrak{p}$ is a resolving subcategory of $\mathsf{mod}\, R_\mathfrak{p}$ with $\mathsf{add}\, R_\mathfrak{p} \neq \mathsf{add}\, \mathcal{X}_\mathfrak{p} \subseteq \mathsf{C}(R_\mathfrak{p})$, Proposition 5.3 implies that $\mathsf{add}\, \mathcal{X}_\mathfrak{p}$ contains $\mathsf{C}_0(R_\mathfrak{p})$. Applying Theorem 3.8 to the subset $\mathrm{NF}(\mathcal{X})$ of $\operatorname{Sing} R$, we obtain an exact sequence $0 \to C \to M \oplus N \to Y \to 0$ with $C \in \mathsf{C}_0(R)$ and $Y \in \mathcal{X}^{\circ n}$, where $n = \dim \mathrm{NF}(M)$. As $\mathcal{X}$ is resolving, $\mathcal{X}^{\circ n}$ is contained in $\mathcal{X}$. Also, we have $\mathfrak{m} \in \mathrm{NF}(M) \subseteq \mathrm{NF}(\mathcal{X})$, which says $\mathcal{X} \neq \mathsf{add}\, R$. As $\Omega^t k \in \mathcal{X}$, we have $\mathsf{C}_0(R) = \mathsf{res}\, \Omega^t k \subseteq \mathcal{X}$ by Remark 3.3(8), and the above exact sequence shows $M \in \mathcal{X}$.

(d) By (a), (b) and (c) we obtain the mutually inverse bijections $(\mathrm{NF}, \mathrm{NF}_{\mathsf{C}}^{-1})$ in the diagram.

(e) It is clear that $\mathcal{U}$ is a resolving subcategory of $\mathsf{mod}\, R$ contained in $\mathsf{C}_\Phi(R)$. Remark 10.2(3) implies that $\mathrm{NF}_{\mathsf{C}}^{-1}(W)$ consists of those R-modules M which belong to $\mathsf{C}(R)$ and satisfies $\mathrm{IPD}(M) \subseteq W$. Thus $\mathrm{NF}_{\mathsf{C}}^{-1}(W)$ is a thick subcategory of $\mathsf{C}(R)$. By virtue of (d), we get the vertical equality in the diagram.

(f) The subset $\mathrm{IPD}(\mathcal{Y})$ of $\operatorname{Spec} R$ is specialization-closed and contained in Φ, while $\mathrm{IPD}^{-1}(W)$ is a thick subcategory of $\mathsf{mod}\, R$ contained in $\mathrm{IPD}^{-1}(\Phi) = \mathsf{mod}_\Phi R$ and containing R. As $k \in \mathcal{Y}$ and R is singular, we have $\mathfrak{m} \in \mathrm{IPD}(k) \subseteq \mathrm{IPD}(\mathcal{Y})$ and $\mathrm{IPD}(\mathcal{Y}) \neq \emptyset$. As W is nonempty and specialization-closed, we get $\mathfrak{m} \in W$ and $k \in \mathrm{IPD}^{-1}(\{\mathfrak{m}\}) \subseteq \mathrm{IPD}^{-1}(W)$. So the maps $\mathrm{IPD}, \mathrm{IPD}^{-1}$ in the diagram are well-defined.

(g) We claim that $\mathrm{IPD}(\mathrm{IPD}^{-1}(W)) = W$. Indeed, ($\subseteq$) is clear. To show ($\supseteq$), we take $\mathfrak{p} \in W$. We have $\mathfrak{p} \in W \subseteq \Phi \subseteq \operatorname{Sing} R$, whence $\mathfrak{p} \in \mathrm{IPD}(R/\mathfrak{p}) \subseteq \mathrm{V}(\mathfrak{p})$. The equality $\mathrm{IPD}(R/\mathfrak{p}) = \mathrm{V}(\mathfrak{p})$ holds, while $\mathrm{V}(\mathfrak{p}) \subseteq W$ as W is specialization-closed. We obtain $R/\mathfrak{p} \in \mathrm{IPD}^{-1}(W)$ and $\mathfrak{p} \in \mathrm{IPD}(R/\mathfrak{p}) \subseteq \mathrm{IPD}(\mathrm{IPD}^{-1}(W))$.

(h) It is evident that $\mathcal{Y}$ is contained in $\mathrm{IPD}^{-1}(\mathrm{IPD}(\mathcal{Y}))$ and $\mathcal{Y} \cap \mathsf{C}(R)$ is a resolving subcategory of $\mathsf{mod}\, R$. Remark 10.2(7) shows that $\mathrm{NF}(\mathcal{Y} \cap \mathsf{C}(R)) = \mathrm{IPD}(\mathcal{Y}) \subseteq \Phi$, and hence $\mathcal{Y} \cap \mathsf{C}(R) \subseteq \mathsf{C}_\Phi(R)$. Since $\mathcal{Y}$ contains k and is closed under syzygies, $\Omega^t k$ belongs to $\mathcal{Y} \cap \mathsf{C}(R)$ by Remark 3.3(8). Let $0 \neq M \in \mathrm{IPD}^{-1}(\mathrm{IPD}(\mathcal{Y}))$ and put $r = \operatorname{Rfd} M$. Remarks 3.3(5) and 10.2(4) imply that $\Omega^r M \in \mathsf{C}(R)$ and $\mathrm{IPD}(M) = \mathrm{NF}(\Omega^r M)$. Hence $\Omega^r M$ belongs to $\mathrm{NF}_{\mathsf{C}}^{-1}(\mathrm{NF}(\mathcal{Y} \cap \mathsf{C}(R)))$. By (d) we get the equality $\mathrm{NF}_{\mathsf{C}}^{-1}(\mathrm{NF}(\mathcal{Y} \cap \mathsf{C}(R))) = \mathcal{Y} \cap \mathsf{C}(R)$, and then M is in $\mathcal{Y}$. We thus have $\mathcal{Y} = \mathrm{IPD}^{-1}(\mathrm{IPD}(\mathcal{Y}))$ and $(\mathrm{NF}_{\mathsf{C}}^{-1} \cdot \mathrm{IPD})(\mathcal{Y}) = (\mathsf{rest}_{\mathsf{C}})(\mathcal{Y})$.

(i) From (f), (g) and (h) we get the mutually inverse bijections $(\mathrm{IPD}, \mathrm{IPD}^{-1})$ in the diagram.

(j) Using (3) and (8) of Remark 10.2, we get $\Phi \supseteq \mathrm{NF}(\mathcal{X}) = \mathrm{IPD}(\mathcal{X}) = \mathrm{IPD}(\mathsf{thick}_{\mathsf{mod}\, R}\, \mathcal{X})$ and observe that $\mathsf{thick}_{\mathsf{mod}\, R}\, \mathcal{X}$ is a thick subcategory of $\mathsf{mod}\, R$ contained in $\mathsf{mod}_\Phi R$ and containing R. As $\mathcal{X}$ contains

$\Omega^t k$, the thick closure $\mathsf{thick}_{\mathsf{mod}\,R}\,\mathcal{X}$ contains k. By virtue of (i), there is an equality $\mathrm{IPD}^{-1}(\mathrm{NF}(\mathcal{X})) = \mathsf{thick}_{\mathsf{mod}\,R}\,\mathcal{X}$, which means that $(\mathrm{IPD}^{-1}\cdot\mathrm{NF})(\mathcal{X}) = (\mathsf{thick}_{\mathsf{mod}})(\mathcal{X})$.

(k) By (d), (e), (h), (i) and (j), we observe that the pair $(\mathsf{thick}_{\mathsf{mod}}, \mathsf{rest}_{\mathsf{C}})$ in the diagram gives mutually inverse bijections and that the top square in the diagram is commutative.

(l) The subcategory $\mathsf{D}^{\mathsf{b}}_{\Phi}(R)$ of $\mathsf{D}^{\mathsf{b}}(R)$ is thick, while $\mathrm{Supp}^{\mathrm{sg}}\,\pi(\mathcal{Z}) = \mathrm{IPD}(\mathcal{Z}) \subseteq \Phi$ and $\mathrm{IPD}(\pi^{-1}(\mathcal{V})) = \mathrm{Supp}^{\mathrm{sg}}(\mathcal{V}) \subseteq \Phi$ by Remark 10.2(9). We get $\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\,\mathcal{Y} \subseteq \mathsf{D}^{\mathsf{b}}_{\Phi}(R)$, $\pi(\mathcal{Z}) \subseteq \mathsf{D}^{\mathsf{sg}}_{\Phi}(R)$, $\mathcal{Z}\cap\mathsf{mod}\,R \subseteq \mathsf{mod}_{\Phi}\,R$ and $\pi^{-1}(\mathcal{V}) \subseteq \mathsf{D}^{\mathsf{b}}_{\Phi}(R)$. Since k belongs to $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{V}$, it belongs to $\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\,\mathcal{Y}$, $\pi(\mathcal{Z})$, $\mathcal{Z}\cap\mathsf{mod}\,R$ and $\pi^{-1}(\mathcal{V})$. Lemma 10.5 yields the bijection pairs $(\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}}, \mathsf{rest}_{\mathsf{mod}})$ and (π, π^{-1}) in the diagram.

(m) Remark 10.2(9) gives $(\pi\cdot\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}}\cdot\mathsf{thick}_{\mathsf{mod}})(\mathcal{U}) = \pi(\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathsf{thick}_{\mathsf{mod}\,R}\,\mathcal{U})) = \pi(\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\,\mathcal{U}) = \mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}(R)}(\pi(\mathcal{U})) = (\mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}})(\mathcal{U})$ and $(\mathsf{rest}_{\mathsf{C}}\cdot\mathsf{rest}_{\mathsf{mod}}\cdot\pi^{-1})(\mathcal{V}) = \pi^{-1}(\mathcal{V})\cap\mathsf{C}(R) = (\mathsf{rest}_{\mathsf{C}})(\mathcal{V})$. By (k) and (l) we get the bijection pair $(\mathsf{thick}_{\mathsf{D}^{\mathsf{sg}}}, \mathsf{rest}_{\mathsf{C}})$ and the bottom commutative square given in the diagram.

Finally, combining (d), (e), (i), (k), (l) and (m) completes the proof of the assertion.

(2) First of all, note that $\mathsf{add}\,R$ is both a resolving subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{C}_{\Phi}(R)$ and a thick subcategory of $\mathsf{C}(R)$ contained in $\mathsf{C}_{\Phi}(R)$ and containing R, $\mathsf{fpd}\,R$ is a thick subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{mod}_{\Phi}\,R$ and containing R, $\mathsf{D}^{\mathsf{perf}}(R)$ is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ contained in $\mathsf{D}^{\mathsf{b}}_{\Phi}(R)$ and containing R, 0 is a thick subcategory of $\mathsf{D}^{\mathsf{sg}}(R)$ contained in $\mathsf{D}^{\mathsf{sg}}_{\Phi}(R)$, and $\emptyset$ is a specialization-closed subset of $\operatorname{Spec} R$ contained in Φ. We call these six elements the *exceptional elements*. As is easy to verify, the exceptional elements correspond by the maps given in the diagram.

Suppose that R is regular. Then $\Phi = \operatorname{Sing} R = \emptyset$. It is seen that $\mathsf{C}_{\Phi}(R) = \mathsf{C}(R) = \mathsf{add}\,R$, $\mathsf{mod}_{\Phi}\,R = \mathsf{mod}\,R = \mathsf{fpd}\,R$, $\mathsf{D}^{\mathsf{b}}_{\Phi}(R) = \mathsf{D}^{\mathsf{b}}(R) = \mathsf{D}^{\mathsf{perf}}(R)$ and $\mathsf{D}^{\mathsf{sg}}_{\Phi}(R) = \mathsf{D}^{\mathsf{sg}}(R) = 0$. Each of the six sets in the diagram given in the assertion is the one-point set consisting of the exceptional element, and we are done.

Next we consider the case where R is singular. Then we can apply assertion (1) to obtain the commutative diagram of mutually inverse bijections in (1). This is actually a commutative diagram of mutually inverse bijections of all the non-exceptional elements of the six sets in (2), since $R = R_{\mathfrak{m}}$ is dominant by assumption and Corollary 10.8 applies. The diagram is thus complemented to the one in (2). ■

Remark 10.11. Let R be a local ring of depth t with maximal ideal $\mathfrak{m}$ and residue field k. Here we consider what Theorem 10.10 asserts in the extreme cases where $\Phi = \emptyset$, $\Phi = \{\mathfrak{m}\}$ and $\Phi = \operatorname{Sing} R$.

(1) Let $\Phi = \emptyset$. Then the six sets in Theorem 10.10(2) are singletons, while the six sets in Theorem 10.10(1) are empty sets. Thus, the theorem says nothing interesting in the case where $\Phi = \emptyset$.

(2) Let $\Phi = \{\mathfrak{m}\}$. Then $\emptyset$ and $\{\mathfrak{m}\}$ are the only specialization-closed subsets of $\operatorname{Spec} R$ contained in Φ. The two assertions of Theorem 10.10 say that the following two statements hold.
   (a) Suppose that R is dominant. Then $\mathsf{add}\,R$ and $\mathsf{C}_{\{\mathfrak{m}\}}(R)(=\mathsf{C}_0(R))$ are the only resolving subcategories of $\mathsf{mod}\,R$ contained in $\mathsf{C}_{\{\mathfrak{m}\}}(R)$, and the only thick subcategories of $\mathsf{C}(R)$ containing R. The only thick subcategories of $\mathsf{mod}\,R$ contained in $\mathsf{mod}_{\{\mathfrak{m}\}}\,R$ and containing R are $\mathsf{fpd}\,R, \mathsf{mod}_{\{\mathfrak{m}\}}\,R$. The only thick subcategories of $\mathsf{D}^{\mathsf{sg}}(R)$ contained in $\mathsf{D}^{\mathsf{sg}}_{\{\mathfrak{m}\}}(R)(=\mathsf{D}^{\mathsf{sg}}_0(R))$ are $0, \mathsf{D}^{\mathsf{sg}}_{\{\mathfrak{m}\}}(R)$. The only thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ contained in $\mathsf{D}^{\mathsf{b}}_{\{\mathfrak{m}\}}(R)$ and containing R are $\mathsf{D}^{\mathsf{perf}}(R), \mathsf{D}^{\mathsf{b}}_{\{\mathfrak{m}\}}(R)$.
   (b) Suppose that R is singular. Then $\mathsf{C}_{\{\mathfrak{m}\}}(R)$ is the only resolving subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{C}_{\{\mathfrak{m}\}}(R)$ and containing $\Omega^t k$, and the only thick subcategory of $\mathsf{C}(R)$ containing $R, \Omega^t k$. The only thick subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{mod}_{\{\mathfrak{m}\}}\,R$ and containing R, k is $\mathsf{mod}_{\{\mathfrak{m}\}}\,R$. The only thick subcategory of $\mathsf{D}^{\mathsf{sg}}(R)$ contained in $\mathsf{D}^{\mathsf{sg}}_{\{\mathfrak{m}\}}(R)$ and containing k is $\mathsf{D}^{\mathsf{sg}}_{\{\mathfrak{m}\}}(R)$. The only thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ contained in $\mathsf{D}^{\mathsf{b}}_{\{\mathfrak{m}\}}(R)$ and containing R, k is $\mathsf{D}^{\mathsf{b}}_{\{\mathfrak{m}\}}(R)$.

(3) Let $\Phi = \operatorname{Sing} R$. Then Theorem 10.10 says that the following two statements hold true.
   (a) If $R_{\mathfrak{p}}$ is dominant for all prime ideals $\mathfrak{p}$ of R, then there are one-to-one correspondences among
      - the resolving subcategories of $\mathsf{mod}\,R$ contained in $\mathsf{C}(R)$,
      - the thick subcategories of $\mathsf{C}(R)$ containing R,
      - the thick subcategories of $\mathsf{mod}\,R$ containing R,
      - the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ containing R,
      - the thick subcategories of $\mathsf{D}^{\mathsf{sg}}(R)$, and
      - the specialization-closed subsets of $\operatorname{Spec} R$ contained in $\operatorname{Sing} R$.
   (b) Assume that R is singular, and suppose that $R_{\mathfrak{p}}$ is dominant for all nonmaximal prime ideals $\mathfrak{p}$ of R. Then there are one-to-one correspondences among
      - the resolving subcategories of $\mathsf{mod}\,R$ contained in $\mathsf{C}(R)$ and containing $\Omega^t k$,
      - the thick subcategories of $\mathsf{C}(R)$ containing R and $\Omega^t k$,

- the thick subcategories of $\mathsf{mod}\, R$ containing R and k,
- the thick subcategories of $\mathsf{D^b}(R)$ containing R and k,
- the thick subcategories of $\mathsf{D^{sg}}(R)$ containing k, and
- the nonempty specialization-closed subsets of $\operatorname{Spec} R$ contained in $\operatorname{Sing} R$.

Remark 10.12. It is definitely worth mentioning that Theorem 10.10 includes all of the known classification theorems of the same type, and moreover, highly generalizes them. More precisely, applying Theorem 10.10 to a Cohen–Macaulay local ring R with $\Phi = \operatorname{Sing} R$ and recalling the sufficient conditions for dominance given in Proposition 5.10 (together with [59, Lemma 5.4]), we immediately and simultaneously recover and refine [22, Theorem 7.10(1)], [46, Theorem 4.5(1)], [57, Theorem 6.8], [59, Theorem 5.6] and [60, Theorem 5.1]. Note that all of those theorems assume that the base ring is a Cohen–Macaulay local ring. Thus, Theorem 10.10 provides classification in the non-Cohen–Macaulay case for the first time. We have already got examples of non-Cohen–Macaulay dominant local rings in Example 8.10.

Remark 10.13. It is possible to formulate a non-local version of Theorem 10.10. Let R be a ring which is not necessarily local. Let Φ be a subset of $\operatorname{Sing} R$. Suppose that $R_\mathfrak{p}$ is a dominant local ring for each $\mathfrak{p} \in \Phi \cup \operatorname{Max} R$. Then one has the same commutative diagram of mutually inverse bijections as in Theorem 10.10(2). Letting $\Phi = \operatorname{Sing} R$ recovers the third one-to-one correspondence in [52, Theorem 6.13].

Indeed, what we should prove is that for every resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ contained in $\mathsf{C}_\Phi(R)$ it holds that $\mathcal{X} = \mathrm{NF}_{\mathsf{C}}^{-1}(\mathrm{NF}(\mathcal{X}))$. For this, it suffices to show that every R-module $M \in \mathsf{C}(R)$ with $\mathrm{NF}(M) \subseteq \mathrm{NF}(\mathcal{X})$ is in $\mathcal{X}$. Fix $\mathfrak{m} \in \operatorname{Max} R$. Let $\Phi_\mathfrak{m}$ be the set of prime ideals P of $R_\mathfrak{m}$ with $P \cap R \in \Phi$. Then $\Phi_\mathfrak{m}$ is a subset of $\operatorname{Sing} R_\mathfrak{m}$, and the local ring $(R_\mathfrak{m})_P = R_{P\cap R}$ is dominant for all $P \in \Phi_\mathfrak{m} \cup \{\mathfrak{m}R_\mathfrak{m}\}$. We see that $\mathsf{add}\, \mathcal{X}_\mathfrak{m}$ is a resolving subcategory of $\mathsf{mod}\, R_\mathfrak{m}$ contained in $\mathsf{C}_{\Phi_\mathfrak{m}}(R_\mathfrak{m})$ (see [24, Lemma 3.2(1)]). Thus we can apply Theorem 10.10(2) to get $\mathsf{add}\, \mathcal{X}_\mathfrak{m} = \mathrm{NF}_{\mathsf{C}}^{-1}(\mathrm{NF}(\mathsf{add}\, \mathcal{X}_\mathfrak{m}))$. Note that $\mathrm{NF}(M_\mathfrak{m})$ is contained in $\mathrm{NF}(\mathcal{X}_\mathfrak{m}) = \mathrm{NF}(\mathsf{add}\, \mathcal{X}_\mathfrak{m})$. It follows that $M_\mathfrak{m} \in \mathsf{add}\, \mathcal{X}_\mathfrak{m}$, and by [24, Proposition 3.3] we obtain $M \in \mathcal{X}$.

To cover the other similar known classification theorems, we introduce a weaker version of dominance.

Definition 10.14. Let R be a d-dimensional Cohen–Macaulay local ring with residue field k and admitting a canonical module ω. We say that R is *quasi-dominant* if $\Omega^d k \in \mathsf{res}\{\omega, M\}$ for every R-module M of infinite projective dimension. It is obvious that if R is dominant, then it is quasi-dominant. When R is Gorenstein, R is dominant if and only if it is quasi-dominant, since $\omega = R$. When R is non-Gorenstein, R is quasi-dominant if and only if $\Omega^d k \in \mathsf{res}\, \omega$, since ω itself has infinite projective dimension.

A Cohen–Macaulay local ring R with a canonical module ω is called *almost Gorenstein* if there exists an exact sequence $0 \to R \to \omega \to C \to 0$ of R-modules with $\mu(C) = \mathrm{e}(C)$. Here are some statements on quasi-dominance; (1) and (2) correspond to Proposition 5.3 and Corollary 10.8, respectively.

Proposition 10.15. *Let R be a d-dimensional Cohen–Macaulay local ring with residue field k. Assume that R possesses a canonical module ω. Then the following five statements hold true.*

(1) *The following are equivalent.*
   (a) *R is quasi-dominant.* (b) *$\mathsf{CM}_0(R) \subseteq \mathsf{res}\{\omega, M\}$ for every R-module M with $\operatorname{pd} M = \infty$.*
   (c) *$\mathsf{CM}_0(R) \subseteq \mathcal{X}$ for every resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ with $\mathsf{add}\, R \neq \mathcal{X} \subseteq \mathsf{CM}(R)$ and $\omega \in \mathcal{X}$.*
(2) *The following are equivalent.*
   (a) *For every $C \in \mathsf{CM}(R)$ that is a nonfree R-module, one has $\Omega^d k \in \mathsf{res}\{\omega, C\}$.*
   (b) *For every $C \in \mathsf{CM}(R)$ that is a nonfree R-module, one has $\Omega^d k \in \mathsf{thick}_{\mathsf{CM}(R)}\{R, \omega, C\}$.*
   (c) *For every $M \in \mathsf{mod}\, R$ with $\operatorname{pd}_R M = \infty$, one has $\Omega^d k \in \mathsf{res}\{\omega, M\}$ (i.e., R is quasi-dominant).*
   (d) *For every $M \in \mathsf{mod}\, R$ with $\operatorname{pd}_R M = \infty$, one has $k \in \mathsf{thick}_{\mathsf{mod}\, R}\{R, \omega, M\}$.*
   (e) *For every $X \in \mathsf{D^b}(R)$ with $\operatorname{pd}_R X = \infty$, one has $k \in \mathsf{thick}_{\mathsf{D^b}(R)}\{R, \omega, X\}$.*
   (f) *For every $Y \in \mathsf{D^{sg}}(R)$ that is a nonzero object, one has $k \in \mathsf{thick}_{\mathsf{D^{sg}}(R)}\{\omega, Y\}$.*
(3) *In the case where R is not Gorenstein, the following are equivalent.*
   (a) *$\Omega^d k \in \mathsf{res}\, \omega$ (i.e., R is quasi-dominant).* (b) *$\Omega^d k \in \mathsf{thick}_{\mathsf{CM}(R)}\{R, \omega\}$.*
   (c) *$k \in \mathsf{thick}_{\mathsf{mod}\, R}\{R, \omega\}$.* (d) *$k \in \mathsf{thick}_{\mathsf{D^b}(R)}\{R, \omega\}$.* (e) *$k \in \mathsf{thick}_{\mathsf{D^{sg}}(R)}\, \omega$.*
(4) *Suppose that R is excellent and has finite CM-representation type. Then R is quasi-dominant.*
(5) *Assume k is infinite. Suppose R is not Gorenstein but almost Gorenstein. Then R is quasi-dominant.*

Proof. An analogous argument as in the proof of Proposition 5.3 shows (1). Letting $E = \omega$ in Proposition 10.7 and $C = \omega$ in Lemma 10.6 imply (2) and (3), respectively.

(4) As R has finite CM-representation type, it has an isolated singularity by [35, Corollary 2]. Since R is excellent, $\widehat{R}$ also has an isolated singularity. By [59, Corollary 6.9] the only resolving subcategories of $\operatorname{mod} R$ contained in $\mathsf{CM}(R)$ and containing ω are $\operatorname{add} R$ and $\mathsf{CM}(R)$. So, if $\mathcal{X}$ is a resolving subcategory of $\operatorname{mod} R$ with $\operatorname{add} R \neq \mathcal{X} \subseteq \mathsf{CM}(R)$ and $\omega \in \mathcal{X}$, then $\mathcal{X} = \mathsf{CM}(R) \supseteq \mathsf{CM}_0(R)$. We are done by (1).

(5) By [30, Theorem 4.3] we get $k \in \mathsf{thick}_{\mathsf{mod}\, R}\{R, \omega\}$. By (3) the ring R is quasi-dominant. ■

Definition 10.16. We denote by $\operatorname{NonGor} R$ the *non-Gorenstein locus* of R, that is, the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is non-Gorenstein. Note that if R is a Cohen–Macaulay local ring with a canonical module ω, then $\operatorname{NonGor} R = \operatorname{NF}(\omega)$ holds.

We obtain a corollary of (the proof of) Theorem 10.10.

Corollary 10.17. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional Cohen–Macaulay local ring with a canonical module ω. Let Φ be a subset of $\operatorname{Sing} R$, and denote by π the canonical functor $\mathsf{D^b}(R) \to \mathsf{D^{sg}}(R)$.*

(1) *Assume that the local ring R is singular. Suppose that the localization $R_\mathfrak{p}$ is a quasi-dominant local ring for every $\mathfrak{p} \in \Phi \setminus \{\mathfrak{m}\}$. Then there is a commutative diagram of mutually inverse bijections:*

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\textit{Resolving subcategories of } \operatorname{mod} R\\ \textit{contained in } \mathsf{CM}_\Phi(R)\\ \textit{and containing } \Omega^t k, \omega\end{array}\right\} & \underset{\operatorname{NF}_{\mathsf{CM}}^{-1}}{\overset{\operatorname{NF}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Nonempty specialization-closed}\\ \textit{subsets of } \operatorname{Spec} R \textit{ contained in } \Phi\\ \textit{and containing } \operatorname{NonGor} R\end{array}\right\} \\
\Big\| & & \mathrm{IPD}\uparrow\downarrow\mathrm{IPD}^{-1} \\
\left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{CM}(R)\\ \textit{contained in } \mathsf{CM}_\Phi(R)\\ \textit{and containing } R, \Omega^t k, \omega\end{array}\right\} & \underset{\mathsf{rest_{CM}}}{\overset{\mathsf{thick_{mod}}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of } \operatorname{mod} R\\ \textit{contained in } \operatorname{mod}_\Phi R\\ \textit{and containing } R, k, \omega\end{array}\right\} \\
\mathsf{rest_{CM}}\uparrow\downarrow\mathsf{thick_{D^{sg}}} & & \mathsf{rest_{mod}}\uparrow\downarrow\mathsf{thick_{D^b}} \\
\left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{D^{sg}}(R)\\ \textit{contained in } \mathsf{D^{sg}_\Phi}(R)\\ \textit{and containing } k, \omega\end{array}\right\} & \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{D^b}(R)\\ \textit{contained in } \mathsf{D^b_\Phi}(R)\\ \textit{and containing } R, k, \omega\end{array}\right\}.
\end{array}
$$

(2) *Suppose that the localization $R_\mathfrak{p}$ is a quasi-dominant local ring for every $\mathfrak{p} \in \Phi \cup \{\mathfrak{m}\}$. Then there is a commutative diagram of mutually inverse bijections:*

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\textit{Resolving subcategories of } \operatorname{mod} R\\ \textit{contained in } \mathsf{CM}_\Phi(R) \textit{ and containing } \omega\end{array}\right\} & \underset{\operatorname{NF}_{\mathsf{CM}}^{-1}}{\overset{\operatorname{NF}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Specialization-closed subsets of } \operatorname{Spec} R\\ \textit{contained in } \Phi \textit{ and containing } \operatorname{NonGor} R\end{array}\right\} \\
\Big\| & & \mathrm{IPD}\uparrow\downarrow\mathrm{IPD}^{-1} \\
\left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{CM}(R)\\ \textit{contained in } \mathsf{CM}_\Phi(R) \textit{ and containing } R, \omega\end{array}\right\} & \underset{\mathsf{rest_{CM}}}{\overset{\mathsf{thick_{mod}}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of } \operatorname{mod} R\\ \textit{contained in } \operatorname{mod}_\Phi R \textit{ and containing } R, \omega\end{array}\right\} \\
\mathsf{rest_{CM}}\uparrow\downarrow\mathsf{thick_{D^{sg}}} & & \mathsf{rest_{mod}}\uparrow\downarrow\mathsf{thick_{D^b}} \\
\left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{D^{sg}}(R)\\ \textit{contained in } \mathsf{D^{sg}_\Phi}(R) \textit{ and containing } \omega\end{array}\right\} & \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} & \left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{D^b}(R)\\ \textit{contained in } \mathsf{D^b_\Phi}(R) \textit{ and containing } R, \omega\end{array}\right\}.
\end{array}
$$

Proof. (1) The proof of Theorem 10.10(1) works except (c), which uses the assumption that $R_\mathfrak{p}$ is dominant for all prime ideals $\mathfrak{p}$ in $\Phi \setminus \{\mathfrak{m}\}$. We have only to replace in (c) dominance and Proposition 5.3 with quasi-dominance and Proposition 10.15(1), respectively, and apply the same argument to a resolving subcategory $\mathcal{X}$ of $\operatorname{mod} R$ with $\omega \in \mathcal{X} \subseteq \mathsf{CM}_\Phi(R)$.

(2) Suppose that the ring R is Gorenstein. Then $\omega = R$ and $\operatorname{NonGor} R = \emptyset$, while quasi-dominance is equivalent to dominance for each localization $R_\mathfrak{p}$. The assertion is none other than Theorem 10.10(2).

Suppose that R is non-Gorenstein. Then, in particular, R is a singular local ring, and we get the commutative diagram of mutually inverse bijections in (1). As $\operatorname{NonGor} R \neq \emptyset$, every subset of $\operatorname{Spec} R$ containing $\operatorname{NonGor} R$ is nonempty. Since $R = R_\mathfrak{m}$ is quasi-dominant by assumption, we see from Proposition 10.15(3) that the six sets in assertion (2) coincide with those in assertion (1). Thus we are done. ■

Remark 10.18. Applying Corollary 10.17 to the set $\Phi = \operatorname{Sing} R$ and invoking Propositions 5.10 and 10.15(4), we recover [22, Theorem 7.10(2)], [46, Theorem 4.5(2)] and [59, Corollary 6.12].

Acknowlegments. The author would like to give his gratitude to an anonymous referee for reading the paper so carefully and making a lot of valuable suggestions. The author also thanks Toshinori Kobayashi and Hiroki Matsui for giving him helpful comments.

References

[1] L. Angeleri Hügel; D. Pospíšil; J. Šťovíček; J. Trlifaj, Tilting, cotilting, and spectra of commutative Noetherian rings, *Trans. Amer. Math. Soc.* **366** (2014), no. 7, 3487–3517.

[2] T. Araya; K.-i. Iima; R. Takahashi, On the structure of Cohen–Macaulay modules over hypersurfaces of countable Cohen–Macaulay representation type, *J. Algebra* **361** (2012), 213–224.

[3] M. Auslander; M. Bridger, Stable module theory, *Mem. Amer. Math. Soc.* **94** (1969).

[4] M. Auslander; R.-O. Buchweitz, The homological theory of maximal Cohen–Macaulay approximations, Colloque en l'honneur de Pierre Samuel (Orsay, 1987), *Mém. Soc. Math. France (N.S.)* **38** (1989), 5–37.

[5] M. Auslander; I. Reiten, On a generalized version of the Nakayama conjecture, *Proc. Amer. Math. Soc.* **52** (1975), 69–74.

[6] M. Auslander; I. Reiten, Applications of contravariantly finite subcategories, *Adv. Math.* **86** (1991), no. 1, 111–152.

[7] L. L. Avramov, Infinite free resolutions, *Six lectures on commutative algebra*, 1–118, Mod. Birkhäuser Class., *Birkhäuser Verlag, Basel*, 2010.

[8] L. L. Avramov; S. B. Iyengar; J. Lipman, Reflexivity and rigidity for complexes, I, Commutative rings, *Algebra Number Theory* **4** (2010), no. 1, 47–86.

[9] L. L. Avramov; S. B. Iyengar; S. Nasseh; S. K. Sather-Wagstaff, Persistence of homology over commutative noetherian rings, `arXiv:2005.10808v1`.

[10] L. L. Avramov; A. Martsinkovsky, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension, *Proc. London Math. Soc. (3)* **85** (2002), no. 2, 393–440.

[11] M. Ballard; D. Favero; L. Katzarkov, Orlov spectra: bounds and gaps, *Invent. Math.* **189** (2012), no. 2, 359–430.

[12] P. Balmer; B. Sanders, The spectrum of the equivariant stable homotopy category of a finite group, *Invent. Math.* **208** (2017), no. 1, 283–326.

[13] D. J. Benson; J. F. Carlson; J. Rickard, Thick subcategories of the stable module category, *Fund. Math.* **153** (1997), no. 1, 59–80.

[14] D. J. Benson; S. B. Iyengar; H. Krause, Stratifying modular representations of finite groups, *Ann. of Math. (2)* **174** (2011), no. 3, 1643–1684.

[15] D. Benson; S. B. Iyengar; H. Krause; J. Pevtsova, Stratification for module categories of finite group schemes, *J. Amer. Math. Soc.* **31** (2018), no. 1, 265–302.

[16] W. Bruns; J. Herzog, Cohen–Macaulay rings, revised edition, Cambridge Studies in Advanced Mathematics **39**, *Cambridge University Press, Cambridge*, 1998.

[17] W. Bruns; U. Vetter, Determinantal rings, Lecture Notes in Mathematics **1327**, *Springer–Verlag, Berlin*, 1988.

[18] R.-O. Buchweitz, Maximal Cohen–Macaulay modules and Tate-cohomology over Gorenstein rings, unpublished manuscript (1986), `http://hdl.handle.net/1807/16682`.

[19] L. W. Christensen, Gorenstein dimensions, Lecture Notes in Mathematics **1747**, *Springer–Verlag, Berlin*, 2000.

[20] L. W. Christensen, Semi-dualizing complexes and their Auslander categories, *Trans. Amer. Math. Soc.* **353** (2001), no. 5, 1839–1883.

[21] L. W. Christensen; H.-B. Foxby; A. Frankild, Restricted homological dimensions and Cohen–Macaulayness, *J. Algebra* **251** (2002), no. 1, 479–502.

[22] H. Dao; T. Kobayashi; R. Takahashi, Burch ideals and Burch rings, *Algebra Number Theory* **14** (2020), no. 8, 2121–2150.

[23] H. Dao; R. Takahashi, The radius of a subcategory of modules, *Algebra Number Theory* **8** (2014), no. 1, 141–172.

[24] H. Dao; R. Takahashi, Classification of resolving subcategories and grade consistent functions, *Int. Math. Res. Not. IMRN* (2015), no. 1, 119–149.

[25] H. Dao; R. Takahashi, The dimension of a subcategory of modules, *Forum Math. Sigma* **3** (2015), Paper No. e19, 31 pp.

[26] E. S. Devinatz; M. J. Hopkins; J. H. Smith, Nilpotence and stable homotopy theory I, *Ann. of Math. (2)* **128** (1988), no. 2, 207–241.

[27] E. M. Friedlander; J. Pevtsova, Π-supports for modules for finite group schemes, *Duke Math. J.* **139** (2007), no. 2, 317–368.

[28] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.

[29] E. S. Golod, G-dimension and generalized perfect ideals, Algebraic geometry and its applications, A translation of *Trudy Mat. Inst. Steklov.* **165** (1984), Proc. Steklov Inst. Math. **1985**, no. 3(165), 67–71, *American Mathematical Society, Providence, RI*, 1986.

[30] S. Goto; R. Takahashi; N. Taniguchi, Almost Gorenstein rings – towards a theory of higher dimension, *J. Pure Appl. Algebra* **219** (2015), no. 7, 2666–2712.

[31] D. Happel, Triangulated categories in the representation theory of finite-dimensional algebras, London Mathematical Society Lecture Note Series **119**, *Cambridge University Press, Cambridge*, 1988.

[32] M. J. Hopkins, Global methods in homotopy theory, *Homotopy theory (Durham, 1985)*, 73–96, London Math. Soc. Lecture Note Ser. **117**, *Cambridge Univ. Press, Cambridge*, 1987.

[33] M. J. Hopkins; J. H. Smith, Nilpotence and stable homotopy theory II, *Ann. of Math. (2)* **148** (1998), no. 1, 1–49.

[34] M. Hovey, Classifying subcategories of modules, *Trans. Amer. Math. Soc.* **353** (2001), no. 8, 3181–3191.

[35] C. Huneke; G. J. Leuschke, Two theorems about maximal Cohen–Macaulay modules, *Math. Ann.* **324** (2002), no. 2, 391–404.

[36] T. Kobayashi; J. Lyle; R. Takahashi, Maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum, *J. Pure Appl. Algebra* **224** (2020), no. 7, 106311, 29 pp.

[37] H. Krause, Thick subcategories of modules over commutative Noetherian rings (with an appendix by Srikanth Iyengar), *Math. Ann.* **340** (2008), no. 4, 733–747.

[38] H. Krause; G. Stevenson, A note on thick subcategories of stable derived categories, *Nagoya Math. J.* **212** (2013), 87–96.

[39] S. Kumashiro, The Auslander–Reiten conjecture for certain non-Gorenstein Cohen–Macaulay rings, `arXiv:1906.02669v2`.

[40] J. Lescot, La série de Bass d'un produit fibré d'anneaux locaux, *Paul Dubreil and Marie-Paule Malliavin algebra seminar, 35th year (Paris, 1982)*, 218–239, Lecture Notes in Math. **1029**, *Springer, Berlin*, 1983.

[41] G. J. Leuschke; R. Wiegand, Cohen–Macaulay representations, Mathematical Surveys and Monographs, **181**, *American Mathematical Society, Providence, RI*, 2012.

[42] J. Lyle; J. Montaño, Extremal growth of Betti numbers and trivial vanishing of (co)homology, *Trans. Amer. Math. Soc.* **373** (2020), no. 11, 7937–7958.

[43] J. Lyle; J. Montaño; S. K. Sather-Wagstaff, Exterior powers and Tor-persistence, `arXiv:2007.09174`.

[44] H. Matsumura, Commutative ring theory, Translated from the Japanese by M. Reid, Second edition, Cambridge Studies in Advanced Mathematics **8**, *Cambridge University Press, Cambridge*, 1989.

[45] S. Nasseh; S. Sather-Wagstaff; R. Takahashi; K. VandeBogert, Applications and homological properties of local rings with decomposable maximal ideals, *J. Pure Appl. Algebra* **223** (2019), no. 3, 1272–1287.

[46] S. Nasseh; R. Takahashi, Local rings with quasi-decomposable maximal ideal, *Math. Proc. Cambridge Philos. Soc.* **168** (2020), no. 2, 305–322.

[47] A. Neeman, The chromatic tower for $D(R)$, With an appendix by Marcel Bökstedt, *Topology* **31** (1992), no. 3, 519–532.

[48] A. Neeman, Triangulated categories, Annals of Mathematics Studies **148**, *Princeton University Press, Princeton, NJ*, 2001.

[49] D. O. Orlov, Triangulated categories of singularities and D-branes in Landau–Ginzburg models (Russian), *Tr. Mat. Inst. Steklova* **246** (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 240–262; translation in *Proc. Steklov Inst. Math.* **246** (2004), no. 3, 227–248.

[50] R. Rouquier, Dimensions of triangulated categories, *J. K-Theory* **1** (2008), no. 2, 193–256.

[51] J. D. Sally, Stretched Gorenstein rings, *J. London Math. Soc. (2)* **20** (1979), no. 1, 19–26.

[52] G. Stevenson, Subcategories of singularity categories via tensor actions, *Compos. Math.* **150** (2014), no. 2, 229–272.

[53] G. Stevenson, Duality for bounded derived categories of complete intersections, *Bull. Lond. Math. Soc.* **46** (2014), no. 2, 245–257.

[54] R. Takahashi, Syzygy modules with semidualizing or G-projective summands, *J. Algebra* **295** (2006), no. 1, 179–194.

[55] R. Takahashi, On G-regular local rings, *Comm. Algebra* **36** (2008), no. 12, 4472–4491.

[56] R. Takahashi, Modules in resolving subcategories which are free on the punctured spectrum, *Pacific J. Math.* **241** (2009), no. 2, 347–367.

[57] R. Takahashi, Classifying thick subcategories of the stable category of Cohen–Macaulay modules, *Adv. Math.* **225** (2010), no. 4, 2076–2116.

[58] R. Takahashi, Contravariantly finite resolving subcategories over commutative rings, *Amer. J. Math.* **133** (2011), no. 2, 417–436.

[59] R. Takahashi, Classifying resolving subcategories over a Cohen–Macaulay local ring, *Math. Z.* **273** (2013), no. 1-2, 569–587.

[60] R. Takahashi, Thick subcategories over Gorenstein local rings that are locally hypersurfaces on the punctured spectra, *J. Math. Soc. Japan* **65** (2013), no. 2, 357–374.

[61] R. Takahashi, Reconstruction from Koszul homology and applications to module and derived categories, *Pacific J. Math.* **268** (2014), no. 1, 231–248.

[62] R. Takahashi, Intersections of resolving subcategories and intersections of thick subcategories, *Eur. J. Math.* (2021), `https://doi.org/10.1007/s40879-021-00470-z`.

[63] R. Takahashi, Classification of dominant resolving subcategories by moderate functions, *Illinois J. Math.* (2021), Advance Publication 1–22, `https://doi.org/10.1215/00192082-9330146`.

[64] R. W. Thomason, The classification of triangulated subcategories, *Compos. Math.* **105** (1997), no. 1, 1–27.

[65] Y. Yoshino, Modules of G-dimension zero over local rings with the cube of maximal ideal being zero, *Commutative algebra, singularities and computer algebra (Sinaia, 2002)*, 255–273, NATO Sci. Ser. II Math. Phys. Chem. **115**, *Kluwer Acad. Publ., Dordrecht*, 2003.

[66] Y. Yoshino, A functorial approach to modules of G-dimension zero, *Illinois J. Math.* **49** (2005), no. 2, 345–367.

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602, Japan
E-mail address: `takahashi@math.nagoya-u.ac.jp`
URL: `https://www.math.nagoya-u.ac.jp/~takahashi/`