GENERATION IN MODULE CATEGORIES AND DERIVED CATEGORIES OF COMMUTATIVE RINGS

RYO TAKAHASHI

Abstract. Let R be a ring, and let M, N be R-modules. It is a natural question to ask whether or how one can build M out of N by iteration of fundamental operations such as direct sums, direct summands and extensions. It is possible to think of this question not only in module categories but also in derived categories. In this article we consider the question in the case where R is a commutative noetherian ring.

Contents

1. Generation problem

In this article, we consider the following problem.

Problem 1.1. Let R be a commutative noetherian ring. Let M, N be objects of the module category $\operatorname{mod} R$ (resp. the derived category $\mathsf{D}^{\mathsf{b}}(R)$). Then:

(1) Clarify whether M can be built out of N by taking short exact sequences (resp. exact triangles) etc.
(2) If M can be built out of N, then compute the number of required short exact sequences (resp. exact triangles).

Problem 1.1 naturally arises for the purpose to understand the structure of the module category $\operatorname{mod} R$ and the derived category $\mathsf{D}^{\mathsf{b}}(R)$. The author has been studying Problem 1.1 for more than ten years. Item (1) of Problem 1.1 will be done by *classifying* the subcategories closed under short exact sequences (resp. exact triangles) etc. The number appearing in item (2) of Problem 1.1 corresponds to *dimensions* of subcategories.

2010 *Mathematics Subject Classification.* 13C60, 13D09, 13H10.
Key words and phrases. module category, derived category, singularity category, contravariantly finite subcategory, resolving subcategory, thick subcategory, dimension, radius, Cohen–Macaulay ring, Gorenstein ring, complete intersection, hypersurface.
The author was partly supported by JSPS Grant-in-Aid for Scientific Research 19K03443.

The organization of this article is as follows. In Section 2, we recall the basic definitions and fundamental properties, which are used later. In Sections 3 and 4, we discuss classification and dimensions of subcategories, respectively.

2. Preliminaries

The following notation is used throughout this article.

Notation 2.1. (1) Let R be a commutative noetherian ring with identity.

(2) We denote by $\operatorname{mod} R$ the category of finitely generated R-modules. We denote by $\mathrm{D^b}(R)$ the bounded derived category of $\operatorname{mod} R$, that is, the derived category of bounded complexes of finitely generated R-modules.

(3) By module, we mean finitely generated module. By subcategory, we mean full subcategory closed under isomorphism.

(4) Recall that an R-module M is called *maximal Cohen–Macaulay* if

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \dim R_\mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. Here, the depth of the zero module over a local ring is ∞ by definition, so an R-module M is maximal Cohen–Macaulay if and only if $\operatorname{depth} M_\mathfrak{p} = \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Supp} M$. Denote by $\mathrm{MCM}(R)$ the subcategory of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay modules.

(5) Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Denote by $\operatorname{Spec}_0 R$ the *punctured spectrum* of R, namely,

$$\operatorname{Spec}_0 R = \operatorname{Spec} R \setminus \{\mathfrak{m}\}.$$

Denote by $\operatorname{Sing} R$ the *singular locus* of R, which is by definition the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is not regular. Denote by $\mu(-)$ the number of elements in the minimal system of generators, that is to say,

$$\mu(M) = \dim_k(M \otimes_R k)$$

for each R-module M. Denote by $\operatorname{edim} R$ the *embedding dimension* of R, i.e.,

$$\operatorname{edim} R = \mu(\mathfrak{m}) = \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Denote by $\operatorname{codim} R$ the *(embedding) codimension* of R, that is,

$$\operatorname{codim} R = \operatorname{edim} R - \operatorname{depth} R.$$

By $\mathrm{e}(-)$ we denote the (Hilbert–Samuel) multiplicity, namely,

$$\mathrm{e}(I) = \lim_{n\to\infty} \frac{d!}{n^d} \ell_R(R/I^{n+1})$$

for an $\mathfrak{m}$-primary ideal I of R, and set $\mathrm{e}(R) = \mathrm{e}(\mathfrak{m})$. By $\ell\ell(-)$ we denote the *Loewy length*, namely,

$$\ell\ell(M) = \inf\{n \geq 0 \mid \mathfrak{m}^n M = 0\}$$

for an R-module M. Note that $\ell\ell(M) < \infty$ if and only if M has finite length.

(6) For an additive category $\mathcal{C}$, the bounded (resp. right bounded) homotopy category is denoted by $\mathrm{K^b}(\mathcal{C})$ (resp. $\mathrm{K^-}(\mathcal{C})$), i.e., the homotopy category of bounded (resp. right bounded) complexes of objects in $\mathcal{C}$.

(7) For an abelian category $\mathcal{A}$, we denote by $\operatorname{proj}\mathcal{A}$ the subcategory of $\mathcal{A}$ consisting of projective objects, and we set $\operatorname{proj} R = \operatorname{proj}(\operatorname{mod} R)$.

(8) The *(first) syzygy* of an object $M \in \mathcal{A}$ is by definition the kernel of an epimorphism from a projective object of $\mathcal{A}$ to M, and denoted by ΩM. For an integer $n \geq 1$ we inductively define the nth syzygy of M by $\Omega^n M = \Omega(\Omega^{n-1}M)$, and set $\Omega^0 M = M$. For each $M \in \mathcal{A}$ and each $n \geq 0$ the object $\Omega^n M$ is uniquely determined up to direct summands which are projective objects.

(9) For an additive category $\mathcal{C}$ and a subcategory $\mathcal{X}$ of $\mathcal{C}$, the *additive closure* of $\mathcal{X}$ is defined as the smallest subcategory of $\mathcal{C}$ containing $\mathcal{X}$ and closed under finite direct sums and direct summands, and denoted by $\operatorname{add}\mathcal{X}$. Note that for an object $M \in \mathcal{A}$ one has

$$M \in \operatorname{add}\mathcal{X} \iff \begin{cases} \text{there exist a finite number of objects} \\ X_1, \dots, X_n \in \mathcal{X} \text{ such that } M \text{ is} \\ \text{(isomorphic to) a direct summand of} \\ \text{the direct sum } X_1 \oplus \cdots \oplus X_n. \end{cases}$$

When $\mathcal{X}$ consists of a single object X, we write $\operatorname{add} X$. Hence, we have

$$\operatorname{add} R = \operatorname{proj} R.$$

Next we recall the definition of a resolving subcategory.

Definition 2.2. Let $\mathcal{A}$ be an abelian category with enough projective objects. A subcategory $\mathcal{X}$ of $\mathcal{A}$ is called *resolving* if it satisfies the following conditions.

(a) $\mathcal{X}$ contains $\operatorname{proj}\mathcal{A}$.

(b) $\mathcal{X}$ is closed under direct summands. That is, every direct summand (in $\mathcal{A}$) of every $X \in \mathcal{X}$ belongs to $\mathcal{X}$.

(c) $\mathcal{X}$ is closed under extensions. That is, for an exact sequence

$$0 \to L \to M \to N \to 0$$

of objects of $\mathcal{A}$, if $L, N \in \mathcal{X}$, then $M \in \mathcal{X}$.

(d) $\mathcal{X}$ is closed under kernels of epimorphisms. That is, for an exact sequence

$$0 \to L \to M \to N \to 0$$

of objects of $\mathcal{A}$, if $M, N \in \mathcal{X}$, then $L \in \mathcal{X}$.

Remark 2.3. (1) Condition (d) in Definition 2.2 can be replaced with the following condition.

(d)' $\mathcal{X}$ is closed under syzygies. That is, for any $X \in \mathcal{X}$ one has $\Omega X \in \mathcal{X}$.

(2) When $\mathcal{A} = \operatorname{mod} R$, condition (a) in Definition 2.2 can be replaced with the following condition.

(a)' R belongs to $\mathcal{X}$.

(3) The subcategory $\operatorname{proj}\mathcal{A}$ is the smallest resolving subcategory of $\mathcal{A}$, while the biggest one is $\mathcal{A}$ itself.

Here are some examples of a resolving subcategory of the abelian category $\operatorname{mod} R$ with enough projective objects.

Example 2.4. (1) If R is a Cohen–Macaulay ring, then $\mathrm{MCM}(R)$ is a resolving subcategory of $\operatorname{mod} R$. (The converse also holds true.)

(2) Set $(-)^* = \mathrm{Hom}_R(-, R)$. Recall that an R-module M is called *totally reflexive* if the canonical map $M \to M^{**}$ is an isomorphism (i.e., M is reflexive) and

$$\mathrm{Ext}^i_R(M, R) = \mathrm{Ext}^i_R(M^*, R) = 0$$

for all positive integers i. The subcategory $\mathcal{G}(R)$ of $\operatorname{mod} R$ consisting of totally reflexive modules is resolving.

(3) Denote by $\operatorname{mod}_0 R$ the subcategory of $\operatorname{mod} R$ consisting of modules which are locally free on the punctured spectrum of R. Then $\operatorname{mod}_0 R$ is a resolving subcatgeory of $\operatorname{mod} R$.

Next we recall the definitions of thick subcategories of an abelian category and a triangulated category.

Definition 2.5. (1) Let $\mathcal{A}$ be an abelian category, and let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. A subcategory $\mathcal{X}$ of $\mathcal{C}$ is called *thick* if it satisfies the following conditions.

(a) $\mathcal{X}$ is closed under direct summands. That is, every direct summand (in $\mathcal{A}$) of every $X \in \mathcal{X}$ belongs to $\mathcal{X}$.

(b) $\mathcal{X}$ is closed under short exact sequences in $\mathcal{C}$. That is, for an exact sequence

$$0 \to L \to M \to N \to 0$$

in $\mathcal{A}$ with $L, M, N \in \mathcal{C}$, if two of L, M, N belong to $\mathcal{X}$, then so does the third.

(2) Let $\mathcal{T}$ be a triangulated category. A subcategory $\mathcal{T}$ of $\mathcal{X}$ is called *thick* if it satisfies the following conditions.

(a) $\mathcal{X}$ is closed under direct summands. That is, every direct summand (in $\mathcal{T}$) of every $X \in \mathcal{X}$ belongs to $\mathcal{X}$.

(b) $\mathcal{X}$ is closed under exact triangles. That is, for an exact triangle

$$L \to M \to N \to \Sigma L$$

in $\mathcal{T}$, if two of L, M, N belong to $\mathcal{X}$, then so does the third.

Remark 2.6. Every thick subcategory of the abelian category $\operatorname{mod} R$ that contains R is a resolving subcategory of $\operatorname{mod} R$.

Here are several examples of a thick subcategory.

Example 2.7. (1) The homotopy category $\mathrm{K}^{\mathsf{b}}(\operatorname{proj} R)$ of projective modules is a thick subcategory of the triangulated category $\mathrm{D}^{\mathsf{b}}(R)$.

(2) The category $\mathcal{G}(R)$ of totally reflexive modules is a thick subcategory of the category $\mathrm{MCM}(R)$ of maximal Cohen–Macaulay modules.

(3) Set

$$\mathrm{MCM}_0(R) = \mathrm{MCM}(R) \cap \operatorname{mod}_0(R).$$

Then $\mathrm{MCM}_0(R)$ is a thick subcategory of $\mathrm{MCM}(R)$.

(4) Denote by $\operatorname{fl} R$ (resp. $\operatorname{fpd} R$) the subcategory of $\operatorname{mod} R$ consisting of modules of finite length (resp. modules of finite projective dimension). Both $\operatorname{fl} R$ and $\operatorname{fpd} R$ are thick subcategories of $\operatorname{mod} R$.

Finally, we recall the definition of a singularity category.

Definition 2.8. The Verdier quotient

$$\mathrm{D}_{\mathrm{sg}}(R) = \frac{\mathrm{D}^{\mathsf{b}}(R)}{\mathrm{K}^{\mathsf{b}}(\operatorname{proj} R)}$$

of the derived category $\mathrm{D}^{\mathsf{b}}(R)$ by the homotopy category $\mathrm{K}^{\mathsf{b}}(\operatorname{proj} R)$ is called the *singularity category* or *stable derived category* of R. Note by definition that $\mathrm{D}_{\mathrm{sg}}(R)$ is a triangulated category as well.

The singularity category has been introduced by Buchweitz [20]. There are many studies on singularity categories by Orlov [45, 46, 47, 48] in connection with the Homological Mirror Symmetry Conjecture.

3. Classification of subcategories

The study of classification of subcategories has started by Gabriel [29] in the 1960s, who classified the Serre subcategories of the module category of a commutative noetherian ring. In the 1990s, Auslander and Reiten [7] classified the contravariantly finite resolving subcategories of the module category of an artin algebra of finite global dimension. In the 2000s, Hovey [32] classified the wide subcategories of the module category of the quotient of a regular coherent ring by a finitely generated ideal.

For triangulated categories, a lot of classification theorems have been obtained for thick subcategories. Devinatz, Hopkins and Smith [27] and Hopkins and Smith [31] classified the thick subcategories of compact objects in the stable homotopy category, and then Hopkins and Neeman [30, 42] classified the thick subcategories of the derived category of perfect complexes over a commutative noetherian ring. Thomason [57] extended this to quasi-compact quasi-separated schemes. Benson, Carlson and Rickard [15] classified the thick tensor ideals of the stable category of finite dimensional representations of a finite group. Benson, Iyengar and Krause [16] extended this to the derived category, while Friedlander and Pevtsova [28] and Benson, Iyengar, Krause and Pevtsova [17] extended it to finite group schemes.

Furthermore, Balmer [10] defined the Balmer spectrum of a tensor triangulated category, and classified the thick tensor ideals of a tensor triangulated category by using the topological structure of the Balmer spectrum. This result is the fundation of *tensor triangular geometry*, which was invented by Balmer himself and introduced in his ICM lecture [12]. This theory spreaded to commutative algebra, algebraic geometry, modular representation theory, stable homotopy theory, motif theory, noncommutative topology, symplectic geometry and so on, and various results have been obtained; see [9, 10, 11, 12, 13, 14] and references therein.

Thus, classification theory of subcategories is a research theme shared by a lot of areas of mathematics, and has been studied actively and widely through the interactions between those areas.

Here, we consider an example to explain how powerful classification of subcategories is.

Example 3.1. Let $R = k[x,y]$ be a polynomial ring in two variables x, y over a field k. For an R-module M we write[1]

$$\langle M \rangle = \left\{ N \in \operatorname{mod} R \;\middle|\; \begin{array}{c} N \text{ can be built out of } M \text{ by taking} \\ \text{direct summands, extensions and syzygies} \end{array} \right\}.$$

(1) There exists an exact sequence

$$0 \to (x,y)/(x^2,y) \to R/(x^2,y) \to R/(x,y) \to 0$$

of R-modules. Note that $(x,y)/(x^2,y)$ is isomorphic to $R/(x,y)$, and (x^2,y) is the first syzygy of $R/(x^2,y)$. Hence

$$R/(x^2,y) \in \langle R/(x,y) \rangle$$

follows.

(2) Suppose that $R/(xy)$ belongs to $\langle R/(x) \rangle$. Then localization at the prime ideal (y) of R shows that $(R/(xy))_{(y)}$ beolongs to $\langle (R/(x))_{(y)} \rangle$. Here, $(R/(xy))_{(y)}$ is isomorphic to the residue field $R_{(y)}/yR_{(y)}$, while we have $(R/(x))_{(y)} = 0$. It is deduced that $R_{(y)}/yR_{(y)}$ is a projective $R_{(y)}$-module, which is a contradiction. Thus,

$$R/(xy) \notin \langle R/(x) \rangle$$

follows.

(3) There exists an exact sequence

$$0 \to R/(xy) \xrightarrow{f} R/(x) \oplus R/(xy^2) \xrightarrow{g} R/(xy) \to 0 \tag{3.1.1}$$

of R-modules, where f and g are defined by

$$f(\overline{a}) = \begin{pmatrix} \overline{a} \\ \overline{ay} \end{pmatrix}, \quad g(\begin{pmatrix} \overline{b} \\ \overline{c} \end{pmatrix}) = \overline{c - by}$$

Thus

$$R/(x) \in \langle R/(xy) \rangle$$

follows.

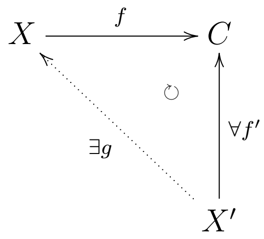
In general, it is quite difficult to find such an exact sequence as (3.1.1), and also there is no way to see at the beginning whether such an exact sequence exists or not. This problem will be settled if we can classify all the subcategories of $\operatorname{mod} R$ closed under direct summands, extensions and syzygies, that is to say, all the resolving subcategories of $\operatorname{mod} R$. We will actually do this later; see Example 3.22.

In what follows, we consider classifications of subcategories of the module category $\operatorname{mod} R$, the derived category $\mathrm{D^b}(R)$ and the singularity category $\mathrm{D_{sg}}(R)$ of a commutative noetherian ring R. We begin with recalling the definition of a contravariantly finite subcategory.

Definition 3.2. Let $\mathcal{C}$ be an additive category, and let $\mathcal{X}$ be a subcategory of $\mathcal{C}$.

[1]The notation $\langle - \rangle$ here is only to simply explain this example, which is different from the one appearing in Definition 4.1

(1) Let $f : X \to C$ be a morphism (in $\mathcal{C}$) from an object $X \in \mathcal{X}$ to an object $C \in \mathcal{C}$. We say that f is a *right $\mathcal{X}$-approximation* of C if for every object $X' \in \mathcal{X}$ and every morphism $f' : X' \to C$ there exists a morphism $g : X' \to X$ such that $f' = fg$.



(2) We say that $\mathcal{X}$ is *contravariantly finite* if every object of $\mathcal{C}$ admits a right $\mathcal{X}$-approximation.

Remark 3.3. (1) The name “contravariantly finite” comes from the fact that for each object $C \in \mathcal{C}$ the contravaiant functor $\mathrm{Hom}_{\mathcal{C}}(-, M)$ from $\mathcal{C}$ to the category of abelian groups is a finitely generated object of the functor category of $\mathcal{C}$.

(2) Dual notions also exist. Namely, a *left $\mathcal{X}$-approximation* and a *covariantly finite* subcategory are defined dually (but we do not use them in this article).

We state a couple of examples of a contravariantly finite subcategory.

Example 3.4. (1) Let X be an R-module. Then the additive closure $\mathrm{add}\, X$ is a contravariantly finite subcategory of $\mathrm{mod}\, R$.

Indeed, take any object $M \in \mathcal{C}$. Then $\mathrm{Hom}_R(X, M)$ is a finitely generated R-module. Choose a system of generators $f_1, \dots, f_n$ of $\mathrm{Hom}_R(X, M)$. Consider the homomorphism

$$f = (f_1, \dots, f_n) : X^{\oplus n} \to M.$$

The module $X^{\oplus n}$ belongs to $\mathrm{add}\, X$. Let $g : Y \to M$ be any homomorphism of R-modules such that $Y \in \mathrm{add}\, X$. Then Y is a direct summands of $X^{\oplus m}$ for some $m \geq 0$. Let

$$\pi = (\pi_1, \dots, \pi_m) : X^{\oplus m} \twoheadrightarrow Y$$

be a splitting of the inclusion map $\theta : Y \hookrightarrow X^{\oplus m}$. Then each $g\pi_i$ belongs to $\mathrm{Hom}_R(X, M)$, and

$$g\pi_i = \sum_{j=1}^{n} a_{ji} f_j$$

for some $a_{ji} \in R$. We have $g\pi = f \cdot A$, where $A = (a_{ij})$ is an $n \times m$ matrix. We get $g = g\pi\theta = fA\theta$, and thus g factors through f. This shows that f is a right $(\mathrm{add}\, X)$-approximation of M.

(2) Let R be a Cohen–Macaulay local ring with a canonical module. Then $\mathrm{MCM}(R)$ is a contravariantly finite subcategory of $\mathrm{mod}\, R$. This is a direct consequence of the so-called *Cohen–Macaulay approximation theorem* due to Auslander and Buchweitz [6].

To be more precise, let M be an R-module. Then the Cohen–Macaulay approximation theorem asserts that there exists an exact sequence

$$0 \to Y \to X \xrightarrow{f} M \to 0$$

of R-modules such that X is maximal Cohen–Macaulay and Y has finite injective dimension. We claim that the map f is a right $\mathrm{MCM}(R)$-approximation of M. In fact, let X' be any maximal Cohen–Macaulay R-module. Applying the functor $\mathrm{Hom}_R(X', -)$ to the above short exact sequence induces an exact sequence

$$\mathrm{Hom}_R(X', X) \xrightarrow{\mathrm{Hom}_R(X', f)} \mathrm{Hom}_R(X', M) \to \mathrm{Ext}^1_R(X', Y).$$

Since X' is maximal Cohen–Macaulay and Y has finite injective dimension, we have $\mathrm{Ext}^1_R(X', Y) = 0$. This implies that the map $\mathrm{Hom}_R(X', f)$ is surjective. Thus the claim follows.

The contravariantly finite resolving subcategories of the module category of a Gorenstein ring can be determined completely, as follows. In view of Remark 2.3 and Example 3.4 and 2.4, we observe that those three subcategories which appear in the theorem are contravariantly finite resolving subcategories.

Theorem 3.5 ([54, Theorem 1.2]). *Let R be a henselian local ring. If R is Gorenstein, then the contravariantly finite resolving subcategories of $\mathrm{mod}\, R$ are the following three subcategories of $\mathrm{mod}\, R$.*

$$\begin{cases} \mathrm{proj}\, R, \\ \mathrm{MCM}(R), \\ \mathrm{mod}\, R. \end{cases}$$

This theorem is a consequence of the following more complicated result. Here, pd_R and id_R stand for the projective dimension and the injective dimension, respectively. A typical example of an R-module G as below is a nonfree totally reflexive R-module, or more generally, an R-module of infinite projective dimension but of finite Gorenstein dimension in the sense of Auslander and Bridger [5].

Proposition 3.6 ([54, Theorem 1.3]). *Let R be a henselian local ring with residue field k. Let $\mathcal{X}$ be a resolving subcategory of $\mathrm{mod}\, R$ such that the R-module k has a right $\mathcal{X}$-approximation. Assume that there exists an R-module $G \in \mathcal{X}$ with $\mathrm{pd}_R G = \infty$ and $\mathrm{Ext}^i_R(G, R) = 0$ for $i \gg 0$. Let M be an R-module such that for each $X \in \mathcal{X}$ satisfies $\mathrm{Ext}^{\gg 0}_R(X, M) = 0$ for $i \gg 0$. Then $\mathrm{id}_R M < \infty$.*

This proposition together with the theorem called "Bass' conjecture" yields the following corollary, which deduces Theorem 3.5.

Corollary 3.7 ([54, Theorem 1.4]). *Let R be a henselian local ring. Let $\mathcal{X} \neq \mathrm{mod}\, R$ be a contravariantly finite resolving subcategory of $\mathrm{mod}\, R$. Assume that there exists an R-module $G \in \mathcal{X}$ with $\mathrm{pd}_R G = \infty$ and $\mathrm{Ext}^i_R(G, R) = 0$ for $i \gg 0$. Then R has to be Cohen–Macaulay, and one obtains an equality $\mathcal{X} = \mathrm{MCM}(R)$.*

This corollary yields as a by product another proof of the following result due to Christensen, Piepmeyer, Striuli and the author [21].

Corollary 3.8 ([54, Corollary 1.5]). *Let R be a complete local ring over an algebraically closed field of characteristic zero. Then the following are equivalent.*

(1) *The local ring R is a simple hypersurface singularity.*

(2) *There exist at least one but only finitely many isomorphism classes of nonfree indecomposable totally reflexive R-modules.*

Sketch of Proof of Corollary 3.8. Suppose that there exist only finitely many isomorphism classes of indecomposable totally reflexive R-modules. Then there exists a totally reflexive R-module G such that $\mathcal{G}(R) = \operatorname{add} G$, and Example 3.4(1) implies that the resolving subcategory $\mathcal{G}(R)$ of $\operatorname{mod} R$ is contravariantly finite. Applying Corollary 3.7, we observe that R is Gorenstein and $\mathcal{G}(R) = \operatorname{MCM}(R)$. Hence R has finite representation type. It is known that a Gorenstein complete local ring of finite representation type over an algebraically closed field of characteristic zero is nothing but a simple hypersurface singularity. ■

To state our next result, we recall the definitions of several notions.

Definition 3.9. (1) Let I be an ideal of R. We say that I is *quasi-decomposable* if I contains an R-regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ such that the R-module $I/(\boldsymbol{x})$ is decomposable.

(2) Let X be a subset of $\operatorname{Spec} R$. We say that X is *specialization-closed* if for every $\mathfrak{p} \in X$ and every $\mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$ one has $\mathfrak{q} \in X$. It is well-known and easy to see that X is specialization-closed if and only if it is a (possibly infinite) union of closed subsets of $\operatorname{Spec} R$ in the Zariski topology.

(3) Let $\mathbb{P}$ be a property of local rings. Let X be a subset of $\operatorname{Spec} R$. We say that X satisfies $\mathbb{P}$ if for all $\mathfrak{p} \in X$ the local ring $R_\mathfrak{p}$ satisfies the property $\mathbb{P}$.

(4) Let $(R, \mathfrak{m})$ be a local ring, and let I be an ideal of R. We say that I is a *Burch ideal* if $\mathfrak{m} I \neq \mathfrak{m}(I : \mathfrak{m})$. We call R a *Burch ring* if there exist a maximal $\widehat{R}$-regular sequence $\boldsymbol{x} = x_1, \dots, x_t$, a regular local ring S and a Burch ideal J of S such that $\widehat{R}/(\boldsymbol{x}) \cong S/J$. Here, $\widehat{R}$ stands for the $\mathfrak{m}$-adic completion of R.

(5) Let R be a Cohen–Macaulay local ring. Then, as is well-known (and easy to see), the inequality

$$\mathrm{e}(R) \geq \operatorname{codim} R + 1$$

holds. We say that R has *minimal multiplicity* if the equality holds. When the residue field of R is infinite, R has minimal multiplicity if and only if there exists a parameter ideal Q of R such that $\mathfrak{m}^2 = Q\mathfrak{m}$.

The following classification theorem on resolving subcategories and thick subcategories holds.

Theorem 3.10 ([22, 41, 53, 55])**.** *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring. Suppose that it satisifies one of the following three conditions.*

(a) *The local ring R is a hypersurface.*
(b) *The maximal ideal $\mathfrak{m}$ of R is quasi-decomposable, and* $\operatorname{Spec}_0 R$ *is either a hypersurface or has minimal multiplicity.*
(c) *The local ring R is a Burch ring, and* $\operatorname{Spec}_0 R$ *is either a hypersurface or has minimal multiplicity.*

Then there are one-to-one correspondences:

$$
\begin{aligned}
&\left\{\begin{matrix}\textit{Thick subcategories of } \mathrm{MCM}(R)\\ \textit{containing } R\end{matrix}\right\}\\
\xleftrightarrow{1-1}\ &\left\{\begin{matrix}\textit{Thick subcategories of } \operatorname{mod} R\\ \textit{containing } R\end{matrix}\right\}\\
\xleftrightarrow{1-1}\ &\left\{\begin{matrix}\textit{Thick subcategories of } \mathrm{D^b}(R)\\ \textit{containing } R\end{matrix}\right\}\\
\xleftrightarrow{1-1}\ &\left\{\begin{matrix}\textit{Resolving subcategories of } \operatorname{mod} R\\ \textit{contained in } \mathrm{MCM}(R)\end{matrix}\right\}\\
\xleftrightarrow{1-1}\ &\left\{\textit{Thick subcategories of } \mathrm{D_{sg}}(R)\right\}\\
\xleftrightarrow{1-1}\ &\left\{\begin{matrix}\textit{Specialization-closed subsets of } \operatorname{Spec} R\\ \textit{contained in } \operatorname{Sing} R\end{matrix}\right\}.
\end{aligned}
$$

A local ring with quasi-decomposable maximal ideal is nothing but a local ring that deforms to a fiber product over the residue field. The class of local rings satisfying conditions (b) and (c) in Theorem 3.10 contains the class of Cohen–Macaulay local rings with minimal multiplicity, so that it contains the class of non-Gorenstein rational singularities of dimension two.

Theorem 3.10 can be thought of as a higher-dimensional version of the theorem of Benson, Carlson and Rickard which is mentioned before. The bijections giving the one-to-one correspondences can be described explicitly.

Key roles are played in the proof of the above theorem by the following two results.

Lemma 3.11 ([22, Proposition 7.6], [41, Lemma 4.4], [53, Proposition 5.9]). *Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring of dimension d. Suppose that it satisifies one of the following three conditions.*

(a) *The local ring R is a hypersurface.*
(b) *The maximal ideal $\mathfrak{m}$ is quasi-decomposable, and $\operatorname{Spec}_0 R$ is either a hypersurface or has minimal multiplicity.*
(c) *The local ring R is a Burch ring, and $\operatorname{Spec}_0 R$ is either a hypersurface or has minimal multiplicity.*

Let M be a nonfree maximal Cohen–Macaulay R-module. Then the d-th syzygy $\Omega^d k$ of the R-module k belongs to the resolving closure of M.

Lemma 3.12 ([53, Theorem 2.4]). *Let R be a Cohen–Macaulay local ring of dimension d. Let M be an R-module of depth t. Assume that M is locally free on the punctured spectrum of R. Then M belongs to the extension closure of the R-module $\bigoplus_{i=t}^{d} \Omega^i k$.*

Here, the *resolving closure* of an R-module M means the smallest resolving subcategory of $\operatorname{mod} R$ containing M. The *extension closure* of M means the smallest subcategory of $\operatorname{mod} R$ which contains M and is closed under direct summands and extensions.

Applying the above lemmas, we can also improve a theorem of Keller, Murfet and Van den Bergh [38] on maximal Cohen–Macaulay modules over a completion, and recover a theorem of Huneke and Wiegand [35] and a theorem of Nasseh and Sather-Wagstaff [40] on rigidity of vanishing of Tor. Recall that a local ring R is said to have an *isolated singularity* if $R_{\mathfrak{p}}$ is a regular local ring for all nonmaximal prime ideals $\mathfrak{p}$ of R.

Corollary 3.13 (Keller, Murfet and Van den Bergh, [53, Corollary 3.8]). *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring whose $\mathfrak{m}$-adic completion $\widehat{R}$ has an isolated singularity (e.g., let R be an excellent Cohen–Macaulay local ring with an isolated singularity). Then the natural functor*

$$\mathrm{D_{sg}}(R) \to \mathrm{D_{sg}}(\widehat{R})$$

is an equivalence up to direct summands.

Corollary 3.14 (Huneke and Wiegand, [53, Corollary 7.3]). *Let R be a hypersurface local ring. Let M and N be R-modules. Suppose that*

$$\mathrm{Tor}_n^R(M, N) = \mathrm{Tor}_{n+1}^R(M, N) = 0$$

for some $n \geq 0$. Then either M or N has finite projective dimension.

Corollary 3.15 (Nasseh and Sather-Wagstaff, [41, Corollary 6.2]). *Let $R = S \times_k T$ be a fiber product, where S and T are local rings with common residue field k and $S \neq k \neq T$. Let M and N be R-modules.*

(1) *Assume that either S or T has depth zero and*

$$\mathrm{Tor}_n^R(M, N) = 0$$

for some $n \geq 5$. Then either M or N is free.

(2) *Assume that*

$$\mathrm{Tor}_n^R(M, N) = \mathrm{Tor}_{n+1}^R(M, N) = 0$$

for some $n \geq 5$. Then either $\mathrm{pd}_R M \leq 1$ or $\mathrm{pd}_R N \leq 1$.

The Ext version of the above corollary is also obtained; see [41, Corollary 6.3]. Furthermore, we can get similar vanishing results on Tor and Ext for local rings with quasi-decomposable maximal ideal and for Burch rings; see [41, Corollaries 6.5 and 6.6] and [22, Corollary 7.13 and Remark 7.14].

Stevenson [51, 52] classified the thick subcategories of the singularity category and the derived category of a complete intersection (more precisely, a quotient of a regular ring by a regular sequence), using Theorem 3.10(a) and a theorem of Orlov [46]. In the following, we explain Stevenson's classification theorem of the thick subcategories of the singularity category.

Let R be the residue ring of a regular local ring $(S, \mathfrak{n})$ by an S-regular sequence $\boldsymbol{x} = x_1, \dots, x_c$. We may assume that the x_i are all in $\mathfrak{n}^2$, so that $c = \operatorname{codim} R$. Then the *generic hypersurface* of R is defined as the graded ring

$$G = \frac{S[y_1, \dots, y_c]}{(x_1 y_1 + \cdots + x_c y_c)},$$

where $y_1, \dots, y_c$ are indeterminates over S with degree 1 and the elements of S have degree 0. The classification theorem of Stevenson is stated as follows.

Theorem 3.16 (Stevenson). *Let R be the quotient of a regular local ring S by an S-regular sequence $\boldsymbol{x} = x_1, \dots, x_c$. Let G be the generic hypersurface of R. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Thick subcategories}\\ \textit{of } \mathrm{D_{sg}}(R)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Specialization-closed subsets}\\ \textit{of the singular locus of } \operatorname{Proj} G\end{matrix}\right\}.$$

To state our next theorem, we need to introduce a certain $\mathbb{N}$-valued function on the set of prime ideals.

Definition 3.17. A function $f : \operatorname{Spec} R \to \mathbb{N}$ is called *grade-consistent* if it satisfies the following two conditions.

(1) For all prime ideals $\mathfrak{p}, \mathfrak{q}$ of R with $\mathfrak{p} \subseteq \mathfrak{q}$, one has $f(\mathfrak{p}) \le f(\mathfrak{q})$.
(2) For all prime ideals $\mathfrak{p}$ of R one has $f(\mathfrak{p}) \le \operatorname{grade} \mathfrak{p}$.

Using grade-consistent functions and specialization-closed subsets of the singular locus of $\operatorname{Proj} G$ where G is the generic hypersurface, we can completely classify the resolving subcategories of the category of finitely generated modules over a local complete intersection.

Theorem 3.18 ([24, Theorem 1.5]). *Let R be a quotient of a regular local ring by a regular sequence. Let G be the generic hypersurface of R. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Resolving subcategories}\\ \textit{of } \operatorname{mod} R\end{matrix}\right\}$$
$$\overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Grade-consistent functions}\\ \textit{on } \operatorname{Spec} R\end{matrix}\right\} \times \left\{\begin{matrix}\textit{Specialization-closed subsets}\\ \textit{of the singular locus of } \operatorname{Proj} G\end{matrix}\right\}.$$

The bijections giving the one-to-one correpondence in the above theorem can be described explicitly.

Let us explain a bit how to obtain Theorem 3.18. It is a a consequence of the combination of the following Propositions 3.19 and 3.20 with Stevenson's Theorem 3.16. One can view Proposition 3.20 as a category version of the Cohen–Macaulay approximation theorem due to Auslander and Buchweitz [6].

Proposition 3.19 ([24, Theorem 1.2]). *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Resolving subcategories}\\ \textit{of } \operatorname{mod} R\\ \textit{contained in } \operatorname{fpd} R\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Grade-consistent functions}\\ \textit{on } \operatorname{Spec} R\end{matrix}\right\}.$$

Proposition 3.20 ([24, Theorem 7.4]). *Let R be a locally complete intersection ring. There exists a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Resolving subcategories}\\ \textit{of } \operatorname{mod} R\end{matrix}\right\}$$
$$\overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Resolving subcategories}\\ \textit{of } \operatorname{mod} R\\ \textit{contained in } \operatorname{fpd} R\end{matrix}\right\} \times \left\{\begin{matrix}\textit{Resolving subcategories}\\ \textit{of } \operatorname{mod} R\\ \textit{contained in } \operatorname{MCM}(R)\end{matrix}\right\}.$$

Applying Proposition 3.19, we also obtain the following corollary.

Corollary 3.21 ([24, Theorem 1.7]). *The following are equivalent for two modules M and N over a regular ring R.*

(1) *One can build N out of M by taking direct summands, extensions and syzygies.*
(2) *One has $\operatorname{pd}_{R_\mathfrak{p}} N_\mathfrak{p} \le \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p}, 0\}$ for each prime ideal $\mathfrak{p}$ of R.*

Corollary 3.21 recovers and categorifies the main theorem of the ICM lecture of Auslander [3] in 1962. Also, it gives an answer to Problem 1.1.

Now Example 3.1 can be explained as follows by using the above corollary.

Example 3.22. Let $R = k[x, y]$ be the polynomial ring in two variables x, y over a field k.

(1) Put $M = R/(x, y)$ and $N = (x^2, y)$. Consider the maximal ideal $\mathfrak{m} = (x, y)$ of R. Then it holds that

$$\operatorname{pd}_{R_\mathfrak{m}} N_\mathfrak{m} = 1 \le 2 = \operatorname{pd}_{R_\mathfrak{m}} M_\mathfrak{m},$$
$$\operatorname{pd}_{R_\mathfrak{p}} N_\mathfrak{p} = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \text{ with } \mathfrak{p} \neq \mathfrak{m}.$$

It is observed that

$$\operatorname{pd}_{R_\mathfrak{p}} N_\mathfrak{p} \le \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p}, 0\}$$

for all prime ideals $\mathfrak{p}$ of R. By virtue of Corollary 3.21, we see that N can be built out of M by taking direct summands, extensions and syzygies.

(2) Put $M = R/(x)$ and $N = R/(xy)$. Consider the prime ideal $\mathfrak{p} = (y)$ of R. We see that $\operatorname{pd}_{R_\mathfrak{p}}(R/(xy))_\mathfrak{p} = 1$, while $\operatorname{pd}_{R_\mathfrak{p}}(R/(x))_\mathfrak{p} = -\infty$ as $\mathfrak{p}$ does not belong to the support of $R/(x)$. It follows that

$$\operatorname{pd}_{R_\mathfrak{p}} N_\mathfrak{p} \not\le \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p}, 0\}.$$

Applying Corollary 3.21, we observe that N cannot be built out of M by taking direct summands, extensions and syzygies.

(3) Put $M = R/(xy)$ and $N = R/(x)$. If $\mathfrak{p}$ is a prime ideal of R with $\operatorname{pd}_{R_\mathfrak{p}}(R/(x))_\mathfrak{p} = 1$, then we must have $\mathfrak{p} = (x)$, and $\operatorname{pd}_{R_\mathfrak{p}}(R/(xy))_\mathfrak{p} = 1$. It is easy to observe from this that

$$\operatorname{pd}_{R_\mathfrak{p}} N_\mathfrak{p} \le \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p}, 0\}$$

for all prime ideals $\mathfrak{p}$ of R. Thanks to Corollary 3.21, we see that N can be built out of M by taking direct summands, extensions and syzygies.

As the final topic of this section, we consider classification of subcategories of the category $\mathrm{D}^-(R)$, the right bounded derived category of $\operatorname{mod} R$, that is, the derived category of right bounded complexes of finitely generated R-modules. This is a tensor triangulated category with tensor product $- \otimes_R^{\mathbf{L}} -$. The category $\mathrm{D}^-(R)$ is equivalent as a tensor triangulated category to the homotopy category $\mathrm{K}^-(\operatorname{proj} R)$. We define a *compact ideal* of $\mathrm{D}^-(R)$ as a thick tensor ideal (i.e., a thick subcategory closed under $X \otimes_R^{\mathbf{L}} -$ for each $X \in \mathrm{D}^-(R)$) generated by bounded complexes. We can completely classify the compact ideals of $\mathrm{D}^-(R)$.

Theorem 3.23 ([39, Theorem A]). *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Compact ideals}\\ \textit{of } \mathrm{D}^{-}(R)\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\textit{Specialization-closed subsets}\\ \textit{of } \operatorname{Spec} R\end{matrix}\right\}.$$

Denote by $\mathrm{D}^{\mathrm{perf}}(R)$ the derived category of *perfect complexes* over R, that is, bounded complexes of finitely generated projective R-modules, or in other words, complexes of finite projective dimension. The category $\mathrm{D}^{\mathrm{perf}}(R)$ is also a tensor triangulated category with tensor product $-\otimes_R^{\mathbf{L}}-$. The category $\mathrm{D}^{\mathrm{perf}}(R)$ is equivalent as a tensor triangulated category to $\mathrm{K}^{\mathsf{b}}(\operatorname{proj} R)$. Restricting the above theorem, we recover the celebrated Hopkins–Neeman theorem [42, Theorem 1.5] stated below.

Corollary 3.24 (Hopkins–Neeman). *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Thick subcategories}\\ \textit{of } \mathrm{D}^{\mathrm{perf}}(R)\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\textit{Specialization-closed subsets}\\ \textit{of } \operatorname{Spec} R\end{matrix}\right\}.$$

The proof of Theorem 3.23 also extends the Hopkins–Neeman smash nilpotence theorem on $\mathrm{K}^{\mathsf{b}}(\operatorname{proj} R) \cong \mathrm{D}^{\mathrm{perf}}(R)$ to $\mathrm{K}^{-}(\operatorname{proj} R) \cong \mathrm{D}^{-}(R)$. For the details, we refer the reader to [39, Theorem 2.7].

4. Dimensions of subcategories

The notion of the dimension of a triangulated category has been introduced by Rouquier [50]. Bondal and Van den Bergh [19] proved that the bounded derived category of coherent sheaves on a smooth proper commutative/noncommutative algebraic variety has finite dimension, and by using it proved that a contravariant cohomological functor of finite type to the category of vector spaces is representable. Rouquier [49] applied the notion of the dimension of a triangulated category to representation dimension. Representation dimension has been introduced by Auslander [4] to measure how far a given artin algebra is from finite representation type, and many representation theorists including Oppermann [44] have investigated it so far. Rouquier computed the dimension of the singularity category of an exterior algebra of a vector space to give the first example of an artinian ring of representation dimension more than three.

On the other hand, Rouquier [50] proved that the bounded derived category of coherent sheaves on a separated scheme of finite type over a perfect field has finite dimension. Recently, Neeman [43] proved that the bounded derived category of coherent sheaves on a separated scheme that is essentially of finite type over a separated excellent scheme of dimension at most two has finite dimension. This clarifies that even in the mixed characteristic case the derived category has finite dimension in many cases.

In what follows, we consider the dimensions of the derived category $\mathrm{D}^{\mathsf{b}}(R)$ and the singularity category $\mathrm{D}_{\mathrm{sg}}(R)$, and analogues for abelian categories. We begin with stating the definitions of the dimension and radius of a subcategory of a triangulated category or an abelian category.

Definition 4.1. Let $\mathcal{T}$ be a triangulated category.

(1) For a subcategory $\mathcal{X}$ of $\mathcal{T}$ we denote by $\langle \mathcal{X} \rangle$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under finite direct sums, direct summands and shifts. That is,
$$\langle \mathcal{X} \rangle = \operatorname{add}\{\Sigma^i X \mid i \in \mathbb{Z},\, X \in \mathcal{X}\}.$$
When $\mathcal{X}$ consists of a single object X, we simply write $\langle X \rangle$.

(2) For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{T}$, we denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects M admitting an exact triangle
$$X \to M \to Y \to \Sigma X$$
with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We set $\mathcal{X} \diamond \mathcal{Y} = \langle \langle \mathcal{X} \rangle * \langle \mathcal{Y} \rangle \rangle$.

(3) Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$, and set
$$\langle \mathcal{C} \rangle_r = \begin{cases} 0 & (r = 0), \\ \langle \mathcal{C} \rangle & (r = 1), \\ \langle \mathcal{C} \rangle_{r-1} \diamond \mathcal{C} = \langle \langle \mathcal{C} \rangle_{r-1} * \langle \mathcal{C} \rangle \rangle & (r \geq 2). \end{cases}$$
When $\mathcal{C}$ consists of a single object C, we simply write $\langle C \rangle_r$.

(4) Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$. We define the *dimension* and *radius* of $\mathcal{X}$ as follows.
$$\dim \mathcal{X} = \inf\{n \geq 0 \mid \mathcal{X} = \langle G \rangle_{n+1} \text{ for some } G \in \mathcal{T}\}$$
$$\operatorname{radius} \mathcal{X} = \inf\{n \geq 0 \mid \mathcal{X} \subseteq \langle G \rangle_{n+1} \text{ for some } G \in \mathcal{T}\}$$

Definition 4.2. Let $\mathcal{A}$ be an abelian category with enough projective objects.

(1) For a subcategory $\mathcal{X}$ of $\mathcal{A}$, we denote by $[\mathcal{X}]$ the smallest subcategory of $\mathcal{A}$ containing $\operatorname{proj} \mathcal{A}$ and $\mathcal{X}$ and closed under finite direct sums, direct summands and syzygies. That is,
$$[\mathcal{X}] = \operatorname{add}(\operatorname{proj} \mathcal{A} \cup \{\Omega^i X \mid i \geq 0,\, X \in \mathcal{X}\}).$$
When $\mathcal{X}$ consists of a single object X, we simply write $[X]$.

(2) For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{A}$, we denote by $\mathcal{X} \circ \mathcal{Y}$ the subcategory of $\mathcal{A}$ consisting of objects $M \in \mathcal{A}$ admitting a short exact sequence
$$0 \to X \to M \to Y \to 0$$
with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We put $\mathcal{X} \bullet \mathcal{Y} = [[\mathcal{X}] \circ [\mathcal{Y}]]$.

(3) Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$, and put
$$[\mathcal{C}]_r = \begin{cases} 0 & (r = 0), \\ [\mathcal{C}] & (r = 1), \\ [\mathcal{C}]_{r-1} \bullet \mathcal{C} = [[\mathcal{C}]_{r-1} \circ [\mathcal{C}]] & (r \geq 2). \end{cases}$$
When $\mathcal{C}$ consists of a single object C, we simply write $[C]_r$.

(4) Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. We define the *dimension* and *radius* of $\mathcal{X}$ as follows.
$$\dim \mathcal{X} = \inf\{n \geq 0 \mid \mathcal{X} = [G]_{n+1} \text{ for some } G \in \mathcal{A}\}$$
$$\operatorname{radius} \mathcal{X} = \inf\{n \geq 0 \mid \mathcal{X} \subseteq [G]_{n+1} \text{ for some } G \in \mathcal{A}\}$$

The following theorem describes the relationship between the dimension of a subcategory and an isolated singularity.

Theorem 4.3 ([25, Theorem 1.1]). *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring. Consider the following four conditions.*

(a) *The subcategory* $\mathrm{MCM}_0(R)$ *of the abelian category* $\operatorname{mod} R$ *has finite dimension.*
(b) *The ideal*

$$\bigcap_{i>0} \bigcap_{M,N \in \mathrm{MCM}_0(R)} \operatorname{Ann}_R \operatorname{Ext}_R^i(M, N)$$

of the local ring R is $\mathfrak{m}$-primary.
(c) *The ideal*

$$\bigcap_{i>0} \bigcap_{M,N \in \mathrm{MCM}_0(R)} \operatorname{Ann}_R \operatorname{Tor}_i^R(M, N)$$

of the local ring R is $\mathfrak{m}$-primary.
(d) *The local ring R is an isolated singularity.*

Then the implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) *hold. If R is equicharacteristic and excellent, then the four conditions* (a), (b), (c), (d) *are equivalent.*

When R is Gorenstein, a similar assertion holds for the stable category $\underline{\mathrm{MCM}_0}(R)$ of $\mathrm{MCM}_0(R)$, which is a triangulated category.

Let $\mathcal{A}$ be an abelian category with enough projective objects. By definition, there is an inequality

$$\operatorname{radius} \mathcal{X} \leq \dim \mathcal{X}$$

for all subcategories $\mathcal{X}$ of $\mathcal{A}$. Applying the above theorem, we see that the equality does not necessarily hold.

Example 4.4. Let $R = k[\![x, y]\!]/(x^2)$ be a homomorphic image of a formal power series ring over a field k. Then for the prime ideal $\mathfrak{p} = (x)$ the local ring $R_\mathfrak{p}$ is not regular, so R does not have an isolated singularity. According to Theorem 4.3, the subcategory $\mathrm{MCM}_0(R)$ of $\operatorname{mod} R$ has infinite dimension. On the other hand, it is observed from [2, Theorem 1.1] that $\mathrm{MCM}(R)$ has dimension (at most) one. Hence $\mathrm{MCM}_0(R)$ has radius (at most) one, and in particular, the strict inequality

$$\operatorname{radius} \mathrm{MCM}_0(R) < \dim \mathrm{MCM}_0(R)$$

holds.

Applying Theorem 4.3 to the case where $\mathrm{MCM}_0(R)$ has dimension zero, we immediately obtain the following corollary.

Corollary 4.5 ([25, Corollary 1.2]). *Let R be a Cohen–Macaulay local ring. Suppose that the number*

$$\# \left\{ M \in \mathrm{MCM}(R) \;\middle|\; \begin{array}{c} M \textit{ is indecomposable, and} \\ M \textit{ is locally free on the puctured spectrum of } R \end{array} \right\} \Big/ \cong$$

is finite. Then R is an isolated singularity.

In fact, under the assumption of the above corollary, we can choose a finite number of modules $M_1, \dots, M_n \in \mathrm{MCM}_0(R)$ whose isomorphism classes form those of the indecomposable maximal Cohen–Macaulay R-modules that are locally free on the puctured spectrum of R. Then setting

$$M = M_1 \oplus \cdots \oplus M_n,$$

we observe that $\mathrm{MCM}_0(R) = [M] = [M]_1$. Hence we obtain $\dim \mathrm{MCM}_0(R) = 0 < \infty$. Applying Theorem 4.3, we deduce that the Cohen–Macaulay local ring R has an isolated singularity.

Corollary 4.5 improves the following celebrated theorem [34].

Corollary 4.6 (Auslander–Huneke–Leuschke–Wiegand)**.** *Let R be a Cohen–Macaulay local ring. Suppose that R has finite representation type. Then R is of an isolated singularity.*

Recall that a Cohen–Macaulay ring R is said to have *finite representation type* provided that there exist only a finite number of isomorphism classes of indecomposable maximal Cohen–Macaulay modules over R.

Concerning the radius of a resolving subcategory, we have the following conjecture.

Conjecture 4.7. Let R be a Cohen–Macaulay local ring. Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. Suppose that $\mathcal{X}$ has finite radius. Then $\mathcal{X}$ is contained in the subcategory $\mathrm{MCM}(R)$ of maximal Cohen–Macaulay modules.

This conjecture holds true in the case where R is a complete intersection.

Theorem 4.8 ([23, Theorem I])**.** *Let R be a local complete intersection. Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$*. If $\mathcal{X}$ has finite radius, then all the modules belonging to $\mathcal{X}$ are maximal Cohen–Macaulay.*

The proof of this theorem is long and contains a lot of ideas. Here we would like to explain roughly how the theorem is proved. Recall that the *(Auslander) transpose* of an R-module M, which is denoted by $\operatorname{Tr} M$, is defined as follows. Take an exact sequence

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

with $P_0, P_1 \in \operatorname{proj} R$. Then $\operatorname{Tr} M$ is by definition the cokernel of the R-dual f^* of the map f. Hence there is an exact sequence

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{Tr} M \to 0.$$

Sketch of Proof of Theorem 4.8. Let $(R, \mathfrak{m})$ be a complete intersection local ring of dimension d. We may assume $d > 0$. Suppose that $\mathcal{X}$ contains an R-module M which is not maximal Cohen–Macaulay. It follows from [18] that M has reducible complexity, and using this, we observe that the resolving closure of M contains an R-module N such that

$$0 < \mathrm{pd}_R N < \infty.$$

Hence N belongs to $\mathcal{X}$. Using a technique given in [56], we may assume that N is locally free on the punctured spectrum of R. Further replacing it with a syzygy, we may assume

$\mathrm{pd}_R N = 1$. Note that $\mathrm{Ext}_R^1(N,R)$ is a nonzero R-module with finite length. We find a nonzero element σ in the socle of $\mathrm{Ext}_R^1(N,R)$. We get a short exact sequence

$$\sigma : 0 \to R \to L \to N \to 0.$$

Since $\mathcal{X}$ is resolving, it contains L. An exact sequence

$$0 \to k \to \mathrm{Ext}_R^1(N,R) \to \mathrm{Ext}_R^1(L,R) \to 0$$

is induced, which shows

$$\ell_R(\mathrm{Ext}_R^1(L,R)) = \ell_R(\mathrm{Ext}_R^1(N,R)) - 1.$$

It is observed that one may assume $\mathrm{Ext}_R^1(N,R) \cong k$. There are isomorphisms $\mathrm{Tr}\, N \cong \mathrm{Ext}_R^1(N,R) \cong k$, and hence $\mathrm{Tr}\, k = N \in \mathcal{X}$. Therefore $\mathrm{Tr}\, K$ belongs to $\mathcal{X}$ for all R-modules K of finite length. In particular,

$$\mathrm{Tr}(R/\mathfrak{m}^i) \in \mathcal{X}$$

for all $i > 0$.

Suppose that $\mathcal{X}$ has finite radius. Then there exist an R-module G and an integer $n > 0$ such that $\mathcal{X} \subseteq [G]_n$. The module $\mathrm{Tr}(R/\mathfrak{m}^i)$ belongs to $[G]_n$ for all $i > 0$. We may assume that R is complete. We see that

$$\begin{aligned} \mathfrak{m}^i &= \mathrm{Ann}_R\, R/\mathfrak{m}^i \\ &= \mathrm{Ann}_R\, \mathrm{Ext}_R^1(\mathrm{Tr}\, R/\mathfrak{m}^i, R) \\ &\supseteq \bigcap_{t>0} \mathrm{Ann}_R\, \mathrm{Ext}_R^t(\mathrm{Tr}(R/\mathfrak{m}^i), R) \\ &\supseteq (\mathrm{Ann}_R\, \mathrm{Ext}_R^j(G,R))^n \text{ for all } 1 \le j \le d. \end{aligned}$$

Applying Krull's intersection theorem, we observe that $\mathrm{Ann}_R\, \mathrm{Ext}_R^j(G,R)$ is nilpotent, and contained in every minimal prime ideal $\mathfrak{p}$ of R. It follows that $\mathrm{Ext}_{R_\mathfrak{p}}^j(G_\mathfrak{p}, R_\mathfrak{p}) \neq 0$ for all $1 \le j \le d$. This contradicts the fact that $R_\mathfrak{p}$ is an artinian Gorenstein ring. ■

The above proof actually shows that Theorem 4.8 holds for every local ring R and an R-module M of finite complete intersection dimension. As a corollary of this statement, we get the following result.

Corollary 4.9. *Let R be a Gorenstein local ring. Consider the following six conditions.*

(1) *The ring R is a hypersurface.*
(2) *The ring R is a complete intersection.*
(3) *Every resolving subcategory in* $\mathrm{MCM}(R)$ *is closed under R-duals.*
(4) *Every resolving subcategory in* $\mathrm{MCM}(R)$ *is closed under cosyzygies.*
(5) *The ring R is AB.*
(6) *The ring R satisfies Conjecture 4.7.*

Then the implications

$$\begin{array}{ccccccc} & & (2) & & & & (5) \\ & \nearrow & & \searrow & & \nearrow & \\ (1) & & & & (4) & & \\ & \searrow & & \nearrow & & \searrow & \\ & & (3) & & & & (6) \end{array}$$

hold true.

Here, a local ring R is called AB if there exists a constant C, depending only on R, such that if $\mathrm{Ext}^i(M,N)=0$ for all $i \gg 0$, then $\mathrm{Ext}^i(M,N)=0$ for all $i > C$. This notion is introduced by Huneke and Jorgensen [33]. The *(first) cosyzygy* $\Omega^{-1}M$ of a maximal Cohen–Macaulay module M over a Gorenstein local ring R is defined by a short exact sequence

$$0 \to M \to F \to \Omega^{-1}M \to 0$$

of maximal Cohen–Macaulay R-modules with F free. For each $n \geq 0$, the nth cosyzygy $\Omega^{-n}M$ is defined similarly to the nth syzygy. For each maximal Cohen–Macaulay R-module M and each integer $n \geq 0$, the nth cosyzygy $\Omega^{-n}M$ is uniquely determined up to free summands.

Applying the theorem of Rouquier [50] stated before to an affine scheme implies that $\mathrm{D^b}(R)$ has finite dimension if R is essentially of finite type over a perfect field. The author [1] proved that the same statement holds true for a complete local ring R over a perfect field. The following theorem improves this.

Theorem 4.10 ([37, Theorem 1.4]). *Let R be either*

(i) *an equicharacteristic excellent local ring, or*
(ii) *a ring that is essentially of finite type over a field.*

Then $\mathrm{D^b}(R)$ *has finite dimension.*

Theorem 4.10 is, as far as the author knows, the strongest result on finite dimension of the derived category of a local ring containing a field.

To prove the above theorem, first we need to make a simplified version of Definition 4.2.

Definition 4.11. Let $\mathcal{A}$ be an abelian category.

(1) For a subcategory $\mathcal{X}$ of $\mathcal{A}$, we denote by $|\mathcal{X}|$ the smallest subcategory of $\mathcal{A}$ containing $\mathcal{X}$ and closed under finite direct sums and direct summands. That is,

$$|\mathcal{X}| = \operatorname{add}\mathcal{X}.$$

When $\mathcal{X}$ consists of a single object X, we simply write $|X|$.

(2) For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{A}$, we put $\mathcal{X} * \mathcal{Y} = ||\mathcal{X}| \circ |\mathcal{Y}||$.

(3) Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$, and put

$$|\mathcal{C}|_r = \begin{cases} 0 & (r=0), \\ |\mathcal{C}| & (r=1), \\ |\mathcal{C}|_{r-1} * \mathcal{C} = ||\mathcal{C}|_{r-1} \circ |\mathcal{C}|| & (r \geq 2). \end{cases}$$

When $\mathcal{C}$ consists of a single object C, we simply write $|C|_r$.

Next, we need to introduce the notion of a cohomology annihilator.

Definition 4.12. For an integer $n \geq 0$ we set

$$\mathrm{ca}^n(R) = \{a \in R \mid a \operatorname{Ext}_R^n(M,N) = 0 \text{ for all } M, N \in \operatorname{mod} R\}$$

and call this the nth *cohomology annihilator* of R.

Also, we need the following two technical lemmas. For an integer $n \geq 0$ we denote by $\Omega^n(\operatorname{mod} R)$ the subcategory of $\operatorname{mod} R$ consisting of nth syzygies of R-modules.

Lemma 4.13 ([37, Theorem 4.3])**.** *Let R have Krull dimension d. Suppose that there exist an R-module G and integers $s, n \geq 0$ such that $\Omega^s(\operatorname{mod} R) \subseteq |G|_n$. Then there is an equality*

$$\operatorname{Sing} R = \mathrm{V}(\mathrm{ca}^{s+d+1}(R)).$$

In particular, $\operatorname{Sing} R$ *is closed.*

Lemma 4.14 ([37, Theorems 5.1 and 5.2])**.** *Let R have Krull dimension d.*

(1) *Suppose that there exists aninteger $s > 0$ such that $\mathrm{ca}^s(R/\mathfrak{p}) \neq 0$ for all prime ideals $\mathfrak{p}$ of R. Then there exist an R-module G and an integer $n \geq 0$ such that*

$$\Omega^{s+d-1}(\operatorname{mod} R) \subseteq |G|_n.$$

(2) *Suppose that for all prime ideals $\mathfrak{p}$ of R there exists an integer $s \leq \dim R/\mathfrak{p} + 1$ such that $\mathrm{ca}^s(R/\mathfrak{p}) \neq 0$. Then there exist an R-module G and an integer $n \geq 0$ such that*

$$\Omega^{d}(\operatorname{mod} R) \subseteq |G|_n.$$

Using the above two lemmas, we can show the following proposition.

Proposition 4.15 ([37, Theorem 5.3])**.** *Let R be a d-dimensional excellent equicharacteristic local ring.*

(1) *There is an equality*

$$\operatorname{Sing} R = \mathrm{V}(\mathrm{ca}^{2d+1}(R)).$$

(2) *There exist an R-module G and an integer $n \geq 0$ such that*

$$\Omega^{3d+1}(\operatorname{mod} R) \subseteq |G|_n.$$

Proof of Proposition 4.15. (1) First we consider the case where R is complete. Fix a prime ideal $\mathfrak{p}$ of R. By virtue of a result of Gabber [36, IV, Théorème 2.1.1], the integral domain $R/\mathfrak{p}$ admits a separable Noether normalization. Then it follows from [58] that $\mathrm{ca}^{\dim R/\mathfrak{p}+1}(R/\mathfrak{p}) \neq 0$, which is shown by using the sum of the Noether differents of $R/\mathfrak{p}$. Lemma 4.14(2) yields $\Omega^d(\operatorname{mod} R) \subseteq |G|_n$ for some R-module G and some integer $n \geq 0$. It follows from Lemma 4.13 that $\operatorname{Sing} R = \mathrm{V}(\mathrm{ca}^{2d+1}(R))$.

Next we consider the case where R is excellent. By the definition of excellence, there exists an ideal I of R such that $\operatorname{Sing} R = \mathrm{V}(I)$. Then

$$I\widehat{R} \subseteq P \iff I \subseteq P \cap R \iff P \cap R \in \operatorname{Sing} R \iff P \in \operatorname{Sing} R$$

as formal fibers are regular. Hence

$$\mathrm{V}(I\widehat{R}) = \operatorname{Sing} \widehat{R} = \mathrm{V}(\mathrm{ca}^{2d+1}(\widehat{R}))$$

by the complete case. Since $\widehat{R}$ is faithfully flat over R, we obtain $\mathrm{V}(I) = \mathrm{V}(\mathrm{ca}^{2d+1}(R))$.

(2) Fix a prime ideal $\mathfrak{p}$ of R. By (1), we have

$$0 \notin \operatorname{Sing} R/\mathfrak{p} = \mathrm{V}(\mathrm{ca}^{2\dim R/\mathfrak{p}+1}(R/\mathfrak{p})).$$

Hence $\mathrm{ca}^{2\dim R/\mathfrak{p}+1}(R/\mathfrak{p}) \neq 0$. Then it is easy to see that $\mathrm{ca}^{2d+1}(R/\mathfrak{p}) \neq 0$. Lemma 4.14(1) yields $\Omega^{3d}(\operatorname{mod} R) \subseteq |G|_n$ for some R-module G and an integer n. ■

Proof of Theorem 4.10. Using Proposition 4.15(2), we easily see that the derived category $\mathrm{D}^{\mathsf{b}}(R)$ has finite dimension. ■

So far, we have stated results on finiteness of the dimension and radius. The following theorem concretely gives an upper bound by using well-known invariants. For a complete local ring

$$R = \frac{k[\![x_1, \dots, x_n]\!]}{(f_1, \dots, f_t)}$$

over a field k, the *Jabobian ideal* of R is by definition the ideal of R generated by the c-minors of the Jacobian matrix of $f_1, \dots, f_t$, where $c = \operatorname{codim} R$.

Theorem 4.16 ([26, Theorem 1.1])**.** *Let R be a complete equicharacteristic Cohen–Macaulay local ring with an isolated singularity. Let J be the Jacobian ideal of R. Then there is an inequality*

$$\dim \mathrm{D}_{\mathrm{sg}}(R) < (\mu(J) - \dim R + 1) \cdot \ell\ell(R/J).$$

If the residue field of R is infinite, the inequality

$$\dim \mathrm{D}_{\mathrm{sg}}(R) < \mathrm{e}(J).$$

holds as well.

In the above theorem, one can replace J with any $\mathfrak{m}$-primary ideal of R contained in the sum of the Noether differents of R.

The first inequality of Theorem 4.16 immediately recovers the following result due to Ballard, Favero and Katzarkov [8, Proposition 4.11].

Corollary 4.17 (Ballard, Favero and Katzarkov, [26, Corollary 1.4])**.** *Let k be a field, and let $R = k[\![x_1, \dots, x_n]\!]/(f)$ be a hypersurface complete local ring. Suppose that R has an isolated singularity. Let*

$$J = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) R$$

be the Jacobian ideal of R. Then the inequality

$$\dim \mathrm{D}_{\mathrm{sg}}(R) < 2\,\ell\ell(R/J)$$

holds true.

We end this article by giving an outline of the proof of Theorem 4.16.

Sketch of Proof of Theorem 4.16. We can show the following statements.

(a) Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $I = (x_1, \dots, x_n)$ be an $\mathfrak{m}$-primary ideal of R. Let M be an R-module. Set $t = \operatorname{depth} R$ and $l = \ell\ell(R/I)$. Then the Koszul complex $\mathrm{K}(\boldsymbol{x}, M)$ belongs to $\langle k \rangle_{(n-t+1)l}$ in $\mathrm{D^b}(R)$.

(b) Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements of R. Let M be an R-module. Suppose that for all $1 \le i \le n$ the multiplication map $M \xrightarrow{x_i} M$ is zero in $\mathrm{D_{sg}}(R)$. Then M is a direct summand of the Koszul complex $\mathrm{K}(\boldsymbol{x}, M)$ in $\mathrm{D_{sg}}(R)$.

(c) Let R be a complete equicharacteristic Cohen–Macaulay local ring. Let x be an element in J. Let M be a maximal Cohen–Macaulay R-module. Then the multiplication map $M \xrightarrow{x} M$ is zero in $\mathrm{D_{sg}}(R)$.

(d) Suppose that the residue field of R is infinite. Then one can choose a minimal reduction Q of J as a parameter ideal of R. It holds that

$$(\nu(Q) - d + 1) \cdot \ell\ell(R/Q) = \ell\ell(R/Q) \le \ell(R/Q) = \mathrm{e}(J).$$

The first inequality in the theorem follows from (a), (b) and (c), while the second one is obtained by (d). ■

References

[1] T. Aihara; R. Takahashi, Generators and dimensions of derived categories of modules, *Comm. Algebra* **43** (2015), no. 11, 5003–5029.

[2] T. Araya; K.-i. Iima; R. Takahashi, On the structure of Cohen–Macaulay modules over hypersurfaces of countable Cohen–Macaulay representation type, *J. Algebra* **361** (2012), 213–224.

[3] M. Auslander, Modules oever unramified regular local rings, *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pp. 230–233, *Inst. Mittag-Leffler, Djursholm*, 1963.

[4] M. Auslander, Representation dimension of Artin algebras, Queen Mary College Math. Notes, Queen Mary College, London, 1971.

[5] M. Auslander; M. Bridger, Stable module theory, Memoirs of the American Mathematical Society, No. 94, *American Mathematical Society, Providence, R.I.*, 1969.

[6] M. Auslander; R.-O. Buchweitz, The homological theory of maximal Cohen–Macaulay approximations, Colloque en l'honneur de Pierre Samuel (Orsay, 1987), *Mém. Soc. Math. France (N.S.)* No. 38 (1989), 5–37.

[7] M. Auslander; I. Reiten, Applications of contravariantly finite subcategories, *Adv. Math.* **86** (1991), no. 1, 111–152.

[8] M. Ballard; D. Favero; L. Katzarkov, Orlov spectra: bounds and gaps, *Invent. Math.* **189** (2012), no. 2, 359–430.

[9] P. Balmer, Presheaves of triangulated categories and reconstruction of schemes, *Math. Ann.* **324** (2002), no. 3, 557–580.

[10] P. Balmer, The spectrum of prime ideals in tensor triangulated categories, *J. Reine Angew. Math.* **588** (2005), 149–168.

[11] P. Balmer, Spectra, spectra, spectra–tensor triangular spectra versus Zariski spectra of endomorphism rings, *Algebr. Geom. Topol.* **10** (2010), no. 3, 1521–1563.

[12] P. Balmer, Tensor triangular geometry, *Proceedings of the International Congress of Mathematicians, Volume II*, 85–112, *Hindustan Book Agency, New Delhi*, 2010.

[13] P. Balmer, Separable extensions in tensor-triangular geometry and generalized Quillen stratification, *Ann. Sci. École Norm. Sup. (4)* **49** (2016), no. 4, 907–925.

[14] P. Balmer; B. Sanders, The spectrum of the equivariant stable homotopy category of a finite group, *Invent. Math.* **208** (2017), no. 1, 283–326.

[15] D. J. Benson; J. F. Carlson; J. Rickard, Thick subcategories of the stable module category, *Fund. Math.* **153** (1997), no. 1, 59–80.
[16] D. J. Benson; S. B. Iyengar; H. Krause, Stratifying modular representations of finite groups, *Ann. of Math. (2)* **174** (2011), no. 3, 1643–1684.
[17] D. Benson; S. B. Iyengar; H. Krause; J. Pevtsova, Stratification for module categories of finite group schemes, *J. Amer. Math. Soc.* **31** (2018), no. 1, 265–302.
[18] P. A. Bergh, Modules with reducible complexity, *J. Algebra* **310** (2007), no. 1, 132–147.
[19] A. Bondal; M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.* **3** (2003), no. 1, 1–36, 258.
[20] R.-O. Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings, Preprint (1986), `http://hdl.handle.net/1807/16682`.
[21] L. W. Christensen; G. Piepmeyer; J. Striuli; R. Takahashi, Finite Gorenstein representation type implies simple singularity, *Adv. Math.* **218** (2008), no. 4, 1012–1026.
[22] H. Dao; T. Kobayashi; R. Takahashi, Burch ideals and Burch rings, `arXiv:1905.02310`.
[23] H. Dao; R. Takahashi, The radius of a subcategory of modules, *Algebra Number Theory* **8** (2014), no. 1, 141–172.
[24] H. Dao; R. Takahashi, Classification of resolving subcategories and grade consistent functions, *Int. Math. Res. Not. IMRN* **2015** (2015), no. 1, 119–149.
[25] H. Dao; R. Takahashi, The dimension of a subcategory of modules, *Forum Math. Sigma* **3** (2015), e19, 31 pp.
[26] H. Dao; R. Takahashi, Upper bounds for dimensions of singularity categories, *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 4, 297–301.
[27] E. S. Devinatz; M. J. Hopkins; J. H. Smith, Nilpotence and stable homotopy theory, I, *Ann. of Math. (2)* **128** (1988), no. 2, 207–241.
[28] E. M. Friedlander; J. Pevtsova, Π-supports for modules for finite group schemes, *Duke Math. J.* **139** (2007), no. 2, 317–368.
[29] P. Gabriel, Des catégories abélienne, *Bull. Soc. Math. France* **90** (1962), 323–448.
[30] M. J. Hopkins, Global methods in homotopy theory, *Homotopy theory (Durham, 1985)*, 73–96, London Math. Soc. Lecture Note Ser., 117, *Cambridge Univ. Press, Cambridge*, 1987.
[31] M. J. Hopkins; J. H. Smith, Nilpotence and stable homotopy theory, II, *Ann. of Math. (2)* **148** (1998), no. 1, 1–49.
[32] M. Hovey, Classifying subcategories of modules, *Trans. Amer. Math. Soc.* **353** (2001), no. 8, 3181–3191.
[33] C. Huneke; D. A. Jorgensen, Symmetry in the vanishing of Ext over Gorenstein rings, *Math. Scand.* **93** (2003), no. 2, 161–184.
[34] C. Huneke; G. J. Leuschke, Two theorems about maximal Cohen–Macaulay modules, *Math. Ann.* **324** (2002), no. 2, 391–404.
[35] C. Huneke; R. Wiegand, Tensor products of modules, rigidity and local cohomology, *Math. Scand.* **81** (1997), no. 2, 161–183.
[36] L. Illusie; Y. Laszlo; F. Orgogozo, Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents, *Séminaire 'a l'École polytechnique 2006–2008* (2013), `arXiv:1207.3648v1`.
[37] S. B. Iyengar; R. Takahashi, Annihilation of cohomology and strong generation of module categories, *Int. Math. Res. Not. IMRN* **2016** (2016), no. 2, 499–535.
[38] B. Keller; D. Murfet; M. Van den Bergh, On two examples by Iyama and Yoshino, *Compos. Math.* **147** (2011), no. 2, 591–612.
[39] H. Matsui; R. Takahashi, Thick tensor ideals of right bounded derived categories, *Algebra Number Theory* **11** (2017), no. 7, 1677–1738.
[40] S. Nasseh; S. Sather-Wagstaff, Vanishing of Ext and Tor over fiber products, *Proc. Amer. Math. Soc.* **145** (2017), no. 11, 4661–4674.
[41] S. Nasseh; R. Takahashi, Local rings with quasi-decomposable maximal ideal, *Math. Proc. Cambridge Philos. Soc.* (to appear).
[42] A. Neeman, The chromatic tower for $D(R)$, With an appendix by Marcel Bökstedt, *Topology* **31** (1992), no. 3, 519–532.

[43] A. Neeman, Strong generators in $D^{perf}(X)$ and $D^b_{coh}(X)$, arXiv:1703.04484.

[44] S. Oppermann, Lower bounds for Auslander's representation dimension, *Duke Math. J.* **148** (2009), no. 2, 211–249.

[45] D. O. Orlov, Triangulated categories of singularities and D-branes in Landau-Ginzburg models, *Proc. Steklov Inst. Math.* **246** (2004), no. 3, 227–248.

[46] D. O. Orlov, Triangulated categories of singularities, and equivalences between Landau–Ginzburg models, *Mat. Sb.* **197** (2006), no. 12, 117–132; translation in *Sb. Math.* **197** (2006), no. 11-12, 1827–1840.

[47] D. Orlov, Derived categories of coherent sheaves and triangulated categories of singularities, *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, 503–531, Progr. Math., 270, *Birkhäuser Boston, Inc., Boston, MA*, 2009.

[48] D. Orlov, Formal completions and idempotent completions of triangulated categories of singularities, *Adv. Math.* **226** (2011), no. 1, 206–217.

[49] R. Rouquier, Representation dimension of exterior algebras, *Invent. Math.* **165** (2006), no. 2, 357–367.

[50] R. Rouquier, Dimensions of triangulated categories, *J. K-Theory* **1** (2008), no. 2, 193–256.

[51] G. Stevenson, Subcategories of singularity categories via tensor actions, *Compos. Math.* **150** (2014), no. 2, 229–272.

[52] G. Stevenson, Duality for bounded derived categories of complete intersections, *Bull. Lond. Math. Soc.* **46** (2014), no. 2, 245–257.

[53] R. Takahashi, Classifying thick subcategories of the stable category of Cohen–Macaulay modules, *Adv. Math.* **225** (2010), no. 4, 2076–2116.

[54] R. Takahashi, Contravariantly finite resolving subcategories over commutative rings, *Amer. J. Math.* **133** (2011), no. 2, 417–436.

[55] R. Takahashi, Thick subcategories over Gorenstein local rings that are locally hypersurfaces on the punctured spectra, *J. Math. Soc. Japan* **65** (2013), no. 2, 357–374.

[56] R. Takahashi, Modules in resolving subcategories which are free on the punctured spectrum, *Pacific J. Math.* **241** (2009), no. 2, 347–367.

[57] R. W. Thomason, The classification of triangulated subcategories, *Compos. Math.* **105** (1997), no. 1, 1–27.

[58] H.-J. Wang, On the Fitting ideals in free resolutions, *Michigan Math. J.* **41** (1994), no. 3, 587–608.

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602, Japan

E-mail address: takahashi@math.nagoya-u.ac.jp

URL: https://www.math.nagoya-u.ac.jp/~takahashi/