REMARKS ON COMPLEXITIES AND ENTROPIES FOR SINGULARITY CATEGORIES

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Abstract. Let R be a commutative noetherian local ring which is singular and has an isolated singularity. Let $\mathsf{D_{sg}}(R)$ be the singularity category of R in the sense of Buchweitz and Orlov. In this paper, we find real numbers t such that the complexity $\delta_t(G,X)$ in the sense of Dimitrov, Haiden, Katzarkov and Kontsevich vanishes for any split generator G of $\mathsf{D_{sg}}(R)$ and any object X of $\mathsf{D_{sg}}(R)$. In particular, the entropy $\mathrm{h}_t(F)$ of an exact endofunctor F of $\mathsf{D_{sg}}(R)$ is not defined for such numbers t.

1. Introduction

In 2014, Dimitrov, Haiden, Katzarkov and Kontsevich [6] have introduced the notions of complexities $\delta_t(G,X)$ and entropies $\mathrm{h}_t(F)$ for a triangulated category. In less than a decade since then, a lot of works on these notions have been done; see [9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 21, 24, 25, 30] for instance.

Let us quickly recall the definition of entropies; see Definition 2.5 for that of complexities. Let F be an exact endofunctor of a triangulated category $\mathcal{T}$. For each $t \in \mathbb{R}$ the *entropy* $\mathrm{h}_t(F)$ of F is defined by

$$\mathrm{h}_t(F) = \lim_{n\to\infty} \tfrac{1}{n} \log \delta_t(G, F^n(G)),$$

where G is a split generator of $\mathcal{T}$. Therefore, as long as the domain of the natural logarithm function is $\mathbb{R}_{>0}$, the entropy $\mathrm{h}_t(F)$ is defined only when the complexity $\delta_t(G,F^n(G))$ is positive for $n \gg 0$.

Let R be a commutative noetherian singular (i.e., nonregular) local ring. Let $\mathsf{D_{sg}}(R)$ be the *singularity category* of R, which is a triangulated category introduced by Buchweitz [4] and Orlov [23]. In this paper, we explore complexities $\delta_t(G,X)$ for $\mathsf{D_{sg}}(R)$. We shall establish and study the following conjecture.

Conjecture 1.1. Let G be a split generator of $\mathsf{D_{sg}}(R)$. Then $\delta_t(G,X)=0$ for all $X \in \mathsf{D_{sg}}(R)$ and $t \neq 0$.

Denote by $\mathrm{e}(R)$, $\operatorname{codepth} R$ and K^R the multiplicity, codepth and Koszul complex of R, respectively. The main results of this paper include the following theorem about the above conjecture. Note that the third assertion says that the conjecture holds if R is a complete intersection with an isolated singularity.

Theorem 1.2 (Theorems 3.8, 3.9 and 3.10)**.** *Let G be a split generator of $\mathsf{D_{sg}}(R)$. Let X be an object of $\mathsf{D_{sg}}(R)$. Assume that X is locally zero on the punctured spectrum of R (this assumption is satisfied whenever R has an isolated singularity). Then the following assertions hold true.*

(1) *Put $c = \operatorname{codepth} R$ and $m = \max_{1\leqslant i\leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$. One has $\delta_t(G,X)=0$ for all $|t| > \frac{\log c + \log m}{2}$.*
(2) *If R is Gorenstein and has infinite residue field, then $\delta_t(G,X)=0$ for all $|t| > \log(\mathrm{e}(R)-1)$.*
(3) *If R is a complete intersection, then $\delta_t(G,X)=0$ for all nonzero real numbers t.*

As an immediate consequence of the above theorem, we obtain the following corollary on entropies.

Corollary 1.3 (Corollary 3.12)**.** *Suppose that the local ring R has an isolated singularity. Let $F : \mathsf{D_{sg}}(R) \to \mathsf{D_{sg}}(R)$ be an exact functor. Then the three statements below hold.*

(1) *Set $c = \operatorname{codepth} R$ and $m = \max_{1\leqslant i\leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$. Then $\mathrm{h}_t(F)$ is not defined for $|t| > \frac{\log c + \log m}{2}$.*
(2) *Assume R is Gorenstein with infinite residue field. Then $\mathrm{h}_t(F)$ is not defined for $|t| > \log(\mathrm{e}(R)-1)$.*
(3) *Suppose that R is a complete intersection. Then the entropy $\mathrm{h}_t(F)$ is defined only for $t=0$.*

The structure of this paper is as follows. In Section 2, we introduce the operation $\star$ for subcategories of a triangulated category, investigate its properties, and recall the definitions of complexities and entropies. In Section 3, after stating some basics from commutative algebra, we explore complexities for the singularity category of a singular local ring, and prove our main results including the ones stated above.

2020 *Mathematics Subject Classification.* 13D09, 13H10, 18G80.
Key words and phrases. complexity, entropy, isolated singularity, Koszul complex, local ring, singularity category.
The author was partly supported by JSPS Grant-in-Aid for Scientific Research 19K03443.

2. Basic properties of complexities

In this section, we work on a general triangulated category, and prove several preliminary results.

Setup 2.1. Throughout this section, let $\mathcal{T}$ be a triangulated category. All subcategories of $\mathcal{T}$ are assumed to be strictly full. We may omit a subscript if it is clear from the context.

We introduce the operation $\star$ for subcategories of $\mathcal{T}$, which plays a central role throughout the paper.

Definition 2.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathcal{T}$.

(1) We denote by $\mathcal{X}\star\mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects $T\in\mathcal{T}$ such that there exists an exact triangle $X\to T\to Y\rightsquigarrow$ in $\mathcal{T}$ such that $X\in\mathcal{X}$ and $Y\in\mathcal{Y}$.

(2) When $\mathcal{X},\mathcal{Y}$ consist of single objects X,Y respectively, we simply write $X\star Y$ to denote $\mathcal{X}\star\mathcal{Y}$.

In the following lemma, we make a list of several fundamental properties of the operation $\star$, which are frequently used later. The first assertion says that the operation $\star$ satisfies associativity. The second and third assertions state that the operation $\star$ is compatible with taking finite direct sums and shifts.

Lemma 2.3. (1) *For subcategories $\mathcal{X},\mathcal{Y},\mathcal{Z}$ of $\mathcal{T}$ one has $(\mathcal{X}\star\mathcal{Y})\star\mathcal{Z}=\mathcal{X}\star(\mathcal{Y}\star\mathcal{Z})$. Hence, there is no ambiguity in writing $\bigstar_{i=1}^{n}\mathcal{X}_i=\mathcal{X}_1\star\cdots\star\mathcal{X}_n$ for subcategories $\mathcal{X}_1,\dots,\mathcal{X}_n$ of $\mathcal{T}$ or $\mathcal{X}^{\star n}=\underbrace{\mathcal{X}\star\cdots\star\mathcal{X}}_{n}$.*

(2) *Let $\{X_{ij}\}_{1\leqslant i\leqslant m,\,1\leqslant j\leqslant n}$ and $\{M_i\}_{1\leqslant i\leqslant m}$ be families of objects of $\mathcal{T}$. Suppose that $M_i\in\bigstar_{j=1}^{n}X_{ij}$ for each $1\leqslant i\leqslant m$. Then it holds that $\bigoplus_{i=1}^{m}M_i\in\bigstar_{j=1}^{n}(\bigoplus_{i=1}^{m}X_{ij})$.*

(3) *Let $X_1,\dots,X_n\in\mathcal{T}$. Then the following statements hold true.*
   (a) *If $M\in\bigstar_{i=1}^{n}X_i$, then $M[s]\in\bigstar_{i=1}^{n}X_i[s]$ for all integers s, $M^{\oplus m}\in\bigstar_{i=1}^{n}X_i^{\oplus m}$ for all positive integers m, and $M\oplus(\bigoplus_{i=1}^{n}Y_i)\in\bigstar_{i=1}^{n}(X_i\oplus Y_i)$ for all objects $Y_1,\dots,Y_n\in\mathcal{T}$.*
   (b) *One has the containment $\bigoplus_{i=1}^{n}X_i\in\bigstar_{i=1}^{n}X_i$.*

Proof. (1) Take any object $T\in(\mathcal{X}\star\mathcal{Y})\star\mathcal{Z}$. Then there exist exact triangles $W\to T\to Z\rightsquigarrow$ and $X\to W\to Y\rightsquigarrow$ with $X\in\mathcal{X}$, $Y\in\mathcal{Y}$ and $Z\in\mathcal{Z}$. The octahedral axiom gives rise to exact triangles $X\to T\to V\rightsquigarrow$ and $Y\to V\to Z\rightsquigarrow$. Hence T is in $\mathcal{X}\star(\mathcal{Y}\star\mathcal{Z})$, and we get $(\mathcal{X}\star\mathcal{Y})\star\mathcal{Z}\subseteq\mathcal{X}\star(\mathcal{Y}\star\mathcal{Z})$.

Conversely, let $T\in\mathcal{X}\star(\mathcal{Y}\star\mathcal{Z})$. There are exact triangles $X\to T\to V\rightsquigarrow$ and $Y\to V\to Z\rightsquigarrow$ with $X\in\mathcal{X}$, $Y\in\mathcal{Y}$ and $Z\in\mathcal{Z}$. The octahedral axiom yields exact triangles $T\to Z\to W\rightsquigarrow$ and $X[1]\to W\to Y[1]\rightsquigarrow$. We get exact triangles $X\to W[-1]\to Y\rightsquigarrow$ and $W[-1]\to T\to Z\rightsquigarrow$, which show that T belongs to $(\mathcal{X}\star\mathcal{Y})\star\mathcal{Z}$. It follows that $(\mathcal{X}\star\mathcal{Y})\star\mathcal{Z}\supseteq\mathcal{X}\star(\mathcal{Y}\star\mathcal{Z})$.

(2) We use induction on n. Let $n=1$. The assumption means that $M_i=X_{i1}$ for each $1\leqslant i\leqslant m$. Hence $\bigoplus_{i=1}^{m}M_i=\bigoplus_{i=1}^{m}X_{i1}$. Let $n\geqslant 2$. Since M_i belongs to $(X_{i1}\star\cdots\star X_{i,n-1})\star X_{in}$ for each $1\leqslant i\leqslant m$, there exists an exact triangle $N_i\to M_i\to X_{in}\rightsquigarrow$ with $N_i\in X_{i1}\star\cdots\star X_{i,n-1}$. Hence there is an exact triangle $\bigoplus_{i=1}^{m}N_i\to\bigoplus_{i=1}^{m}M_i\to\bigoplus_{i=1}^{m}X_{in}\rightsquigarrow$ (see [22, Remark 1.2.2]), and the induction hypothesis implies that $\bigoplus_{i=1}^{m}N_i$ is in $(\bigoplus_{i=1}^{m}X_{i1})\star\cdots\star(\bigoplus_{i=1}^{m}X_{i,n-1})$. It follows that $\bigoplus_{i=1}^{m}M_i\in(\bigoplus_{i=1}^{m}X_{i1})\star\cdots\star(\bigoplus_{i=1}^{m}X_{in})$.

(3a) The second assertion follows by letting $M_i=M$ and $X_{ij}=X_j$ for $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$ in (2). The third assertion is shown by applying (2) to the containments $M\in X_1\star X_2\star\cdots\star X_n$, $Y_1\in Y_1\star 0\star\cdots\star 0$, $Y_2\in 0\star Y_2\star\cdots\star 0$, $\dots$, $Y_n\in 0\star\cdots\star 0\star Y_n$. It remains to prove the first assertion, for which we use induction on n. When $n=1$, we have $M=X_1$ and $M[s]=X_1[s]$. Let $n\geqslant 2$. As M belongs to $(X_1\star\cdots\star X_{n-1})\star X_n$, there exists an exact triangle $N\to M\to X_n\rightsquigarrow$ with $N\in X_1\star\cdots\star X_{n-1}$. Then there is an exact triangle $N[s]\to M[s]\to X_n[s]\rightsquigarrow$, and the induction hypothesis implies $N[s]\in X_1[s]\star\cdots\star X_{n-1}[s]$. We now obtain $M[s]\in X_1[s]\star\cdots\star X_n[s]$, and the first assertion follows.

(3b) Letting $M=X_1=\cdots=X_n=0$ in the third assertion of (3a), we get $Y_1\oplus\cdots\oplus Y_n\in Y_1\star\cdots\star Y_n$ for all objects $Y_1,\dots,Y_n\in\mathcal{T}$. This shows the assertion. ■

Here we recall the definition of split generators, which are used to define complexities and entropies.

Definition 2.4. (1) A *thick subcategory* of $\mathcal{T}$ is by definition a triangulated subcategory of $\mathcal{T}$ closed under direct summands, i.e., a subcategory closed under shifts, mapping cones and direct summands.

(2) For an object $X\in\mathcal{T}$ we denote by $\mathsf{thick}_{\mathcal{T}}X$ the *thick closure* of T, that is to say, the smallest thick subcategory of $\mathcal{T}$ to which X belongs.

(3) A *split generator* of $\mathcal{T}$, which is also called a *thick generator* of $\mathcal{T}$, is defined to be an object of $\mathcal{T}$ whose thick closure coincides with $\mathcal{T}$.

Now we can state the definitions of complexities and entropies introduced in [6].

Definition 2.5 (Dimitrov–Haiden–Katzarkov–Kontsevich)**.**

(1) Let $X, Y \in \mathcal{T}$ and $t \in \mathbb{R}$. We denote by $\delta_t(X,Y)$ the infimum of the sums $\sum_{i=1}^{r} \mathrm{e}^{n_i t}$, where r runs through the nonnegative integers and n_i run through the integers such that there exist a sequence

$$0 = Y_0 \to Y_1 \to \cdots \to Y_{r-1} \to Y_r = Y \oplus Y'$$
$$(X[n_1], \ \dots, \ X[n_r])$$

of exact triangles $\{Y_{i-1} \to Y_i \to X[n_i] \rightsquigarrow\}_{i=1}^{r}$ in $\mathcal{T}$. The function $\mathbb{R} \ni t \mapsto \delta_t(X,Y) \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is called the *complexity*[1] of Y relative to X. When $Y = 0$, one can take $r = 0$, and hence $\delta_t(X,Y) = 0$.

(2) Let $F : \mathcal{T} \to \mathcal{T}$ be an exact functor and $t \in \mathbb{R}$. The *entropy* $\mathrm{h}_t(F)$ of F is defined by

$$\mathrm{h}_t(F) = \lim_{n\to\infty} \tfrac{1}{n} \log \delta_t(G, F^n(G)),$$

where G is a split generator of $\mathcal{T}$. This is independent of the choice of G; see [6, Lemma 2.6].

The following proposition gives an equivalent definition of a complexity. In what follows, whenever we are concerned with a complexity, we regard this equality as its definition, as it is more convenient for us.

Proposition 2.6. *Let $X, Y \in \mathcal{T}$ and $t \in \mathbb{R}$. One then has the equality*

$$\delta_t(X,Y) = \inf\{\textstyle\sum_{i=1}^{r} \mathrm{e}^{n_i t} \mid Y \oplus Y' \in \bigstar_{i=1}^{r} X[n_i] \text{ for some } Y' \in \mathcal{T}\}.$$

Proof. Fix an object $Z \in \mathcal{T}$. It suffices to show that $Z \in \bigstar_{i=1}^{r} X[n_i]$ if and only if there is a sequence

$$0 = Y_0 \to Y_1 \to \cdots \to Y_{r-1} \to Y_r = Z$$
$$(X[n_1], \ \dots, \ X[n_r])$$

of exact triangles. We use induction on r. When $r = 0$, both containments mean that $Z = 0$. Let $r > 0$.

The "only if" part: Since Z is in $(\bigstar_{i=1}^{r-1} X[n_i]) \star X[n_r]$, there exists an exact triangle $\sigma : W \to Z \to X[n_r] \rightsquigarrow$ in $\mathcal{T}$ such that $W \in \bigstar_{i=1}^{r-1} X[n_i]$. Applying the induction hypothesis to W yields a sequence of exact triangles, and splicing it with the exact triangle σ gives rise to a sequence of exact triangles

$$0 = Y_0 \to Y_1 \to \cdots \to Y_{r-2} \to Y_{r-1} = W \to Z.$$
$$(X[n_1], \ \dots, \ X[n_{r-1}], \ X[n_r])$$

The "if" part: By assumption, there exists a sequence of exact triangles

$$0 = Y_0 \to Y_1 \to \cdots \to Y_{r-2} \to Y_{r-1} \to Y_r = Z.$$
$$(X[n_1], \ \dots, \ X[n_{r-1}], \ X[n_r])$$

The induction hypothesis implies that Y_{r-1} belongs to $\bigstar_{i=1}^{r-1} X[n_i]$. The exact triangle $Y_{r-1} \to Z \to X[n_r] \rightsquigarrow$ appearing in the above diagram shows that Z is in $(\bigstar_{i=1}^{r-1} X[n_i]) \star X[n_r] = \bigstar_{i=1}^{r} X[n_i]$. ■

We give a couple of statements concerning complexities. Recall that $\mathcal{T}$ is said to be *periodic* if there exists an integer $n > 0$ such that the nth shift functor $[n]$ is isomorphic to the identity functor $\mathrm{id}_{\mathcal{T}}$ of $\mathcal{T}$.

Proposition 2.7. *Let X and Y be objects of $\mathcal{T}$. Then the following statements hold.*

(1) *Let $t \in \mathbb{R}$. Then $\delta_t(X,Y) < \infty$ if and only if $Y \in \mathsf{thick}_{\mathcal{T}} X$.*

(2) *There is an equality $\delta_0(X,Y) = \inf\{r \in \mathbb{Z}_{\geqslant 0} \mid Y \oplus Y' \in \bigstar_{i=1}^{r} X[n_i] \text{ for some } Y' \in \mathcal{T}\}$.*

(3) *Let $t \in \mathbb{R}$. Suppose that $\mathcal{T}$ is periodic and $\delta_t(X,Y) < \infty$. Then $\delta_t(X,Y) = 0$ unless $t = 0$.*

Proof. (1) The "only if" part: By definition, there exist an object $Y' \in \mathcal{T}$ and integers $n_1, \dots, n_r$ such that $Y \oplus Y'$ belongs to $\bigstar_{i=1}^{r} X[n_i]$. Note that $\bigstar_{i=1}^{r} X[n_i]$ is contained in $\mathsf{thick}\, X$. Hence Y is in $\mathsf{thick}\, X$.

The "if" part: By [2, 2.2.1 and 2.2.4] there are an object $Y' \in \mathcal{T}$ and an integer $s \geqslant 0$ with $Y \oplus Y' \in \mathcal{X}^{\star s}$, where $\mathcal{X}$ is the smallest subcategory of $\mathcal{T}$ which contains X and is closed under finite direct sums and shifts. We can write $Y \oplus Y' \in \bigstar_{i=1}^{s}(\bigoplus_{j=1}^{u_i} X[m_{ij}])$ for some $u_i \in \mathbb{Z}_{\geqslant 0}$ and $m_{ij} \in \mathbb{Z}$. Lemma 2.3(3b) implies $Y \oplus Y' \in \bigstar_{i=1}^{s}(\bigstar_{j=1}^{u_i} X[m_{ij}])$. We get $\delta_t(X,Y) \leqslant \sum_{i=1}^{s} \sum_{j=1}^{u_i} e^{m_{ij} t}$, which shows $\delta_t(X,Y) < \infty$.

(2) The assertion immediately follows from the definition of $\delta_0(X,Y)$.

[1] It may be better for us to call this the *Dimitrov–Haiden–Katzarkov–Kontsevich complexity* because in commutative algebra there is a different notion called complexity, which describes the growth of a minimal free resolution of a finitely generated module over a commutative noetherian local ring. In fact, it does appear implicitly in the proof of Theorem 3.8.

(3) As $\delta_t(X,Y)<\infty$, we have $Y\oplus Y'\in \bigstar_{i=1}^{r}X[n_i]$ for some $Y'\in\mathcal{T}$, $r\in\mathbb{Z}_{\geqslant 0}$ and $n_1,\dots,n_r\in\mathbb{Z}$. Since $\mathcal{T}$ is periodic, we have $[m]\cong \mathrm{id}_{\mathcal{T}}$ for some $m\in\mathbb{Z}_{>0}$. We have $X[n_i]\cong X[n_i+jm]$ for all $j\in\mathbb{Z}$, and get the containment $Y\oplus Y'\in\bigstar_{i=1}^{r}X[n_i+jm]$. We obtain inequalities $0\leqslant\delta_t(X,Y)\leqslant\sum_{i=1}^{r}\mathrm{e}^{(n_i+jm)t}$. It remains to note that $\lim_{j\to-\infty}\sum_{i=1}^{r}\mathrm{e}^{(n_i+jm)t}=0$ if $t>0$ and $\lim_{j\to\infty}\sum_{i=1}^{r}\mathrm{e}^{(n_i+jm)t}=0$ if $t<0$. ■

Remark 2.8. The equality in Proposition 2.7(2) may remind the reader of the notion of a *level* introduced by Avramov, Buchweitz, Iyengar and Miller [2]. Namely, $\delta_0(X,Y)$ looks closely related to the X-level $\mathrm{level}_{\mathcal{T}}^{X}(Y)$ of Y. The difference is that an X-level ignores finite direct sums of copies of X. This is similar to the difference between the lengths of a composition series and a Loewy series of a module over a ring. The complexity $\delta_t(X,Y)$ can also be regarded as a weighted version of $\delta_0(X,Y)$ with respect to shifts.

The following lemma comes from [6, Proposition 2.2]. In this proposition, neither $\delta_t(X,Y)$ nor $\delta_t(Y,Z)$ is assumed to be finite, but in its proof both $\delta_t(X,Y)$ and $\delta_t(Y,Z)$ seem to be assumed to be finite. In fact, without this assumption, we would need to clarify what $0\cdot\infty$ and $\infty\cdot 0$ mean.

Lemma 2.9. *Let t be a real number. Let X, Y and Z be objects of $\mathcal{T}$. Suppose that both $\delta_t(X,Y)$ and $\delta_t(Y,Z)$ are finite. Then there is an inequality $\delta_t(X,Z)\leqslant\delta_t(X,Y)\cdot\delta_t(Y,Z)$.*

3. Main results

In this section, we shall investigate complexities and entropies for the singularity category of a commutative noetherian local ring, which is a triangulated category.

Setup 3.1. Throughout this section, let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. The triangulated category considered in this section is the *singularity category* $\mathsf{D_{sg}}(R)$ of R, which is by definition the Verdier quotient of the bounded derived category of finitely generated R-modules by perfect complexes (i.e., bounded complexes of finitely generated projective R-modules).

For the reader whose expertise is outside of commutative algebra, we collect here several fundamental notions from commutative algebra and their basic properties which are used in the proofs of our results. The details can be found in [1, 3, 20].

Definition 3.2. (1) A sequence $\boldsymbol{x}=x_1,\dots,x_n$ of elements in $\mathfrak{m}$ is called an *R-regular sequence* if the element x_i is regular on $R/(x_1,\dots,x_{i-1})$ (i.e., the multiplication map $R/(x_1,\dots,x_{i-1})\xrightarrow{x_i}R/(x_1,\dots,x_{i-1})$ is injective) for each $1\leqslant i\leqslant n$. Note then that the R-module $R/(\boldsymbol{x})$ has finite projective dimension; see [3, Corollary 1.6.14(b)] or [20, Theorem 16.5(i)].

(2) For a finitely generated R-module M, we denote by $\mu(M)$ the *minimal number of generators* of M, that is, $\mu(M)=\dim_k(M\otimes_R k)$; see [20, Theorem 2.3] and [3, the paragraph before Proposition 1.3.1].

(3) Let $\operatorname{edim}R$ and $\operatorname{depth}R$ stand for the *embedding dimension* of R and the *depth* of R, respectively. These are defined by the equalities $\operatorname{edim}R=\mu(\mathfrak{m})=\dim_k\mathfrak{m}/\mathfrak{m}^2$ and $\operatorname{depth}R=\inf\{i\in\mathbb{N}\mid\operatorname{Ext}_R^i(k,R)\neq 0\}$. The ineqality $\operatorname{depth}R\leqslant\dim R$ always holds by [3, Proposition 1.2.12] or [20, Theorem 17.2], and we say that R is *Cohen–Macaulay* if the equality $\operatorname{depth}R=\dim R$ holds.

(4) We always have the inequality $\dim R\leqslant\operatorname{edim}R$ by [20, the bottom of page 104], and we say that R is *regular* if the equality $\dim R=\operatorname{edim}R$ holds. We say that R is a *singular* local ring if it is not a regular local ring. Note from [20, Theorem 19.2] or [3, Theorem 2.2.7] that R is singular if and only if the category $\mathsf{D_{sg}}(R)$ is nonzero.

(5) The *codimension* and the *codepth* of R are defined by $\operatorname{codim}R=\operatorname{edim}R-\dim R$ and $\operatorname{codepth}R=\operatorname{edim}R-\operatorname{depth}R$. Note that $\operatorname{codim}R=\operatorname{codepth}R$ if (and only if) R is Cohen–Macaulay.

(6) The local ring R is said to be a *hypersurface* provided the inequality $\operatorname{codepth}R\leqslant 1$ holds. According to Cohen's structure theorem (see [3, Theorem A.21]), this condition is equivalent to saying that the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the residue ring $S/(f)$ of some regular local ring S by some principal ideal (f); see [1, §5.1].

(7) The local ring R is called a *complete intersection* if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the residue ring $S/(\boldsymbol{f})$ of a regular local ring $(S,\mathfrak{n})$ by the ideal $(\boldsymbol{f})$ generated by a regular sequence $\boldsymbol{f}=f_1,\dots,f_c$. One can choose $\boldsymbol{f}=f_1,\dots,f_c$ so that $c=\operatorname{codim}R$, and in this case, $f_i\in\mathfrak{n}^2$ for all i; see [3, Corollary 1.6.19 and Theorems 2.3.2(b), 2.3.3(b)].

(8) We say that the local ring R is *Gorenstein* if the R-module R has finite injective dimension.

(9) The *Koszul complex* K^R of R is defined to be the Koszul complex $\mathrm{K}(\boldsymbol{x}, R)$ on R of a minimal system of generators $\boldsymbol{x} = x_1, \dots, x_n$ of $\mathfrak{m}$. This complex is uniquely determined up to isomorphism; see [3, the part following Remark 1.6.20]. Each homology $\mathrm{H}_i(\mathrm{K}^R)$ is a finite-dimensional k-vector space by [20, the paragraph before Theorem 16.5] or [3, Proposition 1.6.5(b)].

(10) We say that R has an *isolated singularity* if $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.

(11) For an R-module M, we denote by $\ell(M)$ the *length* of (a composition series of) M. If the R-module M is finitely generated and $\mathfrak{m}M = 0$, then one has $\ell(M) = \dim_k M < \infty$.

(12) Let $\mathrm{e}(R)$ and $\mathrm{r}(R)$ be the *(Hilbert–Samuel) multiplicity* and *type* of R, respectively. Namely, one has $\mathrm{e}(R) = \lim_{n\to\infty} \frac{d!}{n^d}\ell(R/\mathfrak{m}^{n+1}) \in \mathbb{Z}_{>0}$ and $\mathrm{r}(R) = \dim_k \operatorname{Ext}_R^t(k, R) \in \mathbb{Z}_{\geqslant 0}$, where $d = \dim R$ and $t = \operatorname{depth} R$. When R is Cohen–Macaulay, R is singular if and only if $\mathrm{e}(R) > 1$ by [3, Examples 4.6.3(a) and Exercises 4.6.14(b)]. The ring R is Gorenstein if and only if R is Cohen–Macaulay and $\mathrm{r}(R) = 1$ by [20, Theorem 18.1] or [3, Theorem 3.2.10].

(13) Let M be a finitely generated R-module. Let n be a nonnegative integer. Then we denote by $\Omega_R^n M$ the nth *syzygy* of M over R, that is, the image of the nth differential map in a minimal free resolution of the R-module M. Note by Schanuel's lemma [20, Lemma 4 in §19] that the module $\Omega_R^n M$ is uniquely determined up to isomorphism. We denote by $\beta_n^R(M)$ the nth *Betti number* of M, namely, $\beta_n^R(M) = \mu(\Omega_R^n M)$.

(14) A *system of parameters* of the local ring R is by definition a sequence $\boldsymbol{x} = x_1, \dots, x_d$ of elements of R such that $d = \dim R$ and $R/(\boldsymbol{x})$ is artinian. A system of parameters of R exists; see [20, Theorem 13.4] or [3, Theorem A.3]. If R is Cohen–Macaulay, then a system of parameters of R is an R-regular sequence; see [20, Theorem 17.4(iii)] or [3, Theorem 2.1.2(d)].

(15) The following implications hold true.

$$\begin{aligned} &R \text{ is regular} \Rightarrow R \text{ is a hypersurface} \Rightarrow R \text{ is a complete intersection} \\ &\Rightarrow R \text{ is Gorenstein} \Rightarrow R \text{ is Cohen–Macaulay.} \end{aligned}$$

This follows from [20, the paragraph after Theorem 21.3] or [3, Proposition 3.1.20], and what we have stated above.

(16) The *socle* of the local ring R is defined as the set of elements $x \in R$ such that $\mathfrak{m}x = 0$, and denoted by $\operatorname{Soc} R$. Since there is an isomorphism $\operatorname{Soc} R \cong \operatorname{Hom}_R(k, R)$, one has $\mathrm{r}(R) = \dim_k \operatorname{Soc} R$ when $\operatorname{depth} R = 0$. In general, it holds that $\mathrm{r}(R) = \dim_k \operatorname{Soc} R/(\boldsymbol{x})$ for any R-regular sequence $\boldsymbol{x} = x_1, \dots, x_t$ with $t = \operatorname{depth} R$; see [3, Lemma 1.2.19].

What we want to consider in this section is the following conjecture.

Conjecture 3.3. Let G be a split generator of $\mathsf{D_{sg}}(R)$. Then one has the equality $\delta_t(G, X) = 0$ for all objects X of $\mathsf{D_{sg}}(R)$ and for all nonzero real numbers t.

In the case where R is a hypersurface, it is easy to see that Conjecture 3.3 holds true.

Example 3.4. If R is a hypersurface, then $\delta_t(G, X) = 0$ for all split generators G of $\mathsf{D_{sg}}(R)$, for all $X \in \mathsf{D_{sg}}(R)$ and for all $0 \neq t \in \mathbb{R}$. Indeed, in this case, there exists an isomorphism $\widehat{R} \cong S/(f)$, where S is a regular local ring and $f \in S$. The singularity category $\mathsf{D_{sg}}(\widehat{R})$ of the completion $\widehat{R}$ is equivalent as a triangulated category to the homotopy category of matrix factorizations of f over S, which is periodic of periodicity two; we refer the reader to [4, 7, 8, 23, 29] for the details. It is easy to see that $\mathsf{D_{sg}}(R)$ is also periodic of periodicity two, and the assertion follows from (1) and (3) of Proposition 2.7.

We introduce a condition on an object of the singularity category, which is essential in our theorems.

Definition 3.5. We say that an object X of $\mathsf{D_{sg}}(R)$ is *locally zero on the punctured spectrum of* R if for each nonmaximal prime ideal $\mathfrak{p}$ of R the localized complex $X_\mathfrak{p}$ is isomorphic to 0 in the singularity category $\mathsf{D_{sg}}(R_\mathfrak{p})$ of the local ring $R_\mathfrak{p}$. This condition is equivalent to saying that $X_\mathfrak{p}$ is isomorphic to a perfect complex over $R_\mathfrak{p}$ in the bounded derived category of finitely generated $R_\mathfrak{p}$-modules.

Remark 3.6. Suppose that R has an isolated singularity. Then every object of $\mathsf{D_{sg}}(R)$ is locally zero on the punctured spectrum of R, since $\mathsf{D_{sg}}(R_\mathfrak{p}) = 0$ for all nonmaximal prime ideals $\mathfrak{p}$ of R.

We establish a lemma, which is frequently used in the proofs of our results stated later.

Lemma 3.7. *Let $t \in \mathbb{R}$. Let X be an object of $\mathsf{D_{sg}}(R)$ such that k belongs to $\mathsf{thick}_{\mathsf{D_{sg}}(R)} X$. Let Y be an object of $\mathsf{D_{sg}}(R)$ which is locally zero on the punctured spectrum of R. If $\delta_t(k, k) = 0$, then $\delta_t(X, Y) = 0$.*

Proof. We have that $k \in \mathsf{thick}_{\mathsf{D_{sg}}(R)} X$ by assumption, and that $Y \in \mathsf{thick}_{\mathsf{D_{sg}}(R)} k$ by [28, Corollary 4.3(3)]. It follows from Proposition 2.7(1) that $\delta_t(X,k)$, $\delta_t(k,Y) \in \mathbb{R}$. If $\delta_t(k,k)=0$, then Lemma 2.9 yields

$$0 \leqslant \delta_t(X,Y) \leqslant \delta_t(X,k)\cdot\delta_t(k,Y) \leqslant (\delta_t(X,k)\cdot\delta_t(k,k))\cdot\delta_t(k,Y) = \delta_t(X,k)\cdot 0\cdot\delta_t(k,Y)=0.$$

These inequalities imply that $\delta_t(X,Y)=0$, and the proof of the lemma is completed. ■

Now we shall state and prove three theorems, all of which support Conjecture 3.3.

Theorem 3.8. *Let R be a complete intersection. Let $X \in \mathsf{D_{sg}}(R)$ be such that k belongs to $\mathsf{thick}_{\mathsf{D_{sg}}(R)} X$. Let $Y \in \mathsf{D_{sg}}(R)$ be locally zero on the punctured spectrum of R. Then $\delta_t(X,Y)=0$ for all $t \neq 0$.*

Proof. In view of Lemma 3.7, it suffices to show $\delta_t(k,k)=0$ for all $t\neq 0$. Let $\boldsymbol{x}=x_1,\dots,x_d$ be a system of parameters of R. Then $\boldsymbol{x}$ is an R-regular sequence since R is Cohen–Macaulay, and $R/(\boldsymbol{x})$ is an artinian complete intersection by [3, Theorem 2.3.4(a)]. Fix a nonnegative integer n. It holds in $\mathsf{D_{sg}}(R/(\boldsymbol{x}))$ that $k[-n]\cong\Omega^n_{R/(\boldsymbol{x})}k \in k^{\star h_n}$, where $h_n=\ell(\Omega^n_{R/(\boldsymbol{x})}k)$. Lemma 2.3(3a) implies that

$$k \in k[n]^{\star h_n} \quad \text{in } \mathsf{D_{sg}}(R/(\boldsymbol{x})). \tag{3.8.1}$$

Since $R/(\boldsymbol{x})$ is an artinian Gorenstein ring (or equivalently, selfinjective), $\mathsf{D_{sg}}(R/(\boldsymbol{x}))$ is equivalent (as a triangulated category) to the stable category of finitely generated $R/(\boldsymbol{x})$-modules; see [4, 14, 26]. It is observed that the assignment $X\mapsto X^*:=\mathrm{Hom}_{R/(\boldsymbol{x})}(X,R/(\boldsymbol{x}))$ gives a duality of $\mathsf{D_{sg}}(R/(\boldsymbol{x}))$. Note that k^* is isomorphic to k (as an $R/(\boldsymbol{x})$-module). Applying the above duality functor to (3.8.1), we obtain the containment

$$k \in k[-n]^{\star h_n} \quad \text{in } \mathsf{D_{sg}}(R/(\boldsymbol{x})). \tag{3.8.2}$$

As the R-module $R/(\boldsymbol{x})$ has finite projective dimension, the natural surjection $R\to R/(\boldsymbol{x})$ induces an exact functor $\mathsf{D_{sg}}(R/(\boldsymbol{x}))\to\mathsf{D_{sg}}(R)$. Applying this functor to (3.8.1) and (3.8.2), we get $k\in k[\pm n]^{\star h_n}$ in $\mathsf{D_{sg}}(R)$. The inequality $\delta_t(k,k)\leqslant h_n\cdot \mathrm{e}^{\pm nt}$ follows. The surjection $(R/(\boldsymbol{x}))^{\oplus\beta_n^{R/(\boldsymbol{x})}(k)}\to\Omega^n_{R/(\boldsymbol{x})}k$ shows that $\ell(R/(\boldsymbol{x}))\cdot\beta_n^{R/(\boldsymbol{x})}(k)\geqslant\ell(\Omega^n_{R/(\boldsymbol{x})}k)=h_n$. Since $R/(\boldsymbol{x})$ is a complete intersection, there exists $\alpha\in\mathbb{R}$ such that $\beta_i^{R/(\boldsymbol{x})}(k)\leqslant\alpha i^{c-1}$ for all integers $i\geqslant 0$, where $c=\operatorname{codim}R/(\boldsymbol{x})$; see [1, Theorem 8.1.2]. Thus,

$$0\leqslant\delta_t(k,k)\leqslant h_n\cdot\mathrm{e}^{\pm nt}\leqslant\ell(R/(\boldsymbol{x}))\cdot\beta_n^{R/(\boldsymbol{x})}(k)\cdot\mathrm{e}^{\pm nt}\leqslant\ell(R/(\boldsymbol{x}))\cdot\alpha\cdot\frac{n^{c-1}}{(\mathrm{e}^{\mp t})^n}\quad\text{for all } n\geqslant 0.$$

If $t>0$ (resp. $t<0$), then $\mathrm{e}^t>1$ (resp. $e^{-t}>1$), and $\frac{n^{c-1}}{(\mathrm{e}^t)^n}\to 0$ (resp. $\frac{n^{c-1}}{(\mathrm{e}^{-t})^n}\to 0$) as $n\to\infty$. We conclude that $\delta_t(k,k)=0$ for all nonzero real numbers t, and the assertion of the theorem follows. ■

Theorem 3.9. *Let R be singular and Cohen–Macaulay. Assume that the residue field k is infinite. Let X be an object of $\mathsf{D_{sg}}(R)$ such that $k\in\mathsf{thick}_{\mathsf{D_{sg}}(R)}X$. Let Y be an object of $\mathsf{D_{sg}}(R)$ which is locally zero on the punctured spectrum of R. Put $u=\mathrm{e}(R)$ and $r=\mathrm{r}(R)$. Then $\delta_t(X,Y)=0$ for all $t<-\log(u-1)$ and for all $t>\log(u-r)$. Therefore, $\delta_t(X,Y)=0$ for all $|t|>\log(u-1)$ provided that R is Gorenstein.*

Proof. By Lemma 3.7, it suffices to prove $\delta_t(k,k)=0$ for all $t<-\log(u-1)$ and $t>\log(u-r)$. Since k is infinite and R is Cohen–Macaulay, there exists an R-regular sequence $\boldsymbol{x}=x_1,\dots,x_d$ such that $d=\dim R$ and $u=\mathrm{e}(R)=\ell(R/(\boldsymbol{x}))$; see [3, Corollary 4.6.10 and Theorem 4.7.10]. Since the local ring R is singular, the artinian local ring $R/(\boldsymbol{x})$ is not a field, and there are inclusions $R/(\boldsymbol{x})\supsetneq\mathfrak{m}/(\boldsymbol{x})\supseteq\operatorname{Soc}R/(\boldsymbol{x})\supsetneq 0$, which give rise to the inequalities $u=\ell(R/(\boldsymbol{x}))>\ell(\operatorname{Soc}R/(\boldsymbol{x}))=r\geqslant 1$. Hence $\log(u-r)$ and $\log(u-1)$ are well-defined.

There is an exact sequence $0\to\mathfrak{m}/(\boldsymbol{x})\to R/(\boldsymbol{x})\to k\to 0$ of R-modules. This shows that $\ell(\mathfrak{m}/(\boldsymbol{x}))=u-1$, and that $k\cong(\mathfrak{m}/(\boldsymbol{x}))[1]$ in $\mathsf{D_{sg}}(R)$ as the R-module $R/(\boldsymbol{x})$ has finite projective dimension. Taking a composition series of the R-module $\mathfrak{m}/(\boldsymbol{x})$, we see that $\mathfrak{m}/(\boldsymbol{x})$ belongs to $k^{\star(u-1)}$. Lemma 2.3(3) implies

$$k\in k^{\star(u-1)}[1]\subseteq k[1]^{\star(u-1)}\subseteq(k^{\star(u-1)}[1])[1]^{\star(u-1)}\subseteq k[2]^{\star(u-1)^2}\subseteq\cdots\subseteq k[n]^{\star(u-1)^n}$$

for each integer $n>0$, which yields that $0\leqslant\delta_t(k,k)\leqslant(u-1)^n\mathrm{e}^{nt}=((u-1)\mathrm{e}^t)^n$. If $t<-\log(u-1)$, then $(u-1)\mathrm{e}^t<1$ and $((u-1)\mathrm{e}^t)^n\to 0$ as $n\to\infty$, whence $\delta_t(k,k)=0$.

It holds that $k^{\oplus r} \cong \operatorname{Soc} R/(\boldsymbol{x}) \subseteq R/(\boldsymbol{x})$, which induces an exact sequence $0 \to k^{\oplus r} \to R/(\boldsymbol{x}) \to C \to 0$ of R-modules. Similarly as above, we have $\ell(C) = u - r$, $k^{\oplus r} \cong C[-1]$ in $\mathsf{D_{sg}}(R)$, $C \in k^{\star(u-r)}$, and

$$\begin{aligned}
&k^{\oplus r} \in k^{\star(u-r)}[-1] \subseteq k[-1]^{\star(u-r)},\\
&k^{\oplus r^2} \in k^{\oplus r}[-1]^{\star(u-r)} \subseteq (k[-1]^{\star(u-r)})[-1]^{\star(u-r)} \subseteq k[-2]^{\star(u-r)^2}, \dots,\\
&k^{\oplus r^n} \in k[-n]^{\star(u-r)^n} \text{ for all integers } n > 0.
\end{aligned}$$

Hence, there are inequalities $0 \leqslant \delta_t(k,k) \leqslant (u-r)^n \mathrm{e}^{-nt} = ((u-r)\mathrm{e}^{-t})^n$ for all $n > 0$. If $t > \log(u-r)$, then $(u-r)\mathrm{e}^{-t} < 1$ and $((u-r)\mathrm{e}^{-t})^n \to 0$ as $n \to \infty$, whence $\delta_t(k,k) = 0$. Thus we are done. ■

Theorem 3.10. *Suppose that R is singular. Set $c = \operatorname{codepth} R$ and $m = \max_{1 \leqslant i \leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$. Let X be an object of $\mathsf{D_{sg}}(R)$ such that k belongs to $\mathsf{thick}_{\mathsf{D_{sg}}(R)} X$, and let Y be an object of $\mathsf{D_{sg}}(R)$ which is locally zero on the punctured spectrum of R. Then $\delta_t(X,Y) = 0$ for all $|t| > \frac{\log c + \log m}{2}$.*

Proof. First of all, we remark that the number $\frac{\log c + \log m}{2}$ is well-defined. Indeed, the assumption that R is singular implies $c > 0$. Since the module $\mathrm{H}_c(\mathrm{K}^R)$ is nonzero by [3, Theorem 1.6.17], we have $m > 0$.

According to Lemma 3.7, it is enough to verify that $\delta_t(k,k) = 0$ for all $|t| > \frac{\log c + \log m}{2}$. Put $n = \operatorname{edim} R$ and $K = \mathrm{K}^R$. We have $K = (0 \to K_n \to K_{n-1} \to \cdots \to K_{c+1} \to K_c \to K_{c-1} \to \cdots \to K_1 \to K_0 \to 0)$. For each $i \in \mathbb{Z}$, let Z_i, B_i and H_i be the ith cycle of K, the ith boundary of K and the ith homology of K, respectively. We have $H_i = 0$ for all $i > c$ and $H_c \neq 0$ by [3, Theorem 1.6.17(b)]. For every $i \in \mathbb{Z}$ there exist exact sequences $0 \to B_i \to Z_i \to H_i \to 0$ and $0 \to Z_i \to K_i \to B_{i-1} \to 0$ of R-modules, which induce exact triangles $B_i \to Z_i \to H_i \rightsquigarrow$ and $Z_i \to K_i \to B_{i-1} \rightsquigarrow$ in $\mathsf{D_{sg}}(R)$. Since K_i is a free R-module, we have $K_i \cong 0$ in $\mathsf{D_{sg}}(R)$. Hence $B_{i-1} \cong Z_i[1]$ in $\mathsf{D_{sg}}(R)$, and we get exact triangles

$$B_i[1] \to B_{i-1} \to H_i[1] \rightsquigarrow \qquad \text{and} \qquad H_i[-1] \to B_i \to B_{i-1}[-1] \rightsquigarrow \qquad \text{for each } i \in \mathbb{Z}.$$

It follows that $B_{i-1} \in B_i[1] \star H_i[1]$ and $B_i \in H_i[-1] \star B_{i-1}[-1]$. As $B_{-1} = 0$, we have $H_0 \cong B_0[1]$. Since the R-module B_c has finite projective dimension, we have $B_c \cong 0$ in $\mathsf{D_{sg}}(R)$, so that $H_c \cong B_{c-1}[-1]$. Applying Lemma 2.3(3), we inductively get the containments and inclusions

$$\begin{aligned}
k &= H_0 \in \bigstar_{i=1}^{c} H_{c-i+1}[c-i+2] \subseteq \bigstar_{i=1}^{c} k[c-i+2]^{\star h_{c-i+1}},\\
k^{\oplus s} &= H_c \in \bigstar_{i=1}^{c} H_{c-i}[-i-1] \subseteq \bigstar_{i=1}^{c} k[-i-1]^{\star h_{c-i}},
\end{aligned}$$

where we set $h_j = \dim_k H_j$ for each j and $s = h_c > 0$. Using Lemma 2.3(3) again, for any $r > 0$ we get:

$$\begin{aligned}
k &\in \bigstar_{i=1}^{c} k[c-i+2]^{\star h_{c-i+1}} \subseteq \bigstar_{i=1}^{c} (\bigstar_{j=1}^{c} k[c-j+2]^{\star h_{c-j+1}})[c-i+2]^{\star h_{c-i+1}}\\
&\subseteq \bigstar_{i=1}^{c} (\bigstar_{j=1}^{c} k[2(c+2)-(i+j)]^{\star h_{c-j+1}})^{\star h_{c-i+1}} \subseteq \cdots\\
&\subseteq \bigstar_{i_1=1}^{c} (\bigstar_{i_2=1}^{c} (\cdots (\bigstar_{i_r=1}^{c} k[r(c+2)-(i_1+\cdots+i_r)]^{\star h_{c-i_r+1}}) \cdots)^{\star h_{c-i_2+1}})^{\star h_{c-i_1+1}},\\
k^{\oplus s} &\in \bigstar_{i=1}^{c} k[-i-1]^{\star h_{c-i}},\\
k^{\oplus s^2} &= (k^{\oplus s})^{\oplus s} \in \bigstar_{i=1}^{c} k^{\oplus s}[-i-1]^{\star h_{c-i}}\\
&\subseteq \bigstar_{i=1}^{c} (\bigstar_{j=1}^{c} k[-j-1]^{\star h_{c-j}})[-i-1]^{\star h_{c-i}} \subseteq \bigstar_{i=1}^{c} (\bigstar_{j=1}^{c} k[-(i+j)-2]^{\star h_{c-j}})^{\star h_{c-i}}, \dots,\\
k^{\oplus s^r} &\in \bigstar_{i_1=1}^{c} (\bigstar_{i_2=1}^{c} (\cdots (\bigstar_{i_r=1}^{c} k[-(i_1+\cdots+i_r)-r]^{\star h_{c-i_r}}) \cdots)^{\star h_{c-i_2}})^{\star h_{c-i_1}}.
\end{aligned}$$

We thus obtain the following two inequalities for every positive integer r.

$$\begin{aligned}
\delta_t(k,k) &\leqslant \textstyle\sum_{i_1=1}^{c} h_{c-i_1+1} (\sum_{i_2=1}^{c} h_{c-i_2+1} (\cdots (\sum_{i_r=1}^{c} h_{c-i_r+1} \mathrm{e}^{(r(c+2)-(i_1+\cdots+i_r))t}) \cdots))\\
&= \textstyle\sum_{1 \leqslant i_1,\dots,i_r \leqslant c} h_{c-i_1+1} \cdots h_{c-i_r+1} \mathrm{e}^{((c-i_1+1)+\cdots+(c-i_r+1)+r)t}\\
&= \textstyle\sum_{1 \leqslant i_1,\dots,i_r \leqslant c} h_{i_1} \cdots h_{i_r} \mathrm{e}^{(i_1+\cdots+i_r+r)t},\\
\delta_t(k,k) &\leqslant \textstyle\sum_{i_1=1}^{c} h_{c-i_1} (\sum_{i_2=1}^{c} h_{c-i_2} (\cdots (\sum_{i_r=1}^{c} h_{c-i_r} \mathrm{e}^{(-(i_1+\cdots+i_r)-r)t}) \cdots))\\
&= \textstyle\sum_{1 \leqslant i_1,\dots,i_r \leqslant c} h_{c-i_1} \cdots h_{c-i_r} \mathrm{e}^{((c-i_1)+\cdots+(c-i_r)-r(c+1))t}\\
&= \textstyle\sum_{0 \leqslant i_1,\dots,i_r \leqslant c-1} h_{i_1} \cdots h_{i_r} \mathrm{e}^{(i_1+\cdots+i_r-r(c+1))t}.
\end{aligned}$$

As $h_0 = 1$ and $h_c \geqslant 1$, we have $m = \max\{h_1, \dots, h_c\} = \max\{h_0, h_1, \dots, h_c\}$.

Now, consider the case where $t < -\frac{\log c + \log m}{2}$. Note then that $t < 0$ and $cm\mathrm{e}^{2t} < 1$. We get inequalities

$$0 \leqslant \delta_t(k,k) \leqslant \textstyle\sum_{1 \leqslant i_1,\dots,i_r \leqslant c} h_{i_1} \cdots h_{i_r} \mathrm{e}^{(i_1+\cdots+i_r+r)t} \leqslant \sum_{1 \leqslant i_1,\dots,i_r \leqslant c} m^r \mathrm{e}^{2rt} = (cm\mathrm{e}^{2t})^r.$$

Since $(cm\mathrm{e}^{2t})^r \to 0$ as $r \to \infty$, we obtain the equality $\delta_t(k,k) = 0$. Next, let us deal with the case where $t > \frac{\log c + \log m}{2}$. Then we have $t > 0$ and $cm\mathrm{e}^{-2t} < 1$. Hence the following inequalities hold true.

$$0 \leqslant \delta_t(k,k) \leqslant \textstyle\sum_{0 \leqslant i_1,\dots,i_r \leqslant c-1} h_{i_1} \cdots h_{i_r} \mathrm{e}^{(i_1+\cdots+i_r-r(c+1))t} \leqslant \sum_{0 \leqslant i_1,\dots,i_r \leqslant c-1} m^r \mathrm{e}^{-2rt} = (cm\mathrm{e}^{-2t})^r.$$

Since $(cme^{-2t})^r \to 0$ as $r \to \infty$, we get $\delta_t(k,k) = 0$. Now the proof of the theorem is completed. ■

Remark 3.11. (1) Put $n = \operatorname{edim} R$. Cohen's structure theorem shows that there exist an n-dimensional regular local ring $(S, \mathfrak{n}, k)$ and an ideal I of S such that the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is isomorphic to the residue ring S/I. Choose a minimal system of generators $\boldsymbol{x} = x_1, \dots, x_n$ of $\mathfrak{n}$. It holds that

$$\mathrm{H}_i(\mathrm{K}^R) = \mathrm{H}_i(\boldsymbol{x}, R) \cong \mathrm{H}_i(\boldsymbol{x}, R) \otimes_R \widehat{R} \cong \mathrm{H}_i(\boldsymbol{x}, \widehat{R}) \cong \mathrm{H}_i(\mathrm{K}(\boldsymbol{x}, S) \otimes_S \widehat{R}) \cong \operatorname{Tor}_i^S(k, \widehat{R})$$

for each integer i, where the first isomorphism holds since the R-module $\mathrm{H}_i(\boldsymbol{x}, R)$ has finite length, while the last isomorphism follows from the fact that the Koszul complex $\mathrm{K}(\boldsymbol{x}, S)$ is a free resolution of k over S. Hence, the number $\dim_k \mathrm{H}_i(\mathrm{K}^R)$ is equal to the ith Betti number $\beta_i^S(\widehat{R})$ of $\widehat{R}$ over S.

(2) Let R be a singular hypersurface. Let G be a split generator of $\mathsf{D_{sg}}(R)$, and let X be an object of $\mathsf{D_{sg}}(R)$ which is locally zero on the punctured spectrum of R. The following two statements hold.

(a) As R is a complete intersection, Theorem 3.8 implies that $\delta_t(G, X) = 0$ for all $0 \ne t \in \mathbb{R}$.

(b) Put $c = \operatorname{codepth} R$ and $m = \max_{1 \leqslant i \leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$. Then $c = 1$. We have $\widehat{R} \cong S/(f)$ for some regular local ring $(S, \mathfrak{n})$ and some element $f \in \mathfrak{n}^2$. The sequence $0 \to S \xrightarrow{f} S \to \widehat{R} \to 0$ gives a minimal free resolution of the S-module $\widehat{R}$, and the equalities $\dim_k \mathrm{H}_1(\mathrm{K}^R) = \beta_1^S(\widehat{R}) = 1$ hold by (1). Hence $m = 1$. We get $\frac{\log c + \log m}{2} = 0$, and $\delta_t(G, X) = 0$ for all $t \ne 0$ by Theorem 3.10.

Thus, each of Theorems 3.8 and 3.10 recovers Example 3.4 in the case where X is locally zero on the punctured spectrum of R (e.g., in the case where R has an isolated singularity by Remark 3.6).

Combining the above three theorems with Remark 3.6, we obtain the corollary below on entropies.

Corollary 3.12. *Let R be singular with an isolated singularity. Let F be an exact endofunctor of $\mathsf{D_{sg}}(R)$.*

(1) *Put $c = \operatorname{codepth} R$ and $m = \max_{1 \leqslant i \leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\}$. Then $\delta_t(G, X) = 0$ for all split generators $G \in \mathsf{D_{sg}}(R)$, all $X \in \mathsf{D_{sg}}(R)$ and all $|t| > \frac{\log c + \log m}{2}$. Thus $\mathrm{h}_t(F)$ is not defined if $|t| > \frac{\log c + \log m}{2}$.*

(2) *Assume that R is Gorenstein and k is infinite. Then $\delta_t(G, X) = 0$ for all split generators $G \in \mathsf{D_{sg}}(R)$, all $X \in \mathsf{D_{sg}}(R)$ and all $|t| > \log(\mathrm{e}(R) - 1)$. Thus $\mathrm{h}_t(F)$ is not defined for $|t| > \log(\mathrm{e}(R) - 1)$.*

(3) *Suppose that R is a complete intersection. Then $\delta_t(G, X) = 0$ for all split generators $G \in \mathsf{D_{sg}}(R)$, all $X \in \mathsf{D_{sg}}(R)$ and all nonzero real numbers t. Therefore, the entropy $\mathrm{h}_t(F)$ is defined only for $t = 0$.*

We present three examples below, which say that the bounds $\frac{\log c + \log m}{2}$ and $\log(\mathrm{e}(R) - 1)$ for the real numbers t given in Theorems 3.9, 3.10 and Corollary 3.12(1)(2) are not necessarily best possible.

Example 3.13. Let R be a singular Burch ring in the sense of [5, Definition 2.8]. Then it follows from [5, Proposition 5.10] that $\Omega^u k$ is a direct summand of $\Omega^{u+2} k$, where $u = \operatorname{depth} R$. Hence, in $\mathsf{D_{sg}}(R)$, the object $k[-u]$ is a direct summand of $k[-u-2]$, so that k is a direct summand of $k[-2n]$ for all $n \in \mathbb{Z}_{>0}$. We get $0 \leqslant \delta_t(k, k) \leqslant \mathrm{e}^{-2nt} = (\mathrm{e}^{-2t})^n$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}_{>0}$, which implies $\delta_t(k, k) = 0$ for all $t \in \mathbb{R}_{>0}$. Lemma 3.7 shows $\delta_t(X, Y) = 0$ for all $t \in \mathbb{R}_{>0}$ and all $X, Y \in \mathsf{D_{sg}}(R)$ such that k is in $\mathsf{thick}\, X$ and Y is locally zero on the punctured spectrum of R. By Remark 3.6, if R has an isolated singularity, then $\delta_t(G, X) = 0$ for all $t \in \mathbb{R}_{>0}$, all split generators G of $\mathsf{D_{sg}}(R)$ and all objects X of $\mathsf{D_{sg}}(R)$.

Example 3.14. Let $S = k[\![x, y]\!]$ be a formal power series ring over a field k. Consider the residue ring $R = S/(x^2, xy)$. Then R is not Cohen–Macaulay, but has an isolated singularity and $\operatorname{depth} R = 0$. Fix a split generator $G \in \mathsf{D_{sg}}(R)$ and an object $X \in \mathsf{D_{sg}}(R)$. We have $c := \operatorname{codepth} R = 2$, and the minimal free resolution of R over S is $0 \to S \xrightarrow{\binom{y}{-x}} S^{\oplus 2} \xrightarrow{(x^2, xy)} S \to R \to 0$. We get $m := \max_{1 \leqslant i \leqslant c}\{\dim_k \mathrm{H}_i(\mathrm{K}^R)\} = \max\{\beta_1^S(R), \beta_2^S(R)\} = 2$; see Remark 3.11(1). Hence $\frac{\log c + \log m}{2} = \log 2$, and Corollary 3.12(1) says

$$\delta_t(G, X) = 0 \quad \text{for all} \quad t \in (-\infty, -\log 2) \cup (\log 2, \infty). \tag{3.14.1}$$

We shall give sharper bounds for $t \in \mathbb{R}$ to satisfy $\delta_t(G, X) = 0$. Since the ideal (x) is isomorphic to the residue field k and the element y is regular on $R/(x)$, there exist exact sequences $0 \to k \to R \to R/(x) \to 0$ and $0 \to R/(x) \xrightarrow{y} R/(x) \to k \to 0$. Hence there is an isomorphism $R/(x) \cong k[1]$ in $\mathsf{D_{sg}}(R)$, and we get an exact triangle $k[1] \to k \to k[2] \rightsquigarrow$ in $\mathsf{D_{sg}}(R)$. Applying Lemma 2.3(3a), we obtain

$$k \in k[1] \star k[2] \subseteq (k[1] \star k[2])[1] \star (k[1] \star k[2])[2] \subseteq k[2] \star k[3]^{\star 2} \star k[4] \cdots.$$

We see that k belongs to

$$\begin{cases} \mathcal{X}_1 = 1 \cdot 2, \\ \mathcal{X}_2 = 2 \cdot 3^2 \cdot 4, \\ \mathcal{X}_3 = 3 \cdot 4^2 \cdot 5 \cdot 4 \cdot 5^2 \cdot 6, \\ \mathcal{X}_4 = 4 \cdot 5^2 \cdot 6 \cdot 5 \cdot 6^2 \cdot 7 \cdot 5 \cdot 6^2 \cdot 7 \cdot 6 \cdot 7^2 \cdot 8, \\ \quad \cdots \\ \mathcal{X}_n = n \cdot (n+1)^2 \cdot (n+2) \cdot (n+1) \cdots (2n-1) \cdot (2n-2) \cdot (2n-1)^2 \cdot (2n) \end{cases}$$

where $a_1 \cdots a_r$ denotes $k[a_1] \star \cdots \star k[a_r]$ and $b^2 := b \cdot b$ for $a_1, \dots, a_r, b \in \mathbb{Z}$. Note that $\mathcal{X}_n$ coincides with $\bigstar_{i=0}^{n} k[n+i]^{\star\binom{n}{i}}$ if we ignore the commutativity of the operation $- \star -$. Using the binomial theorem, we get the following inequalities and equalities for all $n > 0$.

$$0 \leqslant \delta_t(k,k) \leqslant \sum_{i=0}^{n} \binom{n}{i} \mathrm{e}^{(n+i)t} = \mathrm{e}^{nt} \sum_{i=0}^{n} \binom{n}{i} (\mathrm{e}^t)^i = \mathrm{e}^{nt}(e^t+1)^n = (\mathrm{e}^t(e^t+1))^n.$$

If $t < \log \frac{\sqrt{5}-1}{2}$, then $\mathrm{e}^t(e^t+1) < 1$, and we see that $\delta_t(k,k) = 0$. In view of [5, Corollary 6.5], the local ring R is a singular Burch ring. It follows from Example 3.13 that

$$\delta_t(G,X) = 0 \quad \text{for all} \quad t \in (-\infty, \log \tfrac{\sqrt{5}-1}{2}) \cup (0, \infty). \tag{3.14.2}$$

Note that $-\log 2 < \log \frac{\sqrt{5}-1}{2} < 0 < \log 2$. Consequently, (3.14.2) yields better upper and lower bounds for $t \in \mathbb{R}$ to satisfy $\delta_t(G,X) = 0$ than (3.14.1).

Example 3.15. Let k be an infinite field, and consider the ring $R = k[x,y,z]/(x^2-y^2, x^2-z^2, xy, xz, yz)$. Then R is an artinian Gorenstein local ring with $\operatorname{edim} R = 3$ and $\mathrm{e}(R) = \ell(R) = 5$, which is not a complete intersection. Let G be a split generator of $\mathsf{D_{sg}}(R)$ and X an object of $\mathsf{D_{sg}}(R)$. Theorem 3.9 implies

$$\delta_t(G,X) = 0 \quad \text{for all} \quad |t| > \log 4. \tag{3.15.1}$$

By [27, Corollary 3] the Poincaré series of the R-module k is $\frac{1}{1-3t+t^2}$, so that $\beta_n^R(k) = \frac{1}{\sqrt{5}}(a^{n+1} - b^{n+1})$ for each $n \geqslant 0$, where $a := \frac{3+\sqrt{5}}{2} > \frac{3-\sqrt{5}}{2} =: b$. The same argument as in the proof of Theorem 3.8 shows

$$0 \leqslant \delta_t(k,k) \leqslant \ell(R) \cdot \beta_n^R(k) \cdot \mathrm{e}^{\pm nt} = \sqrt{5} \cdot (a(\tfrac{a}{\mathrm{e}^{\mp t}})^n - b(\tfrac{b}{\mathrm{e}^{\mp t}})^n)$$

for all $n \geqslant 0$, which implies $\delta_t(k,k) = 0$ for all $|t| > \log a = \log \frac{3+\sqrt{5}}{2}$, and

$$\delta_t(G,X) = 0 \quad \text{for all} \quad |t| > \log \tfrac{3+\sqrt{5}}{2}. \tag{3.15.2}$$

As $\log \frac{3+\sqrt{5}}{2} < \log 4$, (3.15.2) gives better upper/lower bounds for $t \in \mathbb{R}$ with $\delta_t(G,X) = 0$ than (3.15.1).

Acknowlegments. The author would like to thank Akishi Ikeda, Osamu Iyama and Genki Ouchi for giving him valuable and useful comments regarding complexities and entropies.

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