The category of modules of Gorenstein dimension zero

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The contents of this note are entirely contained in the author's own papers [17], [18], and [19].

Let R be a commutative noetherian non-Gorenstein local ring. In this note, we consider the following problem:

(A) There exist infinitely many isomorphism classes of indecomposable R-modules of Gorenstein dimension zero provided there exists at least a non-free R-module of Gorenstein dimension zero.

We shall work on this problem from a categorical point of view. Denote by $\mathrm{mod}R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\mathrm{mod}R$ consisting of all modules of Gorenstein dimension zero. The problem (A) is resolved if we can prove the following:

(B) The category $\mathcal{G}(R)$ is not contravariantly finite in $\mathrm{mod}R$ provided that the ring R has a non-free module in $\mathcal{G}(R)$.

It is proved in this note that the problem (B) holds if R is a henselian local ring of depth at most two.

1 Introduction

Throughout this note, we assume that all rings are commutative noetherian rings and all modules are finitely generated modules.

Gorenstein dimension (G-dimension for short), which is a homological invariant for modules, was defined by Auslander [1] and was deeply studied by him and Bridger [2]. With that as a start, G-dimension has been studied by a lot of algebraists until now.

The notion of modules of finite G-dimension is a common generalization of that of modules of finite projective dimension and that of modules over Gorenstein local rings: over an arbitrary local ring all modules of finite projective dimension are also of finite G-dimension, and all modules over a Gorenstein local ring are of finite G-dimension. (Conversely, a local ring whose residue class field has finite G-dimension is Gorenstein. In the next section, we will introduce several properties of G-dimension.)

Over a Gorenstein local ring, a module has G-dimension zero if and only if it is a maximal Cohen-Macaulay module. Hence it is natural to expect that modules of G-dimension zero over an arbitrary local ring may behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring.

A Cohen-Macaulay local ring is called to be of finite Cohen-Macaulay representation type if it has only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules. This kind of rings have been well researched for a long time. Under a few assumptions, Gorenstein local rings of finite Cohen-Macaulay representation type have been classified completely, and it is known that all isomorphism classes of indecomposable maximal Cohen-Macaulay modules over them can be described concretely; see [25] for the details.

Thus we are interested in non-Gorenstein local rings which have only finitely many isomorphism classes of indecomposable modules of G-dimension zero, especially interested in determining all isomorphism classes of indecomposable modules of G-dimension zero over such rings.

Here, it is natural to ask whether such a ring in fact exists or not. Such a ring does exist. For example, let $(R, \mathfrak{m})$ be a non-Gorenstein local ring with $\mathfrak{m}^2 = 0$. Then every indecomposable R-module of G-dimension zero is isomorphic to R (cf. [27, Proposition 2.4]).

Thus, we would like to know whether there exists a non-Gorenstein local ring which has a non-free module of G-dimension zero and only finitely many isomorphism classes of indecomposable modules of G-dimension zero. Our guess is that such a ring can not exist:

Conjecture 1.1 Let R be a non-Gorenstein local ring. Suppose that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.

Indeed, over a certain artinian local ring having a non-free module of G-dimension zero, Yoshino [27] has constructed a family of modules of G-dimension zero with continuous parameters.

The above conjecture is against our expectation that modules of G-dimension zero over an arbitrary local ring behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring. Indeed, let S be a d-dimensional non-regular Gorenstein local ring of finite Cohen-Macaulay representation type. (Such a ring does exist; see [25].) Then the dth syzygy module of the residue class field of S is a non-free maximal Cohen-Macaulay S-module. Hence the above conjecture does not necessarily hold without the assumption that R is non-Gorenstein.

For a local ring R, we denote by $\mathrm{mod} R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\mathrm{mod} R$ consisting of all R-modules of G-dimension zero. We guess that even the following statement that is stronger than Conjecture 1.1 is true. (It will be seen from Proposition 2.9 that Conjecture 1.2 implies Conjecture 1.1.)

Conjecture 1.2 Let R be a non-Gorenstein local ring. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in $\mathrm{mod} R$.

The purpose of this note is to prove that Conjecture 1.2 is true if R is a henselian local ring of depth at most two:

Theorem 1.3 *Let R be a henselian non-Gorenstein local ring of depth at most two. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in* $\mathrm{mod}R$.

We should remark that the above theorem especially asserts that Conjecture 1.2 holds if R is an artinian local ring and hence that the above theorem extends [27, Theorem 6.1].

In Section 2, we introduce several notions for later use. In Section 3, 4, and 5, we shall prove Theorem 1.3 when R has depth zero, one, and two, respectively.

2 Preliminaries

In this section, we recall the definitions of G-dimension and a (pre)cover, and give several preliminary lemmas involving Wakamatsu's Lemma, which plays a key role for proving Theorem 1.3.

Throughout this section, let R be a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$. Let $k = R/\mathfrak{m}$ be the residue class field of R. All R-modules in this section are assumed to be finitely generated.

First of all, we recall the definition of G-dimension. Put $M^* = \mathrm{Hom}_R(M, R)$ for an R-module M.

Definition 2.1 Let M be an R-module.

(1) If the following conditions hold, then we say that M has *G-dimension zero*, and write $\text{G-dim}_R M = 0$.

i) The natural homomorphism $M \to M^{**}$ is an isomorphism.

ii) $\mathrm{Ext}_R^i(M, R) = 0$ for every $i > 0$.

iii) $\mathrm{Ext}_R^i(M^*, R) = 0$ for every $i > 0$.

(2) If n is a non-negative integer such that there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of R-modules with $\text{G-dim}_R G_i = 0$ for every $i = 0, 1, \cdots, n$, then we say that M has *G-dimension at most n*, and write $\text{G-dim}_R M \leq n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\text{G-dim}_R M = \infty$.

(3) If M has G-dimension at most n but does not have G-dimension at most $n-1$, then we say that M has G-dimension n and write $\text{G-dim}_R M = n$.

For an R-module M, we denote by $\Omega^n M$ the nth syzygy module of M, and set $\Omega M = \Omega^1 M$. G-dimension is a homological invariant for modules sharing a lot of properties with projective dimension. We state here just the properties that will be used later.

Proposition 2.2 (1) *The following conditions are equivalent.*

i) *R is Gorenstein.*

ii) $\text{G-dim}_R M < \infty$ *for any R-module M.*

iii) $\text{G-dim}_R k < \infty$.

(2) *Let M be a non-zero R-module with* $\text{G-dim}_R M < \infty$. *Then*

$$\text{G-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M.$$

(3) *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. If two of L, M, N have finite G-dimension, then so does the third.*

(4) *Let M be an R-module. Then*

$$\text{G-dim}_R(\Omega^n M) = \sup\{\text{G-dim}_R M - n, 0\}$$

for any $n \geq 0$.

(5) *Let M, N be R-modules. Then*

$$\text{G-dim}_R(M \oplus N) = \sup\{\text{G-dim}_R M, \text{G-dim}_R N\}.$$

The proof of this proposition and other properties of G-dimension are stated in detail in [2, Chapter 3,4] and [9, Chapter 1].

We denote by $\text{mod}R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\text{mod}R$ consisting of all R-modules of G-dimension zero. For the minimal free presentation

$$F_1 \xrightarrow{\partial} F_0 \to M \to 0$$

of an R-module M, we denote by $\text{tr}M$ the cokernel of the dual homomorphism $\partial^* : F_0^* \to F_1^*$. The following result follows directly from Proposition 2.2.

Corollary 2.3 *Let M be an R-module. The category $\mathcal{G}(R)$ has the following properties.*

(1) *If M belongs to $\mathcal{G}(R)$, then so do M^*, ΩM, $\text{tr}M$, and any direct summand of M.*

(2) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. If L and N belong to $\mathcal{G}(R)$, then so does M.*

Now we introduce the notion of a cover of a module.

Definition 2.4 Let $\mathcal{X}$ be a full subcategory of $\text{mod}R$.

(1) Let $\phi : X \to M$ be a homomorphism from $X \in \mathcal{X}$ to $M \in \text{mod}R$.

i) We call ϕ an *$\mathcal{X}$-precover* of M if for any homomorphism $\phi' : X' \to M$ with $X' \in \mathcal{X}$ there exists a homomorphism $f : X' \to X$ such that $\phi' = \phi f$.

ii) Assume that ϕ is an $\mathcal{X}$-precover of M. We call ϕ an *$\mathcal{X}$-cover* of M if any endomorphism f of X with $\phi = \phi f$ is an automorphism.

(2) The category $\mathcal{X}$ is said to be *contravariantly finite* if every $M \in \mathrm{mod}R$ has an $\mathcal{X}$-precover.

An $\mathcal{X}$-precover (resp. an $\mathcal{X}$-cover) is often called a right $\mathcal{X}$-approximation (resp. a right minimal $\mathcal{X}$-approximation).

Proposition 2.5 *[26, Lemma (2.2)] Let $\mathcal{X}$ be a full subcategory of* $\mathrm{mod}R$. *Suppose that R is henselian.*

(1) *Let $0 \to N \xrightarrow{\psi} X \xrightarrow{\phi} M$ be an exact sequence of R-modules where ϕ is an $\mathcal{X}$-precover. Then the following conditions are equivalent.*

- i) *ϕ is not an $\mathcal{X}$-cover.*
- ii) *There exists a non-zero submodule L of N such that $\psi(L)$ is a direct summand of X.*

(2) *The following conditions are equivalent for an R-module M.*

- i) *M has an $\mathcal{X}$-precover.*
- ii) *M has an $\mathcal{X}$-cover.*

Proof (1) ii) $\Longrightarrow$ i): Let X' be the complement of $\psi(L)$ in X. Let $\theta : X' \to X$ (resp. $\pi : X \to X'$) be the natural inclusion (resp. the natural projection), and set $f = \theta\pi$. We easily see that $\phi = \phi f$. Suppose that ϕ is an $\mathcal{X}$-cover. Then the endomorphism f of X is an isomorphism, and hence θ and π are isomorphisms. Therefore we have $\psi(L) = 0$. Since ψ is an injection, we have $L = 0$, which is contradiction. Thus ϕ is not an $\mathcal{X}$-cover.

i) $\Longrightarrow$ ii): There exists a non-isomorphism $f \in \mathrm{End}_R X$ such that $\phi = \phi f$. Let $S = R[f]$ be the subalgebra of $\mathrm{End}_R X$ generated by f over R. Note that S is a commutative ring.

Assume that S is a local ring. Let $\mathfrak{n}$ be the unique maximal ideal of S. Noting that S is a finitely generated R-module, we see that the factor ring $S/\mathfrak{m}S$ is an artinian local ring with maximal ideal $\mathfrak{n}/\mathfrak{m}S$. Hence $n^r \subseteq \mathfrak{m}S$ for some integer r. Since $f \in \mathfrak{n}$, we have

$$f^r = a_0 + a_1 f + \cdots + a_s f^s$$

with $a_i \in \mathfrak{m}$, $0 \leq i \leq s$. Noting that $\phi = \phi f$, we get

$$\begin{array}{rcl} \phi & = & \phi f^r \\ & = & (a_0 + a_1 + \cdots + a_s)\phi \\ & \in & \mathfrak{m}\phi. \end{array}$$

It follows from Nakayama's Lemma that $\phi = 0$, i.e., the homomorphism $\psi : N \to X$ is isomorphic. Since f is not an isomorphism, we especially have $X \neq 0$, and hence $N \neq 0$. The module $L := N$ satisfies the condition ii).

Thus, it is enough to consider the case that S is not a local ring. Since R is henselian, the finite R-algebra S is a product of local rings, and hence there is a non-trivial idempotent e in S. Write

$$e = b_0 + b_1 f + \cdots + b_t f^t$$

with $b_i \in R$, $0 \leq i \leq t$, and put $b = b_0 + b_1 + \cdots + b_t$. Taking $1 - e$ instead of e when b is an element in $\mathfrak{m}$, we may assume that b is not an element in $\mathfrak{m}$, i.e., b is a unit of R.

It is easy to see that we have a direct sum decomposition

$$X = \operatorname{Ker} e \oplus \operatorname{Im} e.$$

Since e is not an isomorphism, we have $\operatorname{Ker} e \neq 0$. Noting that $\phi e = b\phi$ and that b is a unit of R, we obtain $\operatorname{Ker} e \subseteq \operatorname{Im} \psi$. Thus $L := \psi^{-1}(\operatorname{Ker} e)$ satisfies the condition ii).

(2) It is obvious that ii) implies i). Let $\phi : X \to M$ be an $\mathcal{X}$-precover of M. Putting $N = \operatorname{Ker} \phi$, we get an exact sequence

$$0 \to N \xrightarrow{\psi} X \xrightarrow{\phi} M,$$

where ψ is the natural inclusion. Suppose that ϕ is not an $\mathcal{X}$-cover. Then it follows from (1) that there exists a non-zero submodule L of N such that $\psi(L)$ is a direct summand of X. Note that L is a direct summand of N. Let N' (resp. X') be the complement of L (resp. $\psi(L)$) in N (resp. X). Then X' belongs to $\mathcal{X}$ by the assumption, and an exact sequence

$$0 \to N' \xrightarrow{\psi'} X' \xrightarrow{\phi'} M$$

is induced. It is easily seen that ϕ' is an $\mathcal{X}$-precover of M. Since the minimal number of generators of X' is smaller than that of X, repeating the same argument, we eventually obtain an $\mathcal{X}$-cover of M. $\Box$

We say that a full subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is closed under direct summands if any direct summand of any R-module belonging to $\mathcal{X}$ also belongs to $\mathcal{X}$. Note by Corollary 2.3(1) that $\mathcal{G}(R)$ is closed under direct summands.

Remark 2.6 Under the assumption of the above proposition, suppose that $\mathcal{X}$ is closed under direct summands. Let $0 \to N \xrightarrow{\psi} X \xrightarrow{\phi} M$ be an exact sequence of R-modules where ϕ is an $\mathcal{X}$-precover. It is seen from the proof of the second statement in the above proposition that there exists a direct summand L of N satisfying the following conditions:

i) $\psi(L)$ is a direct summand of X.

ii) Let N' (resp. X') be the complement of L (resp. $\psi(L)$) in N (resp. X), and let $0 \to N' \xrightarrow{\psi'} X' \xrightarrow{\phi'} M$ be the induced exact sequence. Then ϕ' is an $\mathcal{X}$-cover of M.

We say that a full subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is closed under extensions provided that for any short exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod} R$, if $L, N \in \mathcal{X}$ then $M \in \mathcal{X}$. Note by Corollary 2.3(2) that $\mathcal{G}(R)$ is closed under extensions. The lemma below is so-called Wakamatsu's Lemma, which plays an important role in the notion of a cover. For the proof, see [22] or [23, Lemma 2.1.1].

Lemma 2.7 (Wakamatsu) *Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod} R$ which is closed under extensions, and let $0 \to N \to X \xrightarrow{\phi} M$ be an exact sequence of R-modules where ϕ is an $\mathcal{X}$-cover. Then $\operatorname{Ext}^1_R(Y, N) = 0$ for every $Y \in \mathcal{X}$.*

For R-modules M, N, we define a homomorphism $\lambda_M(N) : M \otimes_R N \to \mathrm{Hom}_R(M^*, N)$ of R-modules by $\lambda_M(N)(m \otimes n)(f) = f(m)n$ for $m \in M,\ n \in N$ and $f \in M^*$.

Proposition 2.8 *[2, Proposition 2.6] Let M be an R-module. There is an exact sequence*

$$\begin{array}{cccccccc} 0 & \longrightarrow & \mathrm{Ext}^1_R(\mathrm{tr}M, -) & \longrightarrow & M \otimes_R - & \xrightarrow{\lambda_M(-)} & \mathrm{Hom}_R(M^*, -) \\ & \longrightarrow & \mathrm{Ext}^2_R(\mathrm{tr}M, -) & \longrightarrow & 0 & & \end{array}$$

of functors from $\mathrm{mod}R$ *to itself.*

We close this section by showing the following proposition, which proves that Conjecture 1.2 implies Conjecture 1.1.

Proposition 2.9 *Suppose that there exist only finitely many isomorphism classes of indecomposable R-modules of G-dimension zero. Then $\mathcal{G}(R)$ is a contravariantly finite subcategory of* $\mathrm{mod}R$.

Proof We can show this by means of the idea appearing in the proof of [4, Proposition 4.2]. Fix an R-module M. Let X be the direct sum of the complete representatives of the isomorphism classes of indecomposable R-modules of G-dimension zero. Note that the R-module X is finitely generated. Taking a system of generators $\phi_1, \phi_2, \cdots, \phi_n$ of the R-module $\mathrm{Hom}_R(X, M)$, we easily see that the homomorphism $(\phi_1, \phi_2, \cdots, \phi_n) : X^n \to M$ is a $\mathcal{G}(R)$-precover of M. It follows that $\mathcal{G}(R)$ is contravariantly finite in $\mathrm{mod}R$. $\Box$

3 The depth zero case

Throughout this section, R is always a local ring with maximal ideal $\mathfrak{m}$ and with residue class field k. We assume that all R-modules in this section are finitely generated.

Theorem 3.1 *Let R be a henselian non-Gorenstein local ring of depth zero. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the residue class field of R does not admit a $\mathcal{G}(R)$-precover. In particular, the category $\mathcal{G}(R)$ is not contravariantly finite in* $\mathrm{mod}R$.

Proof Suppose that the residue field k of R has a $\mathcal{G}(R)$-precover as an R-module. We want to derive contradiction. Proposition 2.5(2) implies that k has a $\mathcal{G}(R)$-cover $\pi : Z \to k$. Since $R \in \mathcal{G}(R)$, every homomorphism $R \to k$ is factored as $R \to Z \xrightarrow{\pi} k$, that is, the homomorphism π is surjective. Hence there exists a short exact sequence

$$0 \to L \xrightarrow{\theta} Z \xrightarrow{\pi} k \to 0$$

of R-modules. Dualizing this sequence, we obtain an exact sequence

$$0 \to k^* \xrightarrow{\pi^*} Z^* \xrightarrow{\theta^*} L^*.$$

Set $C = \mathrm{Im}(\theta^*)$, and let $\alpha : Z^* \to C$ be the map induced by θ^* and $\beta : C \to L^*$ be the natural embedding.

We shall show that the surjective homomorphism $\alpha : Z^* \to C$ is a $\mathcal{G}(R)$-cover of C. Fix $X \in \mathcal{G}(R)$. To prove that any homomorphism $X \to C$ is factored as $X \to Z^* \xrightarrow{\alpha} C$, we may assume that X is non-free and indecomposable. Applying the functor $\mathrm{Hom}_R(X, -)$ to the above exact sequence, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(X, k^*) \xrightarrow{\mathrm{Hom}_R(X,\pi^*)} \mathrm{Hom}_R(X, Z^*) \xrightarrow{\mathrm{Hom}_R(X,\theta^*)} \mathrm{Hom}_R(X, L^*).$$

Here we establish a claim.

Claim *The homomorphism* $\mathrm{Hom}_R(X, \theta^*)$ *is a split epimorphism.*

Proof Note from Corollary 2.3(1) that $\mathrm{tr}X$ and $\Omega\mathrm{tr}X$ belong to $\mathcal{G}(R)$. Applying Lemma 2.7, we see that $\mathrm{Ext}^1_R(\mathrm{tr}X, L) = 0$ and $\mathrm{Ext}^2_R(\mathrm{tr}X, L) \cong \mathrm{Ext}^1_R(\Omega\mathrm{tr}X, L) = 0$. Hence Proposition 2.8 shows that $\lambda_X(L)$ is an isomorphism. On the other hand, noting that X is non-free and indecomposable, we easily see that $\lambda_X(k)$ is the zero map. Thus we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
X \otimes_R L \arrow[r, "\lambda_X(L)", "\cong"'] \arrow[d, "X\otimes_R\theta"'] & \mathrm{Hom}_R(X^*, L) \arrow[d, "\mathrm{Hom}_R(X^*,\theta)"'] \\
X \otimes_R Z \arrow[r, "\lambda_X(Z)"] \arrow[d, "X\otimes_R\pi"'] & \mathrm{Hom}_R(X^*, Z) \arrow[d, "\mathrm{Hom}_R(X^*,\pi)"'] \\
X \otimes_R k \arrow[r, "\lambda_X(k)", "0"'] \arrow[d] & \mathrm{Hom}_R(X^*, k) \\
0 &
\end{tikzcd}$$

with exact columns.

Since $\mathrm{Hom}_R(X^*, \pi) \cdot \lambda_X(Z) = 0$, there exists a homomorphism $\rho : X \otimes_R Z \to \mathrm{Hom}_R(X^*, L)$ such that $\mathrm{Hom}_R(X^*, \theta) \cdot \rho = \lambda_X(Z)$. Therefore we have $\rho \cdot (X \otimes_R \theta) = \lambda_X(L)$ because $\mathrm{Hom}_R(X^*, \theta)$ is an injection. Since $\lambda_X(L)$ is an isomorphism, $X \otimes_R \theta$ is a split monomorphism, and hence $(X \otimes_R \theta)^* : (X \otimes_R Z)^* \to (X \otimes_R L)^*$ is a split epimorphism. There is a commutative diagram

$$\begin{tikzcd}
(X \otimes_R Z)^* \arrow[r, "(X\otimes_R\theta)^*"] \arrow[d, "\cong"] & (X \otimes_R L)^* \arrow[d, "\cong"] \\
\mathrm{Hom}_R(X, Z^*) \arrow[r, "\mathrm{Hom}_R(X,\theta^*)"] & \mathrm{Hom}_R(X, L^*),
\end{tikzcd}$$

where the vertical maps are natural isomorphisms. It follows that $\mathrm{Hom}_R(X, \theta^*)$ is also a split epimorphism, and the claim is proved. $\square$

Since $\mathrm{Hom}_R(X, \theta^*) = \mathrm{Hom}_R(X, \beta) \cdot \mathrm{Hom}_R(X, \alpha)$ and $\mathrm{Hom}_R(X, \beta)$ is an injection, the above claim implies that $\mathrm{Hom}_R(X, \beta)$ is an isomorphism. Therefore $\mathrm{Hom}_R(X, \alpha)$ is a split epimorphism, and hence it is especially a surjection. This means that the homomorphism $\alpha : Z^* \to C$ is a $\mathcal{G}(R)$-precover of C.

Assume that α is not a $\mathcal{G}(R)$-cover. Then Proposition 2.5(1) shows that k^* has some non-zero summand whose image by π^* is a direct summand of Z^*. Since k^* is a k-vector space, the R-module Z^* has a summand isomorphic to the R-module k, and hence $k \in \mathcal{G}(R)$ by Corollary 2.3(1). It follows from Proposition 2.2(1) that R is Gorenstein, which contradicts the assumption of the theorem. Therefore α must be a $\mathcal{G}(R)$-cover of C.

Thus we can apply Lemma 2.7, and get $\mathrm{Ext}^1_R(Y, k^*) = 0$ for every $Y \in \mathcal{G}(R)$. Since R has depth zero, in other words, k^* is a non-zero k-vector space, every R-module in $\mathcal{G}(R)$ is free, which is contrary to the assumption of the theorem. This contradiction completes the proof of the theorem. $\square$

According to Proposition 2.9, we have the following result that gives a corollary of the above theorem:

Corollary 3.2 *Let R be a henselian non-Gorenstein local ring of depth zero. Suppose that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.*

3.1 The depth one case

Throughout this section, R is always a local ring with maximal ideal $\mathfrak{m}$ and with residue class field k. We assume that all R-modules in this section are finitely generated.

We begin with proving the following proposition:

Proposition 3.3 *Suppose that there is a direct sum decomposition $\mathfrak{m} = I \oplus J$ where I, J are non-zero ideals of R and $\text{G-dim}_R I < \infty$. Then R is a Gorenstein local ring of dimension one.*

Proof We proceed the proof step by step.

Step 1 We show that $\operatorname{depth} R \leq 1$. For this, according to [13, Proposition 2.1], it is enough to prove that the punctured spectrum $P = \operatorname{Spec} R - \{\mathfrak{m}\}$ of R is disconnected. For an ideal $\mathfrak{a}$ of R, let $V(\mathfrak{a})$ denote the set of all prime ideals of R containing $\mathfrak{a}$. Now we have $\mathfrak{m} = I + J$, $0 = I \cap J$, and $I, J \neq 0$. Hence P is the disjoint union of the two non-empty closed subsets $V(I) \cap P$ and $V(J) \cap P$. Therefore P is disconnected.

Step 2 We show that $\operatorname{depth} R = 1$. Suppose that $\operatorname{depth} R = 0$. Then note from Proposition 2.2(2) and 2.2(4) that $I, R/I \in \mathcal{G}(R)$ and $\operatorname{depth} R/I = 0$. Dualizing the natural exact sequence $0 \to I \to R \to R/I \to 0$, we obtain another exact sequence

$$0 \to (0 :_R I) \to R \to I^* \to 0.$$

Hence we have an isomorphism $I^* \cong R/(0 :_R I)$. It follows that

$$I \cong I^{**} \cong (R/(0 :_R I))^* \cong (0 :_R (0 :_R I)).$$

It is easy to see from this that $I = (0 :_R (0 :_R I))$. Since $(0 :_R I) \subseteq \mathfrak{m}$, we have $I = (0 :_R (0 :_R I)) \supseteq (0 :_R \mathfrak{m})$. On the other hand, note that $(0 :_{\mathfrak{m}/I} \mathfrak{m}) = (0 :_{R/I} \mathfrak{m}) \neq 0$. Since $J \cong \mathfrak{m}/I$, we see that $(0 :_J \mathfrak{m}) \neq 0$. However, we have $(0 :_J \mathfrak{m}) = (0 :_R \mathfrak{m}) \cap J \subseteq I \cap J = 0$, which is contradiction. This contradiction says that $\operatorname{depth} R = 1$, as desired.

Step 3 We show that $I = (0 :_R J)$ and $J = (0 :_R I)$. Noting that $IJ \subseteq I \cap J = 0$, we have $I \subseteq (0 :_R J)$ and $J \subseteq (0 :_R I)$. Hence $\mathfrak{m} = I + J \subseteq (0 :_R I) + (0 :_R J)$, and therefore $\mathfrak{m} = (0 :_R I) + (0 :_R J)$. Since $(0 :_R I) \cap (0 :_R J) = \operatorname{Soc} R = 0$, we obtain another decomposition

$$\mathfrak{m} = (0 :_R I) \oplus (0 :_R J)$$

of $\mathfrak{m}$. Consider the endomorphism

$$I \subseteq (0 :_R J) \subseteq \mathfrak{m} = I \oplus J \xrightarrow{\delta} I$$

of I, where δ is the projection onto I. It is easy to see that this endomorphism is the identity map of I, and hence I is a direct summand of $(0 :_R J)$. Similarly, J is a direct summand of $(0 :_R I)$. Write $(0 :_R J) = I \oplus I'$ and $(0 :_R I) = J \oplus J'$ for some ideals I', J', and we obtain

$$I \oplus J = \mathfrak{m} = (0 :_R I) \oplus (0 :_R J) = J \oplus J' \oplus I \oplus I'.$$

Thus we see that $I' = J' = 0$, and hence we have $I = (0 :_R J)$ and $J = (0 :_R I)$.

Step 4 We show that R is a Gorenstein local ring of dimension one. Dualizing the natural exact sequence $0 \to J \to R \to R/J \to 0$, we have an exact sequence

$$0 \to (0 :_R J) \to R \xrightarrow{\epsilon} J^*.$$

Since $\operatorname{depth}_R J^* \geq \inf\{2,\ \operatorname{depth} R\} > 0$ by [8, Exercise 1.4.19], the R-module $\operatorname{Im} \epsilon \cong R/I$ has positive depth. Therefore Proposition 2.2(2) implies that $R/I \in \mathcal{G}(R)$. It follows from this and Proposition 2.2(4) that $I \in \mathcal{G}(R)$ and $J = (0 :_R I) \cong (R/I)^* \in \mathcal{G}(R)$. Thus, $\mathfrak{m} = I \oplus J \in \mathcal{G}(R)$ by Proposition 2.2(5), and R is Gorenstein by Proposition 2.2(1) and 2.2(4). Hence we have $\dim R = \operatorname{depth} R = 1$. $\square$

Now we can prove the main theorem of this section.

Theorem 3.4 *Let R be a henselian non-Gorenstein local ring of depth one. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the maximal ideal of R does not admit a $\mathcal{G}(R)$-precover. In particular, the category $\mathcal{G}(R)$ is not contravariantly finite in* $\operatorname{mod} R$.

Proof Suppose that $\mathfrak{m}$ admits a $\mathcal{G}(R)$-precover. We want to derive contradiction. Proposition 2.5(2) implies that $\mathfrak{m}$ admits a $\mathcal{G}(R)$-cover $\pi : X \to \mathfrak{m}$. Since $R \in \mathcal{G}(R)$, any homomorphism from R to $\mathfrak{m}$ factors through π. Hence π is a surjective homomorphism. Setting $L = \operatorname{Ker} \pi$, we get an exact sequence

$$0 \to L \xrightarrow{\theta} X \xrightarrow{\pi} \mathfrak{m} \to 0, \qquad (3.1)$$

where θ is the natural embedding. Lemma 2.7 says that $\mathrm{Ext}_R^1(G, L) = 0$ for every $G \in \mathcal{G}(R)$. According to Corollary 2.3(1), we have $\mathrm{Ext}_R^i(G, L) = 0$ for every $G \in \mathcal{G}(R)$ and every $i > 0$.

Fix $Y \in \mathcal{G}(R)$ which is non-free and indecomposable. Since $\mathrm{tr}Y \in \mathcal{G}(R)$ by Corollary 2.3(1), we have $\mathrm{Ker}\,\lambda_Y(L) = \mathrm{Ext}_R^1(\mathrm{tr}Y, L) = 0$ and $\mathrm{Coker}\,\lambda_Y(L) = \mathrm{Ext}_R^2(\mathrm{tr}Y, L) = 0$ by Proposition 2.8. This means that $\lambda_Y(L)$ is an isomorphism. Hence the composite map $\lambda_Y(X) \cdot (Y \otimes_R \theta) = \mathrm{Hom}_R(Y^*, \theta) \cdot \lambda_Y(L)$ is injective, and therefore so is the map $Y \otimes_R \theta$. Also, we have $\mathrm{Ext}_R^1(Y^*, L) = 0$ because $Y^* \in \mathcal{G}(R)$ by Corollary 2.3(1). Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
Y \otimes_R L \arrow[r, "\lambda_Y(L)", "\cong"'] \arrow[d, "Y\otimes_R\theta"'] & \mathrm{Hom}_R(Y^*, L) \arrow[d, "\mathrm{Hom}_R(Y^*,\theta)"'] \\
Y \otimes_R X \arrow[r, "\lambda_Y(X)"] \arrow[d, "Y\otimes_R\pi"'] & \mathrm{Hom}_R(Y^*, X) \arrow[d, "\mathrm{Hom}_R(Y^*,\pi)"'] \\
Y \otimes_R \mathfrak{m} \arrow[r, "\lambda_Y(\mathfrak{m})"] \arrow[d] & \mathrm{Hom}_R(Y^*, \mathfrak{m}) \arrow[d] \\
0 & 0
\end{tikzcd}$$

with exact columns, and this induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(Y^*, X)^* \arrow[r] \arrow[d] & \mathrm{Hom}_R(Y^*, L)^* \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}_R^1(\mathrm{Hom}_R(Y^*, \mathfrak{m}), R) \arrow[d] \\
(Y \otimes_R X)^* \arrow[r, "\rho"] & (Y \otimes_R L)^* \arrow[r] & \mathrm{Ext}_R^1(Y \otimes_R \mathfrak{m}, R)
\end{tikzcd}$$

with exact rows.

Here, let us examine the module $\mathrm{Ext}_R^1(\mathrm{Hom}_R(Y^*, \mathfrak{m}), R)$. Since Y^* is a non-free indecomposable module, any homomorphism from Y^* to R factors through $\mathfrak{m}$. Therefore we have $\mathrm{Hom}_R(Y^*, \mathfrak{m}) \cong \mathrm{Hom}_R(Y^*, R) \cong Y$, and hence $\mathrm{Ext}_R^1(\mathrm{Hom}_R(Y^*, \mathfrak{m}), R) = 0$ because $Y \in \mathcal{G}(R)$. This means that the homomorphism ρ in the above diagram is surjective.

Dualizing the exact sequence (3.1), we obtain an exact sequence

$$0 \to \mathfrak{m}^* \xrightarrow{\pi^*} X^* \xrightarrow{\theta^*} L^*.$$

Set $C = \mathrm{Im}(\theta^*)$ and let $\sigma : X^* \to C$ be the surjection induced by θ^*. The surjectivity of ρ says that the homomorphism $\mathrm{Hom}_R(Y, \theta^*) : \mathrm{Hom}_R(Y, X^*) \to \mathrm{Hom}_R(Y, L^*)$ is also surjective since the two may be identified, and so is the homomorphism $\mathrm{Hom}_R(Y, \sigma) : \mathrm{Hom}_R(Y, X^*) \to \mathrm{Hom}_R(Y, C)$. This means that σ is a $\mathcal{G}(R)$-precover. According to Remark 2.6, we can take a direct summand Z of $\mathfrak{m}^*$ satisfying the following conditions:

i) $\pi^*(Z)$ is a direct summand of X^*.

ii) Let M (resp. W) be the complement of Z (resp. $\pi^*(Z)$) in $\mathfrak{m}^*$ (resp. X^*), and let $0 \to M \to W \xrightarrow{\tau} C \to 0$ be the induced exact sequence. Then τ is a $\mathcal{G}(R)$-cover.

Lemma 2.7 yields

$$\mathrm{Ext}^1_R(G, M) = 0 \tag{3.2}$$

for any $G \in \mathcal{G}(R)$.

Now, we prove that the maximal ideal $\mathfrak{m}$ is a reflexive ideal. Dualizing the natural exact sequence $0 \to \mathfrak{m} \xrightarrow{\zeta} R \to k \to 0$, we obtain an exact sequence

$$0 \to R \xrightarrow{\mu} \mathfrak{m}^* \to k^r \to 0, \tag{3.3}$$

where r is a positive integer because depth $R = 1$. Dualizing this exact sequence again, we obtain an injection $\nu : \mathfrak{m}^{**} \to R$, which maps $\xi \in \mathfrak{m}^{**}$ to $\xi(\zeta) \in R$. Let $\eta : \mathfrak{m} \to \mathfrak{m}^{**}$ be the natural homomorphism. Then we easily see that $\zeta = \nu\eta$. Thus, we can regard $\mathfrak{m}^{**}$ as an ideal of R containing the maximal ideal $\mathfrak{m}$, and hence either ν or η is an isomorphism.

Assume that ν is an isomorphism. Then there exists $\xi \in \mathfrak{m}^{**}$ such that $\xi(\zeta) = 1$. This means that the composition $\xi\mu$ is the identity map of R, and hence the exact sequence (3.3) splits. Therefore we have $R \oplus k^r \cong \mathfrak{m}^* = Z \oplus M$. Noting that Z is isomorphic to $\pi^*(Z)$ which is a direct summand of X^*, we see from Corollary 2.3(1) that $X^* \in \mathcal{G}(R)$ and that $Z \in \mathcal{G}(R)$. Since $k \notin \mathcal{G}(R)$ by Proposition 2.2(1), we see from the Krull-Schmidt Theorem that $Z \cong R$ and $M \cong k^r$. According to (3.2), every R-module in $\mathcal{G}(R)$ is free, and we obtain contradiction. Hence η must be an isomorphism, which says that $\mathfrak{m}$ is a reflexive ideal, as desired.

Thus, we have $\mathfrak{m} \cong \mathfrak{m}^{**} \cong Z^* \oplus M^*$. It follows that the module $Z^* \in \mathcal{G}(R)$ can be regarded as a subideal of $\mathfrak{m}$. Since R is not Gorenstein, Proposition 3.3 implies that either $Z^* = 0$ or $M^* = 0$. But if $M^* = 0$, then $\mathfrak{m} \cong Z^* \in \mathcal{G}(R)$, which would imply that R was Gorenstein by Proposition 2.2(1) and 2.2(4). Thus, $Z^* = 0$. Hence $Z \cong Z^{**} = 0$, and therefore $M = \mathfrak{m}^*$. By (3.2) and (3.3), for every $G \in \mathcal{G}(R)$, we have $\mathrm{Ext}^1_R(G, k^r) = 0$, and see that G is a free R-module, which is contrary to the assumption of our theorem. This contradiction completes the proof of our theorem. □

According to Proposition 2.9, we have the following result that gives a corollary of the above theorem:

Corollary 3.5 *Suppose that R is a henselian non-Gorenstein local ring of depth one and that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.*

3.2 The depth two case

Throughout this section, R is always a local ring with maximal ideal $\mathfrak{m}$ and with residue class field k. We assume that all R-modules in this section are finitely generated.

Theorem 3.6 *Let R be a henselian non-Gorenstein local ring of depth two. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in* mod$R$.

Proof Since $\mathrm{Ext}_R^1(\mathfrak{m}, R) \cong \mathrm{Ext}_R^2(k, R) \neq 0$, we have a non-split exact sequence

$$\sigma : 0 \to R \to M \to \mathfrak{m} \to 0. \tag{3.4}$$

Dualizing this, we obtain an exact sequence

$$0 \to \mathfrak{m}^* \to M^* \to R^* \xrightarrow{\eta} \mathrm{Ext}_R^1(\mathfrak{m}, R).$$

Note from definition that the connecting homomorphism η sends $\mathrm{id}_R \in R^*$ to the element $s \in \mathrm{Ext}_R^1(\mathfrak{m}, R)$ corresponding to the exact sequence σ. Since σ does not split, s is a non-zero element of $\mathrm{Ext}_R^1(\mathfrak{m}, R)$. Hence η is a non-zero map. Noting that $\mathrm{Ext}_R^1(\mathfrak{m}, R) \cong \mathrm{Ext}_R^2(k, R)$, we see that the image of η is annihilated by $\mathfrak{m}$. Also noting that $\mathfrak{m}^* \cong R^* \cong R$, we get an exact sequence

$$0 \to R \to M^* \to \mathfrak{m} \to 0. \tag{3.5}$$

Claim 1 *The modules* $\mathrm{Hom}_R(G, M)$ *and* $\mathrm{Hom}_R(G, M^*)$ *belong to $\mathcal{G}(R)$ for every non-free indecomposable module $G \in \mathcal{G}(R)$.*

Proof Applying the functor $\mathrm{Hom}_R(G, -)$ to the exact sequence (3.4) gives an exact sequence

$$0 \to G^* \to \mathrm{Hom}_R(G, M) \to \mathrm{Hom}_R(G, \mathfrak{m}) \to \mathrm{Ext}_R^1(G, R).$$

Since G is non-free and indecomposable, any homomorphism from G to R factors through $\mathfrak{m}$, and hence $\mathrm{Hom}_R(G, \mathfrak{m}) \cong G^*$. Also, since $G \in \mathcal{G}(R)$, we have $\mathrm{Ext}_R^1(G, R) = 0$. Thus Corollary 2.3(2) implies that $\mathrm{Hom}_R(G, M) \in \mathcal{G}(R)$. The same argument for the exact sequence (3.5) shows that $\mathrm{Hom}_R(G, M^*) \in \mathcal{G}(R)$. □

We shall prove that the module M can not have a $\mathcal{G}(R)$-precover. Suppose that M has a $\mathcal{G}(R)$-precover. Then M has a $\mathcal{G}(R)$-cover $\pi : X \to M$ by Proposition 2.5(2). Since $R \in \mathcal{G}(R)$, any homomorphism from R to M factors through π. Hence π is a surjective homomorphism. Setting $N = \mathrm{Ker}\,\pi$, we get an exact sequence

$$0 \to N \xrightarrow{\theta} X \xrightarrow{\pi} M \to 0, \tag{3.6}$$

where θ is the natural embedding. We see from Corollary 2.3 and Lemma 2.7 that $\mathrm{Ext}_R^i(G, N) = 0$ for any $G \in \mathcal{G}(R)$ and any $i > 0$. Dualizing the exact sequence (3.6), we obtain an exact sequence

$$0 \to M^* \xrightarrow{\pi^*} X^* \xrightarrow{\theta^*} N^*.$$

Put $C = \mathrm{Im}(\theta^*)$ and let $\mu : X^* \to C$ be the surjection induced by θ^*.

Claim 2 *The homomorphism μ is a $\mathcal{G}(R)$-precover of C.*

Proof Fix a non-free indecomposable module $G \in \mathcal{G}(R)$. Applying the functors $G \otimes_R -$ and $\mathrm{Hom}_R(G^*, -)$ to the exact sequence (3.6) yields a commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] \\
G \otimes_R N \arrow[r, "\lambda_G(N)"] \arrow[d, "G\otimes_R\theta"'] & \mathrm{Hom}_R(G^*, N) \arrow[d, "\mathrm{Hom}_R(G^*,\theta)"'] \\
G \otimes_R X \arrow[r, "\lambda_G(X)"] \arrow[d, "G\otimes_R\pi"'] & \mathrm{Hom}_R(G^*, X) \arrow[d, "\mathrm{Hom}_R(G^*,\pi)"'] \\
G \otimes_R M \arrow[r, "\lambda_G(M)"] \arrow[d] & \mathrm{Hom}_R(G^*, M) \arrow[d] \\
0 & \mathrm{Ext}^1_R(G^*, N)
\end{tikzcd}
$$

with exact columns. Noting that $\mathrm{tr}G \in \mathcal{G}(R)$ by Corollary 2.3(1), we see from Proposition 2.8 that $\mathrm{Ker}\, \lambda_G(N) = \mathrm{Ext}^1_R(\mathrm{tr}G, N) = 0$ and $\mathrm{Coker}\, \lambda_G(N) = \mathrm{Ext}^2_R(\mathrm{tr}G, N) = 0$. This means that $\lambda_G(N)$ is an isomorphism. It follows from the commutativity of the above diagram that the homomorphism $G \otimes_R \theta$ is injective. Also, we have $\mathrm{Ext}^1_R(G^*, N) = 0$ because $G^* \in \mathcal{G}(R)$ by Corollary 2.3(1). Thus we obtain a commutative diagram

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
G \otimes_R N \arrow[r, "\lambda_G(N)", "\cong"'] \arrow[d, "G\otimes_R\theta"'] & \mathrm{Hom}_R(G^*, N) \arrow[d] \\
G \otimes_R X \arrow[r] \arrow[d] & \mathrm{Hom}_R(G^*, X) \arrow[d] \\
G \otimes_R M \arrow[r] \arrow[d] & \mathrm{Hom}_R(G^*, M) \arrow[d] \\
0 & 0
\end{tikzcd}
$$

with exact columns. Dualizing this diagram induces a commutative diagram

$$
\begin{tikzcd}
\mathrm{Hom}_R(G^*, X)^* \arrow[r] \arrow[d] & \mathrm{Hom}_R(G^*, N)^* \arrow[r] \arrow[d, "(\lambda_G(N))^*"', "\cong"] & \mathrm{Ext}^1_R(\mathrm{Hom}_R(G^*, M), R) \arrow[d] \\
(G \otimes_R X)^* \arrow[r, "(G\otimes_R\theta)^*"] & (G \otimes_R N)^* \arrow[r] & \mathrm{Ext}^1_R(G \otimes_R M, R)
\end{tikzcd}
$$

with exact rows. Since $\mathrm{Hom}_R(G^*, M) \in \mathcal{G}(R)$ by Claim 1, we have $\mathrm{Ext}^1_R(\mathrm{Hom}_R(G^*, M), R) = 0$. From the above commutative diagram, it is seen that $(G \otimes_R \theta)^*$ is a surjective homomorphism. Note that there is a natural commutative diagram

$$\begin{array}{ccc} (G \otimes_R X)^* & \xrightarrow{(G \otimes_R \theta)^*} & (G \otimes_R N)^* \\ \big\downarrow \cong & & \big\downarrow \cong \\ \mathrm{Hom}_R(G, X^*) & \xrightarrow{\mathrm{Hom}_R(G,\theta^*)} & \mathrm{Hom}_R(G, N^*) \end{array}$$

with isomorphic vertical maps. Therefore the homomorphism $\mathrm{Hom}_R(G, \theta^*)$ is also surjective, and so is the homomorphism $\mathrm{Hom}_R(G, \mu) : \mathrm{Hom}_R(G, X^*) \to \mathrm{Hom}_R(G, C)$. It is easy to see from this that μ is a $\mathcal{G}(R)$-precover of C. $\square$

According to Claim 2 and Remark 2.6, we have direct sum decompositions $M^* = Y \oplus L,\ X^* = \pi^*(Y) \oplus Z$, and an exact sequence

$$0 \to L \to Z \xrightarrow{\nu} C \to 0$$

where ν is a $\mathcal{G}(R)$-cover of C. Since Y is isomorphic to the direct summand $\pi^*(Y)$ of X^*, Corollary 2.3(1) implies that $Y \in \mathcal{G}(R)$. Lemma 2.7 yields $\mathrm{Ext}^1_R(G, L) = 0$ for any $G \in \mathcal{G}(R)$.

Claim 3 *The module* $\mathrm{Hom}_R(G, Y)$ *belongs to* $\mathcal{G}(R)$ *for any* $G \in \mathcal{G}(R)$.

Proof We may assume that G is non-free and indecomposable. The module $\mathrm{Hom}_R(G, Y)$ is isomorphic to a direct summand of $\mathrm{Hom}_R(G, M^*)$. Since the module $\mathrm{Hom}_R(G, M^*)$ is an object of $\mathcal{G}(R)$ by Claim 1, so is the module $\mathrm{Hom}_R(G, Y)$ by Corollary 2.3(1). $\square$

Here, by the assumption of the theorem, we have a non-free indecomposable module $W \in \mathcal{G}(R)$. There is an exact sequence

$$0 \to \Omega W \to F \to W \to 0$$

of R-modules such that F is a free module. Applying the functor $\mathrm{Hom}_R(-, Y)$ to this exact sequence, we get an exact sequence

$$0 \to \mathrm{Hom}_R(W, Y) \to \mathrm{Hom}_R(F, Y) \to \mathrm{Hom}_R(\Omega W, Y) \to \mathrm{Ext}^1_R(W, Y) \to 0.$$

Since $\mathrm{Hom}_R(W, Y)$, $\mathrm{Hom}_R(F, Y)$, and $\mathrm{Hom}_R(\Omega W, Y)$ belong to $\mathcal{G}(R)$ by Claim 3, the R-module $\mathrm{Ext}^1_R(W, Y)$ has G-dimension at most two, especially it has finite G-dimension.

On the other hand, there are isomorphisms

$$\begin{array}{rcl} \mathrm{Ext}^1_R(W, Y) & \cong & \mathrm{Ext}^1_R(W, Y) \oplus \mathrm{Ext}^1_R(W, L) \\ & \cong & \mathrm{Ext}^1_R(W, M^*) \\ & \cong & \mathrm{Ext}^1_R(W, \mathfrak{m}), \end{array}$$

where the last isomorphism is induced by the exact sequence (3.5). Applying the functor $\mathrm{Hom}_R(W, -)$ to the natural exact sequence

$$0 \to \mathfrak{m} \to R \to k \to 0$$

and noting that $\mathrm{Hom}_R(W, \mathfrak{m}) \cong W^*$ because W is a non-free indecomposable module, we obtain an isomorphism $\mathrm{Ext}^1_R(W, \mathfrak{m}) \cong \mathrm{Hom}_R(W, k)$, and hence

$\mathrm{Ext}^1_R(W, Y)$ is a non-zero k-vector space. Therefore Proposition 2.2(1) and 2.2(5) say that R is Gorenstein, contrary to the assumption of our theorem. This contradiction proves that the R-module M does not have a $\mathcal{G}(R)$-precover, which establishes our theorem. $\Box$

According to Proposition 2.9, we have the following result that gives a corollary of the above theorem:

Corollary 3.7 *Suppose that R is a henselian non-Gorenstein local ring of depth two and that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.*

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