Cubic NLS on $\mathbb{T}^2$

joint with Colliander, Keel, Staffilani, Tao

(NLS) $\quad i u_t - \Delta u = -|u|^2 u \qquad x \in \mathbb{T}^2$

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$$

"$-$" defocusing NLS

"$+$" focusing NLS

- Scaling in $\mathbb{R}^d$. nonlinear term of the form $|u|^{p-1}u \sim u^p$

$$u_\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \quad \text{in } H^s$$

$$s = s_c \equiv \frac{d}{2} - \frac{2}{p-1} \quad \text{Scale invariant}$$

$s_c = 0$ -- mass critical

$s_c = 1$ -- energy critical

(NLS) is L^2-critical.

Conservation laws

$\int |u|^2 dx$ mass

$\int \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 dx$ energy

$\int \mathrm{Im}(\nabla u\, \bar{u})\, dx$ momentum

Initial Value Problem in $H^s(\mathbb{T}^2)$

$s \geqq 0$ locally well-posed for small data

$s > 0$ — large data

Bourgain (1993)

$s < 0$ ill-posed

$s \geqq 1$ globally well-posed (energy)

$s > \frac{2}{3}$ —— (I-method)

Silva, Pavlovic, Staffilani, Tzriakis (2007)

c.f. Line case $\mathbb{R}^d$

Cazenave, Ginibre-Velo, Colliander-Roy

$s > \frac{1}{3}$

Behavior of sol. ($\mathbb{T}^d$)

$+|u|^{p-1}u$ focusing NLS in H^1

$1 < p < 1 + \frac{4}{d}$ … globally w.p.

$p \geq 1 + \frac{4}{d}$ … blow-up if "energy ≤ 0"

Kavian (1987)

Planchon - Raphael (2007)

c.f Glassey (1977) $\mathbb{R}^d$-case

(Viral identity)

$-|u|^{p-1}u$ de-focusing NLS

in H^1

$$\underbrace{1 + \frac{4}{d} \leq}_{L^2\text{-subcritical}} \overbrace{p < 1 + \frac{4}{d-2}}^{H^1 \text{ subcritical}} \quad \cdots \quad \text{globall w.p}$$

(NLS) … $p = 3$, $d = 2$

$\forall$ $\forall u_0 \in H^\infty \Rightarrow \sup_t \|u(t)\|_{H^s} < \infty$

$(0 \leq s \leq 1)$

$\forall$ $s > 1$ $\Rightarrow$ No bound for $\|u(t)\|_{H^s}$ provided by conservation laws

Conjectur. If $s > 1$, $\sup_t \|u(t)\|_{H^s} = \infty$.

c.f Bourgain (1996)

$$\|u(t)\|_{H^s} \leq C(1 + |t|)^{2(s-1)+} \quad (s \geq 4)$$

proof Fourier decomposition + H^1 conservation

$$\to \quad \|u(t)\|_{H^s} \le \|u(0)\|_{H^s} + C\|u(0)\|_{H^s}^{1-\delta}$$

$$0<\delta<1 \quad , \quad |t| \le {}^{\exists}T$$

$$a_{k+1} \le a_k + C a_k^{1-\delta} \Rightarrow a_k \le C k^{\frac{1}{\delta}}$$

Question $\quad \|u(t)\|_{H^s} \to \infty \,?\quad (s>1)$

Thm Let $s>1$ $\quad 0<\sigma \ll 1, \quad K \gg 1$

There exists a global H^∞ solution $u(t)$, ${}^{\exists}T>0$

s.t

$$\|u(0)\|_{H^s} < \sigma$$

$$\|u(T)\|_{H^s} > K$$

Thm $\Rightarrow$. $|\hat{u}(0,n)|$: small , but $|\hat{u}(t,n)|$ grows for long time.

mass cons.

. Shift the mass to increasingly high freq. mode.

c.f Bourgain's example for 1d - cubic NLW.

Bourgain's example

NLW , 1-d, cubic, defocusing.

$u_{tt} - u_{xx} + u^3 = 0$ + periodic b.c.

$s \gg 1$, $\forall T > 0$, $\exists$ data $u(0) \in H^\infty$ (small)

strong result

s.t $\| u(T) \|_{H^s} \gg 1$

proof

$$v(t) = u(t) + i(-\Delta)^{-\frac{1}{2}} u'(t)$$

$$(\mathrm{NLW}) \Leftrightarrow \quad i v_t = -i(-\Delta)^{\frac{1}{2}} v - (-\Delta)^{-\frac{1}{2}} (\mathrm{Re}\, v)^3$$

$$v(t) = \sum_{n \in \mathbb{Z}} v_n(t) \underbrace{e^{inx - i|n|t}}_{\text{plane-wave}} \rightarrow \text{substitute}$$

$\{ v_n(t) \}$ satisfies

$$i v_n' = -\frac{1}{8|n|} \sum_{*} v_{n_1}^{\varepsilon_1} v_{n_2}^{\varepsilon_2} v_{n_3}^{\varepsilon_3} e^{-i(\varepsilon_1 |n_1| + \varepsilon_2 |n_2| + \varepsilon_3 |n_3| - |n|)}$$

$$* : \quad \varepsilon_1 n_1 + \varepsilon_2 n_2 + \varepsilon_3 n_3 = n$$

$$\varepsilon_j = \pm 1, \quad v^{-1} = \overline{v}$$

Approx. ODE

$$i v_n' = -\frac{1}{8|n|} \sum_{\text{Resonant.}} v_{n_1}^{\varepsilon_1} v_{n_2}^{\varepsilon_2} v_{n_3}^{\varepsilon_3}$$

Resonant : $\varepsilon_1 n_1 + \varepsilon_2 n_2 + \varepsilon_3 n_3 = n$

$\underline{\varepsilon_1 |n_1| + \varepsilon_2 |n_2| + \varepsilon_3 |n_3| = |n|}$

Assume $1 \le n \le N$. $1 \le k \ll N$

single mode $|v_k| \sim 1$, $|v_j| \sim 0$ $(j \neq k)$ at $t = 0$

Leading term of $\sum_{\text{Resonant}}$ is $\left\{ c\, v_{n_*} \right.$ if $n \neq k$ $(n_* \sim n)$ $(n \gg 1)$

2d - NLS

$$i u_t - \Delta u = -|u|^2 u$$

$$u = \sum_{n \in \mathbb{Z}^2} u_n(t)\, e^{i n\cdot x + i|n|^2 t}$$

$$i u_n' = - \sum_{n_1 - n_2 + n_3 = n} u_{n_1} \overline{u_{n_2}} u_{n_3} e^{i\omega t}$$

$$\omega = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$$

$$= 2\,(n_1 - n_2)\cdot(n_2 - n_3)$$

$$\to \; = -\sum_{\substack{n_1 \neq n \\ n_3 \neq n}} \underbrace{- \sum_{n_1 = n} - \sum_{n_3 = n} + \overbrace{\sum_{n_1 = n_2 = n_3 = n}}^{|u_n|^2 u_n}}_{\omega = 0}$$

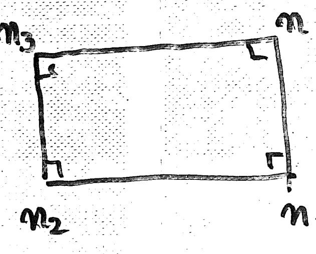
Canceled by $u \mapsto u e^{-i\|u(0)\|_{L^2}^2 t}$

$\omega \neq 0$ ... non-resonant ... harmless (c.f. smoothing effect)

$\omega = 0$ ... resonant ... leading term

$\omega = 0$ under $n_1 - n_2 + n_3 = n$

$\Leftrightarrow$



rectangle in $\mathbb{Z}^2$

accept the permutation $n_1 \leftrightarrow n_3$, $n \leftrightarrow n_2$

Approx. NLS $\quad n \in \mathbb{Z}^2 \quad u_n = u_n(t)$

$$i u_n' = |u_n|^2 u_n - \underbrace{\sum_{(n_1, n_2, n_3) \in \Gamma(n)}}_{\text{Resonant}} u_{n_1} \overline{u}_{n_2} u_{n_3}$$

← only resonance contribution.

⇐ perturbation Lemma

Strategy : Construct

(I) bounded subsets $\Lambda \subset \mathbb{Z}^2$ of $(n_1, n_2, n_3) \in \Gamma(n)$

(II) finite dim. ver. of "Approx. NLS"

… Toy model $\{b_j\}_{j \in \mathbb{Z}^1}$

(III) choose data for which $b_j(t)$ grows for $|j| \gg 1$ as $|t| \to \infty$

(I) $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \oplus \cdots \oplus \Lambda_N$ disjoint-union

- Rectangle (n_1, n_2, n_3, n_4) is nuclear family

$\Leftrightarrow$ $\underbrace{n_1, n_3}_{\text{parents}} \in {}^{\exists}\Lambda_j$ $\underbrace{n_2, n_4}_{\text{children.}} \in \Lambda_{j+1}$

- ${}^{\forall} j$ ${}^{\forall} n_1 \in \Lambda_j$ ${}^{\exists 1}$ nuclear family Λ_j, Λ_{j+1}
s.t. ${}^{\exists} n_3 \in \Lambda_j$, ${}^{\exists} n_2, n_4 \in \Lambda_{j+1}$

- ${}^{\forall} j$ ${}^{\forall} n_2 \in \Lambda_{j+1}$ ${}^{\exists 1}$ nuclear family Λ_j, Λ_{j+1}
s.t. $n_4 \in \Lambda_{j+1}$, $n_1, n_3 \in \Lambda_j$

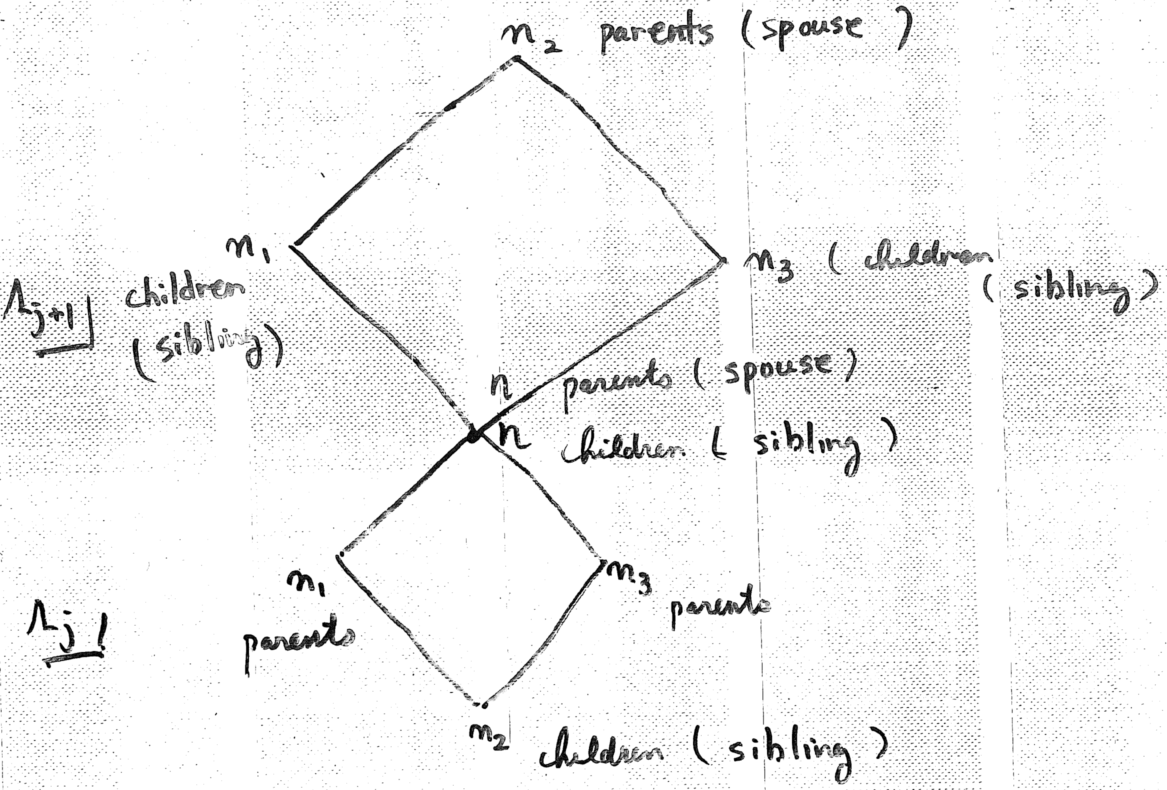
next generation of Λ_j is Λ_{j+1}

- The sibling of a freq. is never its spouse

- Λ contains only the nuclear family

- $\# \Lambda_j = 2^{N-1}$
- $\sum_{n \in \Lambda_j} |n|^2 = 2^{N-1}$
- Λ_N has ~~one~~ single n s.t $|n| = 2^{\frac{N-1}{2}}$
the other is $n = 0$.

$$i\, u_n' = |u_n|^2 u_n - \sum_{(n_1, n_2, n_3) \in \Gamma(n)} u_{n_1} \overline{u_{n_2}} u_{n_3}$$



$$\sum_{(n_1, n_2, n_3) \in \Gamma(n)} u_{n_1} \overline{u_{n_2}} u_{n_3} \rightarrow u_{\text{parents}} \overline{u}_{\text{children}} u_{\text{parents}} \quad \llcorner \Lambda_j$$

$$+ u_{\text{children}} \overline{u}_{\text{parents}} u_{\text{children}} \quad \ulcorner \Lambda_{j+1}$$

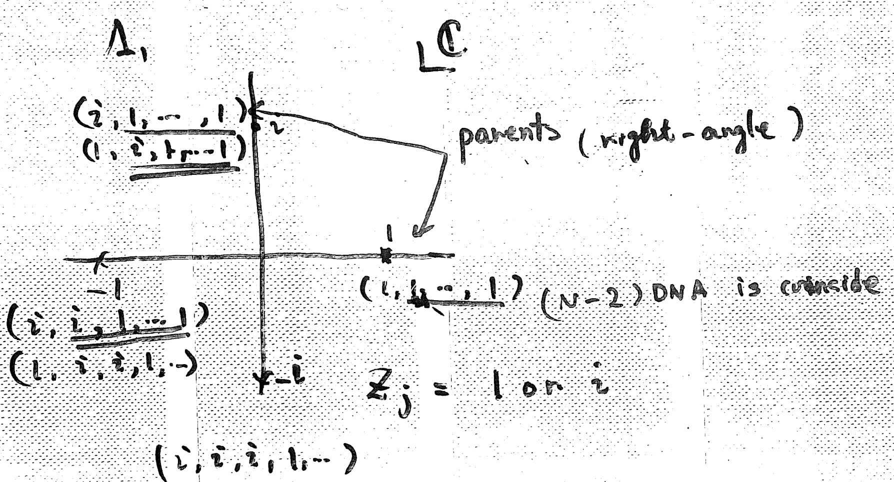
Make it for $|n| \gg 1$

$\mathbb{Z}^2 \subsetneq \mathbb{C}$ ← address with $(N-1)$-tuple numbers

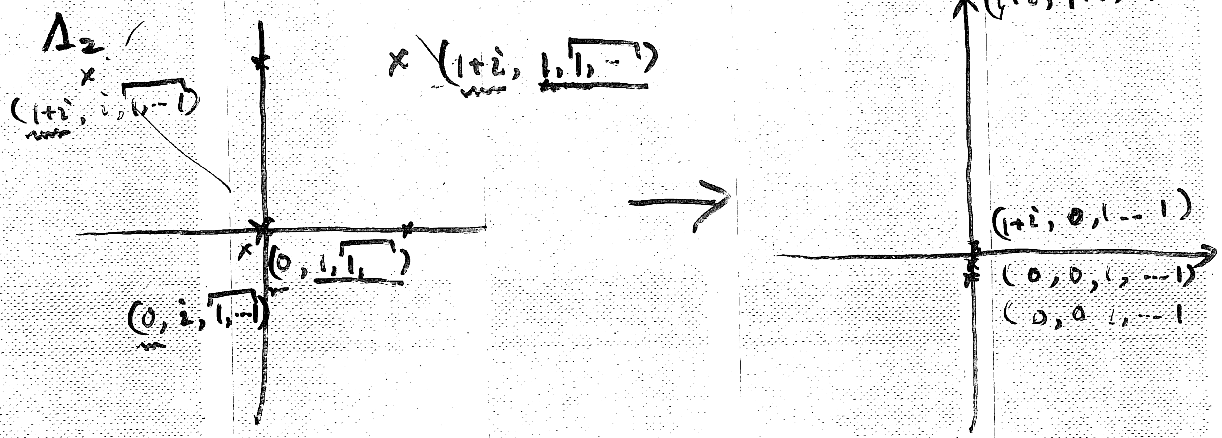
$\mathbb{C} \ni P = P(z_1, z_2, \cdots z_{N-1})$

$z_j = 0, 1, i, 1+i$

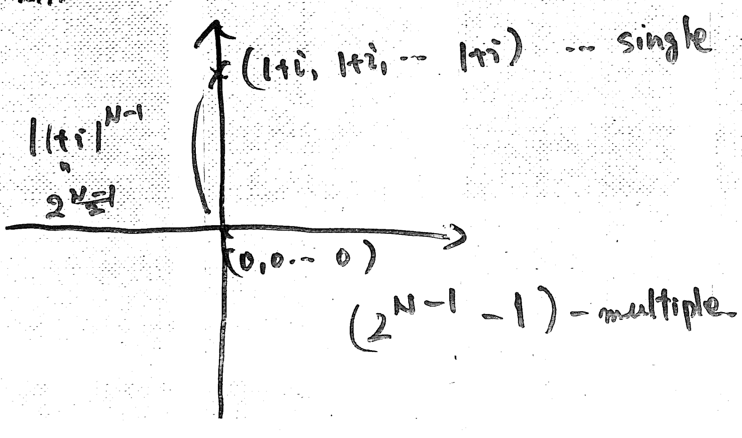
Δ_1



Δ_2



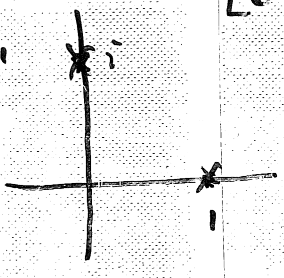
Δ_N



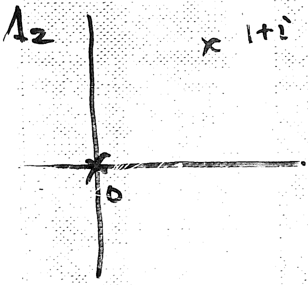
ex $N=2$

LC

Λ_1

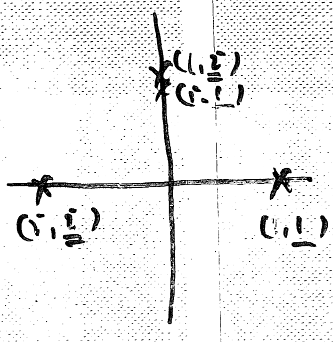


Λ_2

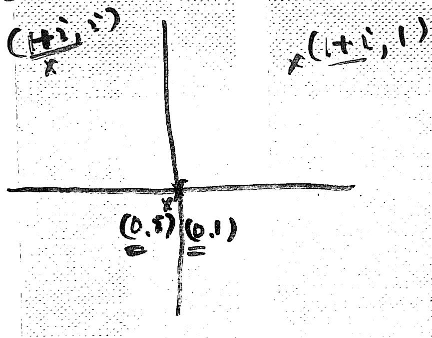


$N=3$

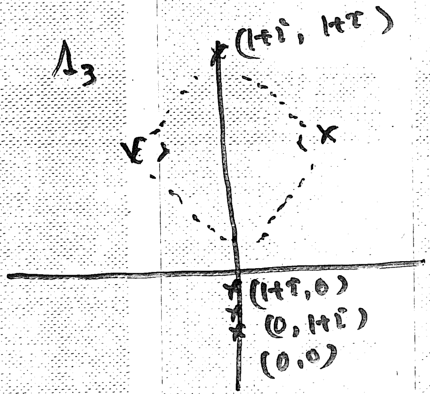
Λ_1



Λ_2



Λ_3



(II) Model

$(n_1, n_2, n_3) \in \Gamma(n)$ ← $\boxed{10}$

n: parent ⇒ n_2 : spouse, n_1, n_3 children

n: children ⇒ n_2 : sibling n_1, n_3 parents

Toy Model ≈ (Approx. NLS)

$$i\, b_j'(t) = |b_j|^2 b_j - 2\, \overline{b_j}\, (b_{j-1}^2 + b_{j+1}^2)$$

$$1 \le j \le N$$

$$b_0 = b_{N+1} = 0$$

$$\underline{b_j = u_n \qquad n \in \Lambda_j}$$

This assumption is justified $u_n = u_{n'}$ for $n, n' \in \Lambda_j$

Properties

- $\sum_{j=1}^{N} |b_j(t)|^2$ conserved (mass)
- $\frac{1}{2} \sum_{j=1}^{N} |b_j(t)|^2 \left(\sum_{n \in \Lambda_j} |n|^2 \right) + \frac{1}{2} \left(\sum_j |b_j(t)|^2 |\Lambda_j| \right)^2$
  $- \frac{1}{4} \sum_j |b_j(t)|^4 |\Lambda_j| + \mathrm{Re}\, b_j^2\, \overline{b_{j+1}}^2$
  energy
- $b_j(0) = 0 \Rightarrow b_j(t) = 0 \quad \forall t$
- $\| u(t) \|_{H^s} \sim \left(\sum |b_j(t)|^2 \sum_{n \in \Lambda_j} |n|^{2s} \right)^{\frac{1}{2}}$

$$\sum_{n \in \Lambda_j} |n|^2 = 2^{N-1}, \quad \sum_{n \in \Lambda_j} |n|^{2s} \gg 2^{N-1}$$

for $s > 1$

(III) Initial placement of $\{b_j(0)\}$ on Λ

ex. $N=2$

<u>Thm.</u> There is initial data $b_1(0)$, $b_2(0)$ s.t.

$$|b_1(t)| \sim 1, \quad |b_2(t)| \sim 0 \quad \text{for } t \ll -1$$

$$|b_1(t)| \sim 0, \quad |b_2(t)| \sim 1 \quad \text{for } t \gg +1$$

<u>proof</u> Replace

$$b_1(t) \longrightarrow b_1(t) = r e^{i\theta}$$

$$b_2(t) \longrightarrow b_2(t) = c\, e^{i\theta}$$

$\underbrace{r}_{\text{real}}, \underbrace{c}_{\text{complex}}, \underbrace{\theta}_{\text{real}}$ are fuctions w.r.t t

Mass cons. $r^2 + |c|^2 = 1$

Suppose r : const.

Toy model $\Rightarrow$
$$\begin{cases} \theta' = -r^2 + 2\,\mathrm{Re}\, c^2 \\ c' + i c \theta' = -i |c|^2 c + 2 i r^2 \bar{c} \end{cases}$$

$c = \alpha(t) + i \beta(t)$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix}}_{\text{diagonalization}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \text{"nonlinear"}$$

$$c(t) = f(t)\, \underline{\underline{\omega}} + g(t)\, \underline{\underline{\omega^2}}$$

ω, ω^2 basis of $\mathbb{C}$ $\qquad \omega = e^{\frac{2\pi}{3} i}$

Under the above observation, we let

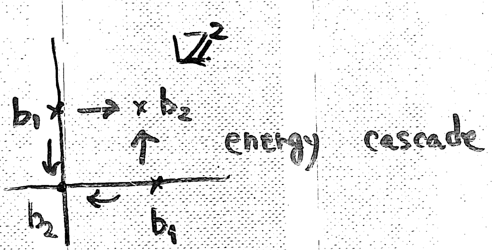
$$\theta = -t$$

$$\begin{cases} b_1(t) = e^{-i\theta}\,\omega\sqrt{1-c^2} \\ b_2(t) = e^{-i\theta}\,\omega^2\, c \end{cases}$$

$$\Rightarrow \quad c'(t) = \sqrt{3}\, c(t)\left(1 - c(t)^2\right)$$

$$\Rightarrow \quad c(t) = \frac{1}{\sqrt{1+e^{-\sqrt{3}t}}}, \qquad \sqrt{1-c(t)^2} = \frac{1}{\sqrt{1+e^{\sqrt{3}t}}}$$

$$b_1(t) = \frac{e^{-i\theta}\,\omega}{\sqrt{1+e^{+\sqrt{3}t}}}, \qquad b_2(t) = \frac{e^{-i\theta}\,\omega^2}{\sqrt{1+e^{-\sqrt{3}t}}}$$



N - tuple version

Thm There is initial data $b_1(0), \cdots, b_N(0)$ and $T>0$, s.t

$$|b_1(0)| \sim 1 \qquad |b_j(0)| \simeq 0 \quad (j \neq 1)$$

$$|b_j(T)| \sim 0 \quad (j \neq N) \qquad |b_N(T)| \sim 1$$