# Geometry of Calabi-Yau Moduli Space and Flux Vacua

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## ♣ Moduli Problem

We consider string theory compactified on CY manifolds. CY manifolds in general have a number of moduli associated with the freedom of changing their complex and Kähler structures.

CY manidfols are characterized by the Ricci flatness condition

$$R_{IJ}(g) = 0, \qquad I, J = 1, 2, \cdots, 6$$

and the existence of a holomorphic 3-form $\Omega_{ijk}$, $i, j, k = 1, 2, 3$. Deformation of the metric obeys the condition

$$R_{IJ}(g + \delta g) = 0 \Longrightarrow \Delta(g)\delta g = 0$$

There are two types of deformations in CY manifolds

$\delta g_{i\bar{j}}$ : Kähler deformation, $(1, 1)$ type
$\delta g_{ij}$ : complex structure deformation, $(1, 2)$ type
$(\delta g_{ij} g^{j\bar{k}} \bar{\Omega}_{\bar{k}\bar{\ell}\bar{m}} = \delta g_{i,\bar{k}\bar{m}})$

These degrees of freedom appear as massless scalar fields in 4 dimensions. Existence of massless scalars is in direct conflict with phenomenology. One has to generates a potential $V$ for moduli fields so that they are fixed at the extremum of the potential.

In the following we consider type IIB theory and concentrate on stabilizing the complex structure moduli $z_a$ $(a = 1, \ldots, h_{2,1})$.

A superpotential becomes generated when RR or NS fluxes are turned on,

$$H^{RR} \equiv dB^{RR}, \;\; H^{NSNS} \equiv dB^{NSNS}, \;\; \tau = C_0 + ie^{-\phi}$$

$$\begin{aligned} W(z_a) &= \int_M (H^{RR} - \tau H^{NSNS}) \wedge \Omega(z_a) \\ &= \sum N_I X_I(z_a) - \sum M_I F_I(z_a) \end{aligned}$$

where

$$M_I = \int_{A_I} (H^{RR} - \tau H^{NSNS}), \;\; N_I = \int_{B_I} (H^{RR} - \tau H^{NSNS})$$
$$I = 0, 1, \cdots, h_{2,1}$$

are fluxes through $A_I$ and $B_I$ cycles. $\{A_a, B_a\}$ denote a symplectic basis of 3-cycles

$$A_a \cup B_b = \delta_{ab}, \;\; A_a \cap A_b = B_a \cap B_b = 0$$

And their periods are given by

$$X_I(z_a) = \int_{A_I} \Omega(z_a), \qquad F_I(z_a) = \int_{B_I} \Omega(z_a) = \frac{\partial F}{\partial X_I}(z_a)$$

Gukov-Vafa-Witten

It is then possible to fix all complex structure moduli.

$$\frac{\partial W}{\partial z_a} = 0 \Longrightarrow \{z_a\} \text{ all fixed}$$

• Kähler potential on Calabi-Yau moduli space is given by

$$K = -\log i \int_M \Omega \wedge \bar{\Omega} = -\log i \sum_I \left( X_I \bar{F}_I - \bar{X}_I F_I \right)$$

**Freedom of Kähler transformation:**

$$K(z_a, \bar{z}_a) \longrightarrow K(z_a, \bar{z}_a) + f(z_a) + \bar{f}(\bar{z}_a),$$

$$\Omega(z_a) \longrightarrow e^{-f(z_a)}\Omega(z_a), \;\; W(z_a) \longrightarrow e^{-f(z_a)}W(z_a)$$

**Periods $(X_I, F_I)$ are holomorphic sections of a line bundle $L$. Metric is invariant under Kähler transformation**

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$$

## ♣ Number of vacua in string theory

- **Fix CY mfd M**
- **number of 3-cycles: 100∼200**

- **Upper bound on fluxes:**

$$\int H_1 \wedge H_2 \leq \text{const. depending on geometry of M}$$
$$\approx 1000 - 5000 \text{ (tadpole condition)}$$

- **possible choice of fluxes:**

$$10^{100} \sim 10^{200}$$

**Altogether there exist an enormous number of string vacua $\mathcal{O}(10^{100})$**

## ♣ Statistical treatment

Douglas, Ashok, Denef, ...

**Vacua distribution function on moduli space $\mathcal{M}$**

$$\rho(z) = \delta(D_a W)\delta(D_{\bar{b}} W^*) \times \left| \det \begin{pmatrix} \partial_a D_b W & \partial_a D_{\bar{b}} W^* \\ \partial_{\bar{a}} D_b W & \partial_{\bar{a}} D_{\bar{b}} W^* \end{pmatrix} \right|$$

Simplify $\Downarrow$

$$\tilde{\rho}(z) = \delta(D_a W)\delta(D_{\bar{b}} W^*) \times \det \begin{pmatrix} \partial_a D_b W & \partial_a D_{\bar{b}} W^* \\ \partial_{\bar{a}} D_b W & \partial_{\bar{a}} D_{\bar{b}} W^* \end{pmatrix}$$

This is an index counting the number of vacua with $\pm$ signs.

Further simplifying assumption:
Fluxes obey Gaussian distribution $\Longrightarrow$ $W$ itself obeys Gaussian distribution.

It follows

$$\tilde{\rho}(z) \prod dz^a \wedge d\bar{z}^{\bar{a}} = \det \frac{1}{2\pi} \underbrace{\left(R^a_{\ b} + \delta^a_{\ b}\ \omega\right)}_{\text{curvature and Kähler form on } \mathcal{M}}$$

This depends only on the geometry of CY moduli space and is the Euler

**number of the bundle $T\mathcal{M} \otimes L$. Flux vacua should be concentrated around singular points in CY moduli space where the curvature $R^a{}_b$ is peaked.**

## ♣ Singular loci in Calabi-Yau moduli space

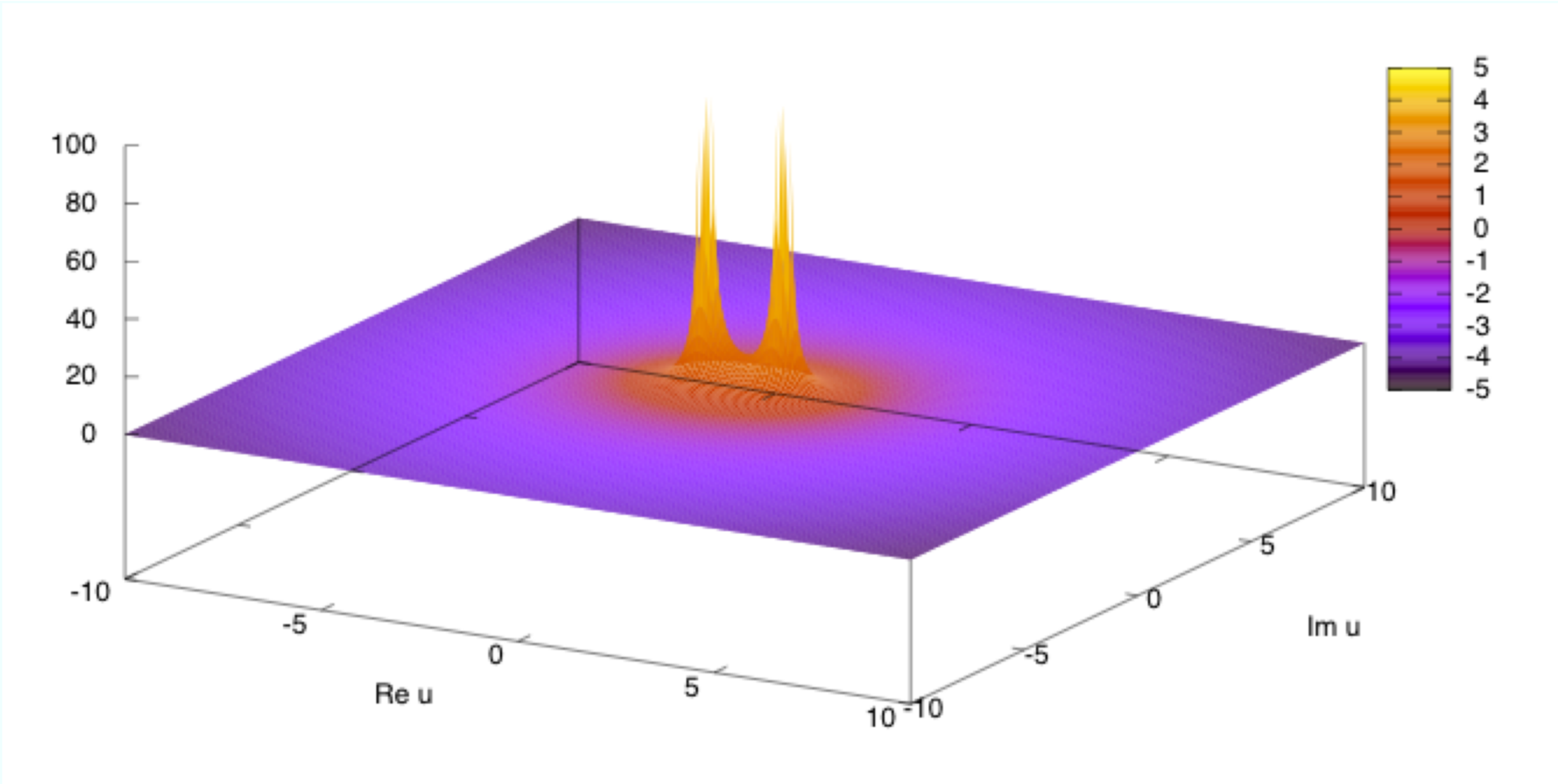
String vacua are not distributed uniformly but concentrated around singular locus of $\mathcal{M}$. We study the distribution of vacua around singular loci in CY moduli space where interesting non-perturbative phenomena take place.

Types of singularities:

- conifolds: generation of massless matter multiplets.
- ADE singularities and rigid limit: gauge symmetry enhancements and decoupling of gravity
- Argyres-Douglas points: massless electrons and monopoles.
- Large complex structure limit: Mirror of the large radius limit.
- $\cdots$

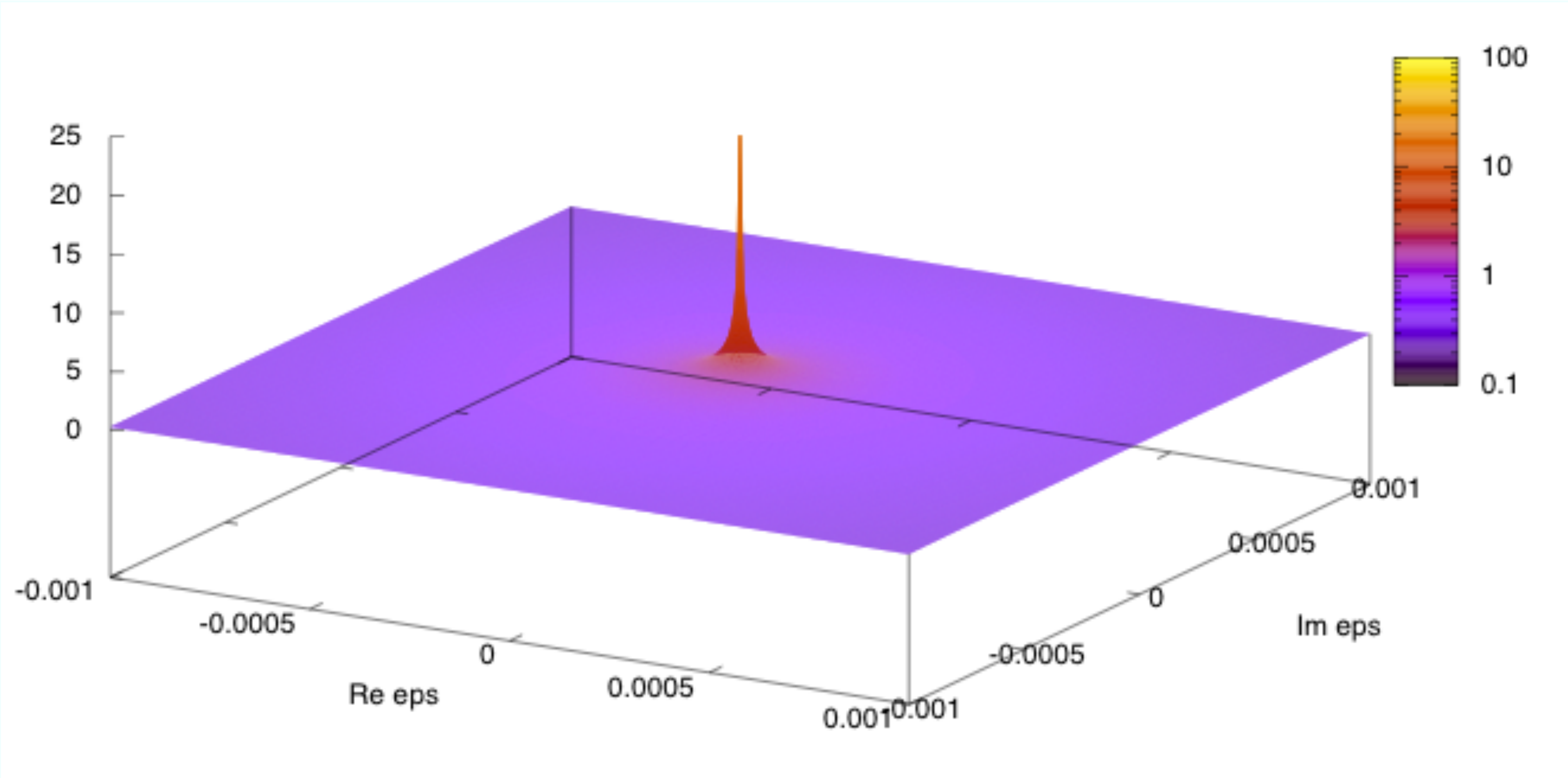
**Examples: on the Seiberg-Witten $u$ plane.**

$$\tilde{\rho} \sim \frac{1}{|u \mp 1|^2 (\log |u \mp 1|)^2} \qquad \textbf{near} \quad u \sim \pm 1$$



**Near the rigid limit.**

$$\tilde{\rho} \sim \frac{1}{|\epsilon|^2(\log|\epsilon|)^2} \qquad \textbf{near} \quad \epsilon \sim 0$$



We claim that the vacuum density behaves as

$$\text{vacuum density} \approx \frac{dz d\bar{z}}{|z|^2 \log|z|^2}$$

around each of these singular points. Note that the integral around $z = 0$ is finite

$$\int d^2 z \frac{dz d\bar{z}}{|z|^2 \log|z|^2} < \infty$$

so that there exist a finite number of vacua around these singular loci.

## ♣ Special geometry relations

- metric

  $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$

- Yukawa coupling
  $F_{ijk} = \sum_I X_I \partial_i \partial_j \partial_k F_I - (X \leftrightarrow F)$

- curvature
  $R_{i\bar{j}k\bar{\ell}} = g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}} - e^{2K} g^{m\bar{n}} F_{ikm} \bar{F}_{\bar{j}\bar{\ell}\bar{n}}$

## ♣ Nilpotent Orbit Theorem

Assemble periods $(X_I, F_I) \Longrightarrow \Omega_I \ \ (I = 1, \cdots, h_{2,1} + 2)$
Under monodromy transformation

$$\Omega_I \to M \Omega_I$$

eigenvalues of $M$ are roots of unity ($1/k$-th power,say). Then $N \equiv M^k - 1$ becomes a nilpotent matrix and after a change of variable $a = z^k$ one finds

$$\Omega_I = e^{\frac{N}{2\pi i}\log a}\left[\Omega_I^{(0)} + a\Omega_I^{(1)} + a^2\Omega_I^{(2)} + \cdots\right]$$

There is an integer $p$ so that

$$N^p\Omega_I \neq 0 \qquad \text{but} \qquad N^{p+1}\Omega_I = 0$$

Then we find $X_I, F_I \approx \log^p a$.

## ♣ Conifolds

singularity at $z = 0$

$$z = \int_A \Omega, \qquad \partial_z F = \int_B \Omega$$

Under monodromy transformation

$$A \to A, \;\; B \to B + A$$

Thus

$$\begin{cases} \Omega_1 = z \\ \Omega_2 = \dfrac{1}{2\pi i} z \log z \\ \vdots \end{cases}$$

Hence

$$K \approx \log(\textbf{const} + |z|^2 \log |z|) \approx \textbf{const}' + |z|^2 \log |z|$$

$$\Downarrow$$

$$g_{z\bar{z}} \approx \log |z|$$

$$\Downarrow$$

$$R_{z\bar{z}} \approx -\partial_z \partial_{\bar{z}} \log \det g_{z\bar{z}} = \frac{1}{|z|^2 \log^2 |z|}$$

One can generalize the discussion to many variable cases and present a general analysis. Instead we would like to present more specific examples in the following.

## ♣ Large Complex Structure Limit

$$\begin{cases} \Omega_1 = \text{const} \\ \Omega_2 \approx \log z \\ \Omega_3 \approx (\log z)^2 \\ \Omega_4 \approx (\log z)^3 \\ \cdot \\ \cdot \end{cases}$$

$$K \approx \log(\log |z|^3) \Longrightarrow g_{z\bar{z}} \approx \frac{1}{|z|^2 \log |z|^2} \Longrightarrow R_{z\bar{z}} \approx \frac{1}{|z|^2 \log |z|^2}$$

We again find the same distribution.

Probably the most interesting cases are the non-compact limit of Calabi-Yau manifolds where $K_3$ fibration develops ADE singularities. In this limit gravitational degrees of freedom become decoupled and string theory is reduced to SUSY gauge theories.

## ♣ Decoupling (Rigid) Limit: SU(2) Example

$$X_8[1,1,2,2,2]:$$
$$W = \frac{B}{8}x_1^8 + \frac{B}{8}x_2^8 + \frac{1}{4}x_3^4 + \frac{1}{4}x_4^4 + \frac{1}{4}x_5^4 - \psi_0 x_1x_2x_3x_4x_5 - \frac{1}{4}\psi_2(x_1x_2)^4$$

By a change of variable $x_0 = x_1x_2, \zeta = (x_1/x_2)^4$, $W$ may be written as

$$W(x;B',\psi_0) = \frac{1}{4}(B'x_0^4 + x_3^4 + x_4^4 + x_5^4) - \psi_0 x_0x_3x_4x_5$$

with

$$B' = \frac{1}{2}(B\zeta + \frac{B}{\zeta} - 2\psi_2)$$

This is a K3 fibration over $\mathrm{P}^1$. Decoupling limit is given by

$$B \longrightarrow 0$$

When we parametrize

$$B = \epsilon\Lambda^2, \qquad \psi_2 + \psi_0^4 = \epsilon u,$$

and make a suitable redefinition of variables we obtain an $A_1$ singularity fibered over $\mathrm{P}^1$:

$$W = \frac{\epsilon}{2}\left[\frac{1}{2}(\zeta + \frac{\Lambda^4}{\zeta}) + y_1^2 + y_2^2 + y_3^2 - u\right]$$

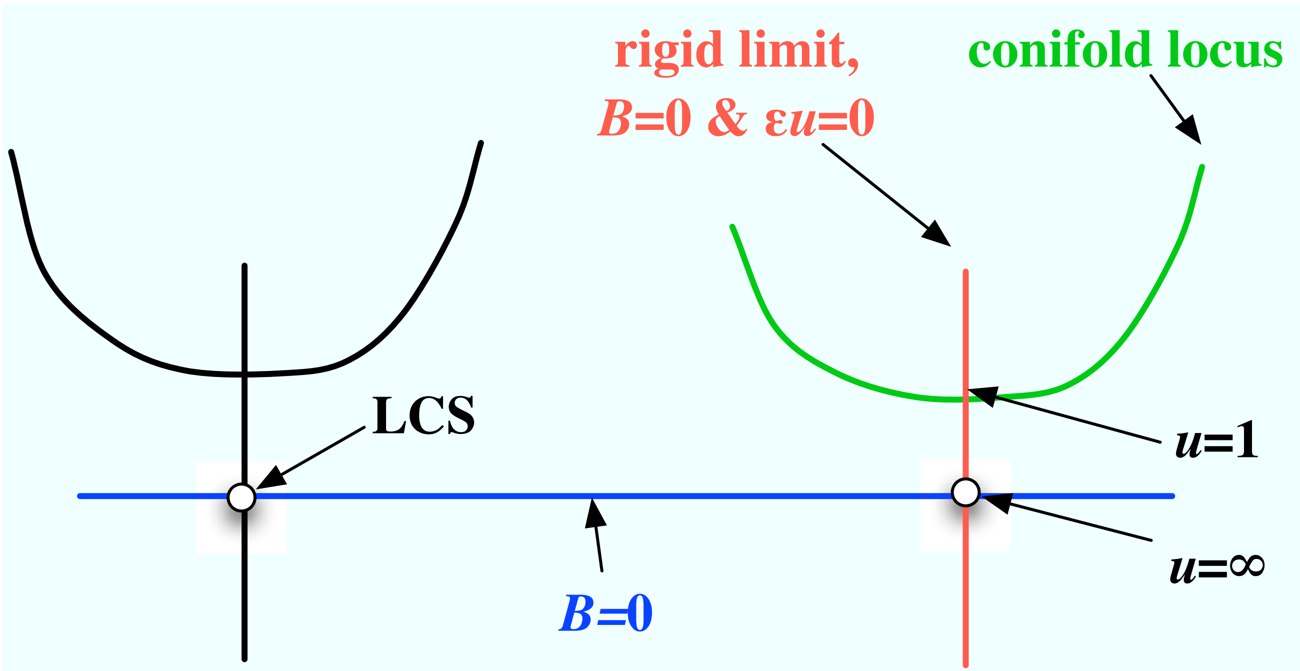
Discriminant of CY manifold is given by

$$\Delta_{CY} \propto B^2 (B^2 - \psi_2^2)(B^2 - (\psi_2 + \psi_0^4))$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

decoupling LCS $SU(2)$

where LCS is the large complex structure limit.



## ♣ Decoupling (Rigid) Limit: $SU(3)$ Example

$$X_{24}[1,1,2,8,12]:$$

$$W = \frac{B}{24}(x_1^{24}+x_2^{24}) - \frac{\psi_2}{12}(x_1x_2)^{12} + \frac{1}{12}x_3^{12} + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^2$$
$$-\psi_0 x_1x_2x_3x_4x_5 - \frac{1}{6}\psi_1(x_1x_2x_3)^6$$

This space again has a K3 fibration. By a change of variable $x_0 = x_1x_2, \ \zeta = (x_1/x_2)^{12}$ it is rewritten as

$$W = \frac{B'}{12}x_0^{12} + \frac{1}{12}x_3^{12} + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^2 - \psi_0 x_0x_3x_4x_5 - \frac{1}{6}\psi_1(x_0x_3)^6$$
$$B' = \frac{1}{2}(B\zeta + \frac{B}{\zeta}) - \psi_2$$

Discriminant is given by

$$\Delta_{CY} = B^2(B^2 - (\psi_1^2 + \psi_2)^2)(B^2 - ((\psi_1 + \psi_0^6)^2 + \psi_2)^2)(B^2 - \psi_2^2)$$

$$\begin{array}{ccccc} \Downarrow & \searrow & & \swarrow & \Downarrow \\ \text{Decoupling} & & \text{SU(3)} & & \text{LCS} \end{array}$$

Decoupling limit is taken as

$$B = \epsilon\Lambda^3, \quad \psi_0^6\psi_1 = \epsilon u^{3/2}, \quad \psi_1^2 + \psi_2 = \epsilon(v - u^{3/2}), \qquad \epsilon \longrightarrow 0$$

By a suitable redefinition of variables we obtain an $A_2$ singularity fibered over $\mathrm{P}^1$

$$W = \epsilon\left[\frac{1}{12}(\zeta + \frac{\Lambda^6}{\zeta}) + \frac{y_3^2}{2} + \frac{y_4^2}{2} + \frac{y_5^3}{3} - \frac{u}{4}y_5 - \frac{v}{12}\right]$$

Billó-Denef-Frè-Pesando-Troost-Van Proyen and Zanon, hep-th/9803228 made a detailed analysis of these models: they explicitly constructed

3-cycles and evaluated the behavior of the periods in the decoupling limit.

In the $X_{24}[1,1,2,8,12]$ model there exist 8 cycles

$$\left(V_{v_a}, V_{v_b}, V_{t_a}, V_{t_b}, T_{v_a}, T_{v_b}, T_{t_a}, T_{t_b}\right)$$

out of which $V_{v_a}, V_{v_b}, T_{v_a}, T_{v_b}$ are the periods of $SU(3)$ gauge theory. Other cycles are needed when one embeds gauge theory into supergravity. Periods behave as

$$V_{v_a}, V_{v_b}, T_{v_a}, T_{v_b} \sim \epsilon^{1/3} : \qquad \text{gauge theory periods}$$

$$V_{t_a}, V_{t_b}, T_{t_a}, T_{t_b} \sim \text{const} + \text{const}' \cdot \log \epsilon : \quad \text{gravity periods}$$

In the case of $SU(2)$ example, gauge theory periods behave as $\epsilon^{1/2}$ and gravity periods as const+const'$\cdot \log \epsilon$.

## • Vacuum density near decoupling point

We have the behavior of periods

$$\begin{cases} \Omega_1 \approx \log \epsilon \\ \Omega_2 \approx \epsilon^{1/N} \\ \cdot \\ \cdot \end{cases}$$

and the Kähler potential

$$K \approx \log \left[ \log |\epsilon| + |\epsilon|^{2/N} K(u, u^*, \cdots) + \cdots \right]$$

Here $K(u, u^*, \cdots)$ denotes the Kähler potential of the gauge theory. We then have the behavior

$$g_{\epsilon\bar{\epsilon}} \approx \frac{1}{|\epsilon|^2 \log |\epsilon|^2}, \qquad R_{\epsilon\bar{\epsilon}} \approx \frac{1}{|\epsilon|^2 \log |\epsilon|^2}$$

We again find the same enhancement of vacuum concentration near decoupling point.

## • Heterotic Duals and RG Flow

Presence of two length scales $|\epsilon|^{2/N}$, $\log 1/|\epsilon|$ suggest a ratio of mass scales $\Lambda$ of gauge theory and that of ambient supergravity

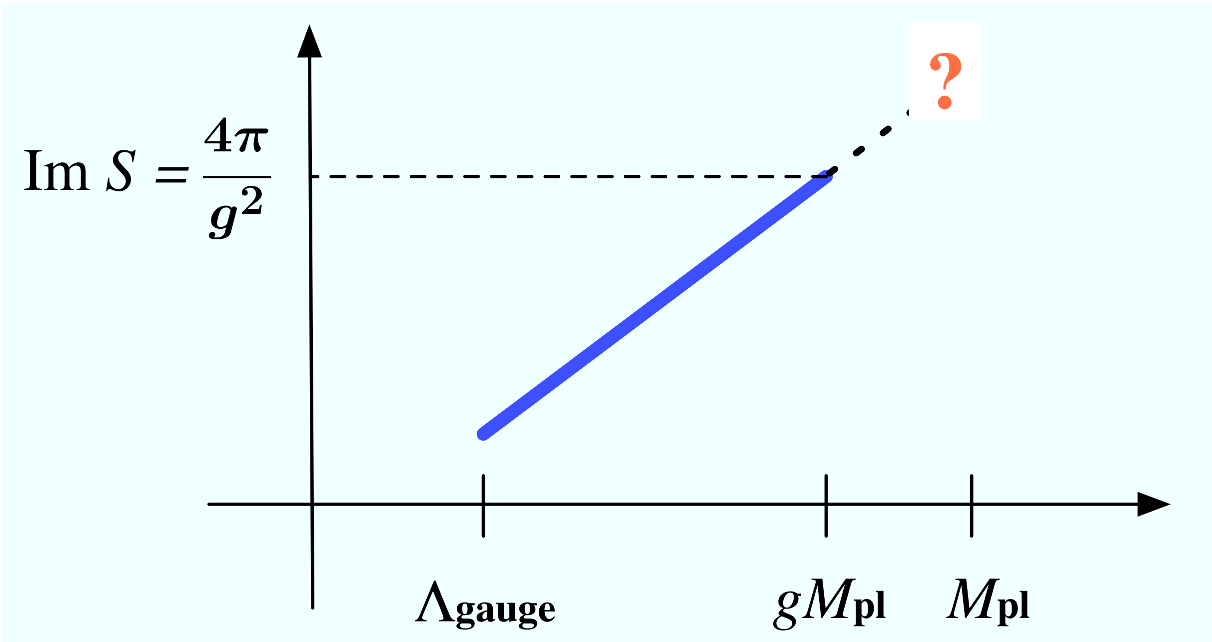
$$\frac{|\epsilon|^{2/N}}{\log 1/|\epsilon|} \approx \left(\frac{\Lambda_{gauge}}{M_{pl}}\right)^2$$

On the other hand, it is well-known that the above models have a dual heterotic description: the model $X_{24}[1,1,2,8,12]$, for instance, coincides with the $(S,T,U)$ model of heterotic string compactified on $K_3 \times T^2$.

Here $\log 1/\epsilon$ corresponds to the variable $S$ where $S = \dfrac{4\pi}{g^2}$ is the heterotic dilaton and $\epsilon \approx e^{-8\pi^2/g^2}$ . Then the above mass ratio can be written as

$$\Lambda_{gauge} \approx e^{-8\pi^2/Ng^2}\; gM_{pl}$$

As compared with the standard (one-loop) RG formula, there exists an extra factor of $g$ in front of the right-hand-side. This is in fact the length scale of heterotic string theory and has the form of a proposal by Arkani-Hamed-Motl-Nicolis-Vafa of an anomalously small mass scale $\Lambda = gM$ in field theory embedded in gravity.



## ♠ Cosmological Constants and SUSY Breaking Scales

**Scenario of KKLT :**

Kachru-Kallosh-Linde-Trivedy

**Recall $\mathcal{N} = 1$ local SUSY has a potential**

$$V = e^{K}(g^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - 3W\bar{W})$$

**First choose a SUSY vacuum,**

$$D_i W = 0$$

**Then**

$$V = -3e^{K}|W|^2$$

**This is an AdS space. Then break SUSY by introducing $\bar{D}$ branes. SUSY breaking energy is always positive and may convert AdS into a dS space.**

dS space is unstable and eventually tunnel into Minkowski space. One needs a fine-tuning in converting AdS into dS with a small positive cosmological constant.

Most of the vacua in string landscape have comological constants and SUSY breaking scale of order $M_{Planck}$. We need a novel idea in the vacuum selection so that one finds a realistic universe from string theory without resourse to anthropic principle.

Concentration of vacua near symmetry enhacement point may help us in this search.