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Schrödinger symmetry and gravity duals for NRCFTs

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INTRODUCTION

Our interest

Non-relativistic (NR) limit in AdS/CFT

Two motives:

1) Applications to condensed matter physics (CMP) i.e., AdS/CMP

Most of condensed matter systems are **non-relativistic (NR)**

2) A new arena to test AdS/CFT duality

Does AdS/CFT work even in NR limit?

We will discuss how to realize NR limit in AdS/CFT

AdS/NRCFT

Gravity (string) on AdS space ⟺ CFT

↓

What is the gravity dual ? ⟺ (???) NRCFT

NR conformal symmetry

Schrödinger symmetry → fermions at unitarity [Nishida-Son,2007]

Related to real condensed matter systems

Our aim

Look for the gravity duals for NRCFTs

by using the Schrödinger symmetry as a clue

Plan of the talk

1. Schrödinger symmetry (and its relatives)
2. Schrödinger spacetime and Lifshitz spacetime
3. Coset construction of NR spacetime
4. String theory embedding
5. A new approach to AdS/NRCFT

1. Schrödinger symmetry

What is Schrödinger symmetry ?

Non-relativistic analog of the relativistic conformal symmetry

Conformal $\supset$ Poincare $\Rightarrow$ Galilei

[Hagen, Niederer,1972]

Schrödinger symmetry

= Galilean symm. + dilatation + special conformal

Dilatation (in NR theories)

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i \qquad\qquad z: \text{ dynamical exponent}$$

EX $z = 2$ (Schrödinger), $z = 1$ (relativistic)

Free Schrödinger eq. $$\left(i\partial_t + \frac{1}{2m}\partial_x^2\right)\psi = 0$$ (scale inv.)

Special conformal trans.

$$t \to \frac{t}{1+\lambda t}, \qquad x^i \to \frac{x^i}{1+\lambda t}$$

Only for $z = 2$ and λ has no spacetime index.

$\Rightarrow$ a generalization of mobius tras.

The generators of Schrödinger algebra $i, j = 1, \ldots, d$ ($\sharp$ of spatial directions)

$$\left.\begin{array}{l} H \ \text{(Hamiltonian)}, \quad P^i \ \text{(spatial translation)}, \\ J^{ij} \ \text{(spatial rotation)}, \quad G^i \ \text{(Galilean boost)}, \\ M \ \text{(mass operator)} \end{array}\right\} = \text{Galilean algebra}$$

(Bargmann alg.)

D (dilatation), C (special conformal trans.) C has no index

The Schrödinger algebra

$i, j = 1, \ldots, d$ ($\sharp$ of spatial directions)

$$[H, G_i] = -iP_i, \quad [P_i, G_j] = i\delta_{ij}M, \quad [M, \text{any}] = 0,$$

$$[J_{ij}, J_{kl}] = i(\delta_{ik}J_{jl} - \delta_{jk}J_{il} + \delta_{il}J_{kj} - \delta_{jl}J_{ki}),$$

$$[J_{ij}, P_k] = i(\delta_{ik}P_j - \delta_{jk}P_i), \quad [J_{ij}, G_k] = i(\delta_{ik}G_j - \delta_{jk}G_i)$$

Galilean algebra

Dilatation

$$[D, P_i] = iP_i, \quad [D, G_i] = -iG_i, \quad [D, H] = 2iH,$$

Dynamical exponent

$z = 2$

Special conformal

$$[C, H] = -iD, \quad [C, P_i] = iG_i, \quad [C, D] = 2iC$$

$H, \ C, \ D$ ➡ $SL(2)$ subalgebra

Algebra with arbitrary z

Galilean algebra

+

Dilatation

$$[D, P_i] = iP_i, \quad [D, G_i] = \underline{(1-z)}iG_i, \quad [D, H] = \underline{z}iH,$$

$$[D, M] = i\underline{(2-z)}M$$

Dynamical exponent

- *M* is not a center any more.
- special conformal trans. *C* is not contained.

FACT

A Schrödinger algebra in ***d+1*** D is embedded into a ``relativistic'' conformal algebra in ***(d+1)+1*** D as a subalgebra.

EX Schrödinger algebra in 2+1 D can be embedded into **SO(4,2)** in 3+1 D

This is true for arbitrary z.

Note: Schrödinger algebra cannot be obtained as Inonu-Wigner contraction

Galilean algebra can be obtained from Poincare algebra as IW contraction.

But it doesn't work for the commutators including the dilatation.

c.f. relativistic conformal algebra $\xrightarrow{\text{IW contraction}}$ conformal Galilean algebra

$\neq$

Schrödinger algebra

A relativistic conformal algebra in (*d+1*)+1 D

The generators: $\tilde{P}^\mu, \quad \tilde{J}^{\mu\nu}, \quad \tilde{D}, \quad \tilde{K}^\mu \qquad (\mu = 0, \ldots, d+1)$

$$[\tilde{J}^{\mu\nu}, \tilde{J}^{\rho\sigma}] = i\eta^{\mu\rho}\tilde{J}^{\nu\sigma} \pm \text{permutations}, \quad [\tilde{J}^{\mu\nu}, \tilde{D}] = 0,$$

$$[\tilde{J}^{\mu\nu}, \tilde{P}] = i(\eta^{\mu\rho}\tilde{P}^\nu - \eta^{\nu\rho}\tilde{P}^\mu), \quad [\tilde{J}^{\mu\nu}, \tilde{K}^\rho] = i(\eta^{\mu\rho}\tilde{K}^\nu - \eta^{\nu\rho}\tilde{K}^\mu),$$

$$[\tilde{D}, \tilde{P}^\mu] = -i\tilde{P}^\mu, \quad [\tilde{D}, \tilde{K}^\mu] = i\tilde{K}^\mu, \quad [\tilde{P}^\mu, \tilde{K}^\nu] = -2i(\tilde{J}^{\mu\nu} + \eta^{\mu\nu}\tilde{D})$$

The embedding of the Schrödinger algebra in *d+1* dim. spacetime

$$H = \tilde{P}^-, \quad M = \tilde{P}^+, \quad P^i = \tilde{P}^i, \quad J^{ij} = \tilde{J}^{ij}$$

$$G^i = \tilde{J}^{i+}, \quad D = \tilde{D} + (z-1)\tilde{J}^{+-}, \quad \boxed{C = \frac{\tilde{K}^+}{2}}$$

(Not contained for z>2)

Light-cone : $\tilde{P}^\pm \equiv \frac{1}{\sqrt{2}}(\tilde{P}^0 \pm \tilde{P}^{d+1}), \qquad (i, j = 1, \ldots, d)$

# of spatial directions

This embedding law has a nice interpretation in the z=2 case (at free level).

→ Light-like compactification of massless Klein-Gordon equation

Massless KG eq. $$(-2\partial_+\partial_- + \partial_i^2)\phi(x^+, x^-, x^i) = 0$$ *(d+1)+1* D

Here we decompose the field like $\phi(x^+, x^-, x^i) = \Phi(x^+, x^i)\mathrm{e}^{-iMx^-}$

↓

Schrödinger eq. $$\left(i\partial_+ + \frac{1}{2M}\partial_i^2\right)\Phi(x^+, x^i) = 0$$ *d+1* D

The dimension is lowered by one!

$\boxed{0 \leq x^- \leq 2\pi/M \quad \text{(compactified)} \;\Rightarrow\; \text{The difference of dimensionality}}$

Schrödinger mass *M* is the inverse of compactification radius

Note: This is different from the usual NR limit of the field theory

The usual NR limit

We should start from a massive theory: e.g. “massive” KG equation

$$S = c \int dt \int d^d x \left[\frac{1}{c^2} \partial_t \phi^* \partial_t \phi - \partial_i \phi^* \partial_i \phi - m^2 c^2 |\phi|^2 \right]$$

Field decomposition: $\phi = \frac{1}{\sqrt{2mc}} \left[\mathrm{e}^{-imc^2 t} \Phi + \mathrm{e}^{imc^2 t} \hat{\Phi}^* \right]$

particle anti-particle

Then we usually set $\hat{\Phi} = 0$ (only particles are kept)

By taking $c \to \infty$

$$S = \int dt \int d^d x \left[i \Phi^* \partial_t \Phi - \frac{1}{2m} \partial_i \Phi^* \partial_i \Phi \right]$$

Note: The starting action should be “massive”

Two possible ways to realize NR field theory

1. Light-like compactification for massless theory [D.Son]

   It is directly applicable to AdS/CFT and studied intensively in the recent.

   (CFT is massless)

2. Usual NR limit for massive theory

   In order to apply it to AdS/CFT, we have to realize massive theories somehow.

   e.g. taking a mass deformation, moving to Higgs branch

   This is not so straightforward.

Mainly we will discuss along the 1st direction.

We will discuss the 2nd direction in the final section.

A simple scenario based on light-like compactification

CFT

The field theory is compactified on the light-like circle: $0 \le x^- \le \dfrac{2\pi}{M}$

A certain sector of DLCQ of a relativistic CFT

Schrödinger symmetry $\Leftarrow$ LC Hamiltonian

Gravity

$$ds^2 = \frac{-2dx^+dx^- + (dx^i)^2 + dz^2}{z^2}$$

with x^--compactification

[Goldberger,Barbon-Fuertes]

SO(2,d+2) is broken to Sch(d) symmetry

But the problem is not so easy as it looks.

What is the dimensionally reduced theory in the DLCQ limit?

(The DLCQ interpretation is possible only for z=2 case)

The next questions are

1) Are there other gravity solutions except for DLCQ AdS?

Rel. conformal symm. is broken to the Schrödinger symm.,

so DLCQ AdS is not maximally symmetric.

There may be some deformations of DLCQ AdS $\Longrightarrow$ YES!

2) Are there other NR scaling symmetries?

YES! Lifshitz fixed point $\Longrightarrow$ scale invariance with z

[Takayanagi's talk] (Not Schrödinger symm.)

Lifshitz field theory: $$\mathcal{L} = \int d^2x \left[\underbrace{(\partial_t \phi)^2}_{\text{2nd order}} - \kappa \underbrace{(\partial_x^2 \phi)^2}_{\text{4th order}} \right]$$ (z=2 case)

2. Schrödinger spacetime and Lifshitz spacetime

There may be various Schrödinger inv. gravity solutions other than DLCQ AdS.

→ Deformations of DLCQ AdS spacetime preserving the Schrödinger symmetry

Schrödinger spacetime

[Son, Balasubramanian-McGreevy]

σ : const.

$$ds^2 = \underbrace{r^2[-2dx^+dx^- + (dx^i)^2] + \frac{dr^2}{r^2}}_{\text{AdS space}} \underbrace{- \sigma^2 r^{2z}(dx^+)^2}_{\textbf{deformation term}}$$

z=2 case: Son, z≠2 case: Balasubramanian-McGreevy

AdS pp-wave geometry

Some comments on the metric

Son-BM 's solution vs. DLCQ AdS

Both solutions are not distinguished from the point of view of the symmetry.

➡ Another criterion: causal structure

Son-BM 's solution has the same causal structure as NRCFT.

DLCQ AdS does not [Herzog-Rangamani-Ross,0807.1099]

Boundary behavior: x^--circle shrinks or not?

shrink [Maldacena-Martelli-Tachikawa, 0807.1100]

Not shrink [Horava-Melby Thompson, 0909.3841]

Anisotropic conformal infinity

Tidal force at the origin:

divergent for $1 < z < 2$

Not divergent for $z \geq 2, \; z = 1$

[Blau-Hartong-Rollier, 0904.3304]

How did they find the deformation term?

Start from the conformal invariance of AdS metric.

AdS metric in Poincare coordinates: $$ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}$$

This metric is invariant under the transformation laws:

$$\begin{aligned}
&\tilde{P}^\mu : \ x^\mu \to x^\mu + a^\mu, \quad \tilde{J}^{\mu\nu} : \ x^\mu \to x^\mu + a\epsilon^{\mu\nu} x_\nu, \\
&\tilde{D} : \ x^\mu \to (1-a)x^\mu, \ z \to (1-a)z, \\
&\tilde{K}^\mu : \ x^\mu \to x^\mu + a^\mu(z^2 + x \cdot x) - 2x^\mu(a \cdot x), \\
&\qquad\quad z \to (1 - 2a \cdot x)z \qquad [a \cdot b = \eta_{\mu\nu} a^\mu b^\nu]
\end{aligned}$$

From this, we can obtain the transformation laws under the Schrödinger group.

The transformation laws under Schrödinger group (z=2 case)

$$
\begin{aligned}
&H: \; x^+ \to x^+ + a^+, \qquad M: \; x^- \to x^- + a^-, \\
&P^i: \; x^i \to x^i + a^i, \qquad J^{ij}: \; x^i \to x^i + a\epsilon^{ij} x_j, \\
&G^i: \; x^i \to x^i - a^i x^+, \quad x^- \to x^- - a^i x^i, \\
&D: \; x^+ \to (1-a)^2 x^+, \quad x^- \to x^-, \quad x^i \to (1-a) x^i, \\
&\qquad z \to (1-a) z, \\
&C: \; x^+ \to (1 - a x^+) x^+, \quad x^- \to x^- - \frac{a}{2}(x^i x^i + z^2), \\
&\qquad x^i \to (1 - a x^+) x^i, \quad z \to (1 - a x^+) z
\end{aligned}
$$

By using them, Son found a deformation term somehow.

$$
-\frac{(dx^+)^2}{z^4}
$$

(sign is fixed from other condition)

This term itself is invariant under the Schrödinger group.

It is possible to derive the transformation laws for z≠2 case.

What theory supports the Schrödinger spacetime?

Einstein gravity coupled with a massive vector field

$$S = \int d^{d+2}x\, dr\, \sqrt{-g}\left[R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu\right]$$

The ansatz $A_+ \propto r^z$ solves the e.o.m., provided

$$\Lambda = -\frac{1}{2}(d+1)(d+2) \qquad m^2 = z(z+d)$$

(negative cosmological const.) (mass of vector field)

Originally, it was proposed as a solution of Einstein gravity
with negative cosmological constant and a massive vector field. (Not in string theory)

➡ Phenomenological approach to condensed matter physics

Phenomenological approach in AdS/CMP

1. Look for an interesting fixed point in CMP

2. Fix the metric in gravity theory from the symmetry argument

3. Construct a theory which supports the metric as a solution

   an effective theory for the fixed point

4. Study the theory and reproduce the well known results

   try to make non-trivial predictions

Lifshitz spacetime

It is proposed as the gravity dual for Lifshitz fixed point.

Symmetry of Lifshitz fixed point

Scale invariance: $t \to \lambda^z t, \quad x^i \to \lambda x^i,$

time translation H, spatial translation P_i, spatial rotation J_{ij}

Note: No Galilean symmetry

Lifshitz field theory: $\mathcal{L} = \int d^2x \left[\underbrace{(\partial_t \phi)^2}_{\text{2nd order}} - \kappa \underbrace{(\partial_x^2 \phi)^2}_{\text{4th order}} \right]$ (z=2 case)

Metric: $ds^2 = \underbrace{-r^{2z}}_{\text{anisotropy}} dt^2 + r^2 (dx^i)^2 + \frac{dr^2}{r^2}$

[Kachru-Liu-Mulligan, 0808.1725]

What theory supports Lifshitz spacetime?

$$S = \int d^4x\sqrt{-g}\,(R-2\Lambda) - \frac{1}{2}\int \Big(\quad F_{(2)}\wedge *F_{(2)} + F_{(3)}\wedge *F_{(3)}\Big) - c\int B_{(2)}\wedge F_{(2)}.$$

$$F_{(2)} = dA_{(1)}, F_{(3)} = dB_{(2)}$$

Sol. The following fluxes support the Lifshitz spacetime

$$F_{(2)} = A\ \theta_r \wedge \theta_t, \quad F_{(3)} = B\ \theta_r \wedge \theta_x \wedge \theta_y\ .$$

$$\theta_t = L\, r^z dt, \quad \theta_{x^i} = L\, r\, dx^i, \quad \theta_r = L\, dr/r$$

$$\Lambda = -\frac{z^2+z+4}{2L^2}, \quad A^2 = \frac{2z(z-1)}{L^2}, \quad B^2 = \frac{4(z-1)}{L^2}, \quad c^2 = \frac{2z}{L^2}$$

(negative cosmological const.)

From the reality of the flux, $z \geq 1$

So far, we have seen the gravity solutions preserving Schrödinger symm. and Lifshitz symm.

In these examples, the metrics have been fixed somehow
after repeated trial and error.

In principle, it is possible to do that but inconvenient!

We want a systematic way to fix the metric from the symmetry

⬇

Coset construction of metric

[Sakura Schäfer-Nameki, M. Yamazaki, K.Y., 0903.4245]

3. Coset construction of non-relativistic spacetime

Coset construction of NR spacetime

Coset construction is applicable to only homogeneous spaces.

A homogeneous space is represented by a coset $M = G/L$

EX $$S^2 = SO(3)/SO(2), \quad AdS_5 = SO(2,4)/SO(1,4)$$

G : isometry, L : local Lorentz symmetry

Our aim

Consider a variety of deformations of the DLCQ AdS background within the class of homogeneous space by using the coset construction

Note: Schrödinger and Lifshitz spacetimes are homogeneous.

Coset construction of the metric

vielbeins spin connections

1. MC 1-from $J = g^{-1}dg = e^m \underline{T_m} + e^p \underline{T_p}$ $(\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{l})$

→ vielbeins $(g \in M \equiv G/L)$

2. Contraction of the vielbeins: $ds^2 = e^m e^n \Omega_{mn}$

Look for the symmetric 2-form

↑ symm. 2-form

The possible deformations are contained in the 2-form as parameters.

If *G* is semi-simple → straightforward (Use the Killing form)

But if *G* is non semi-simple, step 2 is not so obvious. (No non-deg. Killing form)

Nappi-Witten's argument for non semi-simple case

[Nappi-Witten, hep-th/9310112]

2D Poincare with a central extension

$$[J, P_i] = \epsilon_{ij} P_j, \quad [P_i, P_j] = \epsilon_{ij} T, \quad [T, J] = [T, P_i] = 0 \qquad (i, j = 1, 2)$$

Killing form (degenerate)

$$\Omega_{AB} = \begin{array}{cc} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} P_1 \\ P_2 \\ J \\ T \end{array} \end{array}$$

right-G inv.

Most general symmetric 2-form

$$\Omega_{AB} = k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The condition for the symm. 2-form

$$f_{AB}{}^{D} \Omega_{CD} + f_{AC}{}^{D} \Omega_{BD} = 0$$

PP-wave type geometry

NW-like interpretation for Schrödinger spacetime ?

[Sakura Schäfer-Nameki, M. Yamazaki, K.Y., 0903.4245]

G : Schrödinger group is non-semisimple $\longrightarrow$ Killing form is degenerate

Q1. What is the corresponding coset ?

Physical assumptions

Schrödinger coset

$(\alpha, \beta :$ const.)

$$G/L = \{H, M, P_i, D\}, \quad L = \{J_{ij}, G_i, \alpha C + \beta M\}$$

Q2. What is the symm. 2-form ?

The above Schrödinger coset is NOT reductive.

The construction of symm. 2-form for the non-reductive coset [Fels-Renner, 2006]

$$[m],\ [n] = \{H, P_1, P_2, M, D\}$$

(3 parameters)

2-form:

$$\Omega_{[m][n]} = \begin{pmatrix} \Omega_{HH} & 0 & 0 & -\Omega_{PP} & 0 \\ 0 & \Omega_{PP} & 0 & 0 & 0 \\ 0 & 0 & \Omega_{PP} & 0 & 0 \\ -\Omega_{PP} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2(\beta/\alpha)\Omega_{PP} \end{pmatrix}, \quad (\alpha, \beta \neq 0)$$

vielbeins: $e_H = e^{2x_D} dx_H, \quad e_M = dx_M, \quad e_i = e^{x_D} dx^i, \quad e_D = dx_D$

where $g = e^{x_H H} e^{x_M M} e^{x_i P^i} e^{x_D D}$ ⬅ coordinate system

metric: $$ds^2 = r^2(-2dx^+ dx^- + dx^i dx^i) + \frac{dr^2}{r^2} - \sigma r^4 (dx^+)^2$$

where $\Omega_{PP} = 1, \quad \sigma \equiv \Omega_{HH}, \quad e^{x_D} = r, \quad x_H = x^+, \quad x_M = x^-$

↑ AdS radius

(β/α has been absorbed by rescaling r .)

Similarly, we can derive the metric for arbitrary z case.

Gravity dual for Lifshitz fixed point

Let's consider $G = \{H, P^1, P^2, D, J\}$: a subalgebra of Sch(2)

algebra: $[J, P^i] = -\epsilon^{ij} P^j, \quad [D, H] = zH, \quad [D, P^i] = P^i$

Take $G/L = \{H, P^1, P^2, D\}, \quad L = \{J\}$ (reductive)

2-form: (4 parameters)

$$\Omega_{[m][n]} = \begin{pmatrix} \Omega_{HH} & 0 & 0 & \Omega_{HD} \\ 0 & \Omega_{PP} & 0 & 0 \\ 0 & 0 & \Omega_{PP} & 0 \\ \Omega_{HD} & 0 & 0 & \Omega_{DD} \end{pmatrix}$$

vielbeins:

$$e_H = e^{zx_D} dx_H, \quad e_i = e^{x_D} dx^i, \quad e_D = dx_D$$

$$g = e^{x_H H} e^{x_i P^i} e^{x_D D}$$

metric: [Kachru-Liu-Mulligan]

$$ds^2 = -r^{2z} dt^2 + r^2 dx^i dx^i + \frac{dr^2}{r^2}$$

Unique!

4. String theory embedding

String theory embedding

So far, Schrödinger spacetime and Lifshitz spacetime have not been discussed in the context of string theory.

Schrödinger spacetime can be embedded into string theory:

Two known methods:

1. null Melvin Twist (NMT) TsT transformation

   [Herzog-Rangamani-Ross] [Maldacena-Martelli-Tachikawa] [Adams-Balasubramanian-McGreevy]

2. brane-wave deformation [Hartnoll-K.Y.]

c.f. Lifshitz spacetime has not been embedded into string theory.

For the Lifshitz spacetime with spatial anisotropy, it is possible. [Azeyanagi-Li-Takayanagi]

Hereafter we will focus upon the string theory embedding of Schrödinger spacetime.

1. null Melvin Twist - TsT transformation

[Herzog-Rangamani-Ross]
[Maldacena-Martelli-Tachikawa]
[Adams-Balasubramanian-McGreevy]

EX $AdS_5 \times S^5$ background

$$ds^2 = r^2\left[-dx^+dx^- + (dx^i)^2\right] + \frac{dr^2}{r^2} + \underbrace{ds^2(CP^2) + \eta^2}_{S^5}$$

$$\eta = d\phi + P$$

Steps of NMT

1) Do T-duality on ϕ into $\tilde{\phi}$

2) Take the shift: $x^- \to x^- + \sigma\tilde{\phi}$

3) Do T-duality again on $\tilde{\phi}$ into ϕ

$$ds^2 = r^2\left[-dx^+dx^- + (dx^i)^2\right] + \frac{dr^2}{r^2} \underline{- \sigma^2 r^4 (dx^+)^2}$$

$$+ds^2(CP^2) + \eta^2$$

The only 2 terms appear!

$$F_5 = 4(1+*)\mathrm{vol}(X_5) \qquad \boxed{B_2 = \sigma r^2 dx^+ \wedge \eta}$$

➡ Non-SUSY

Schrödinger Black Hole

[Herzog-Rangamani-Ross]
[Maldacena-Martelli-Tachikawa]
[Adams-Balasubramanian-McGreevy]

It is also possible to apply NMT to non-extremal D3-brane solution.

Start from the metric: (in the near horizon region)

$$ds^2 = \frac{1}{1-\frac{r_0^4}{r^4}}\frac{dr^2}{r^2} + r^2\left[-dx^+dx^- + \frac{r_0^4}{4r^4}(\lambda^{-1}dx^+ + \lambda dx^-)^2 + (dx^i)^2\right]$$
$$+ds^2(CP^2) + \eta^2$$

Here we took the boost: $x^+ \to \lambda^{-1}x^+, \quad x^- \to \lambda x^-$

Temperature & Chemical potential: $$\frac{1}{T} = \frac{\pi\lambda}{r_0}, \qquad \frac{\mu_N}{T} = \frac{\pi}{r_0 r^- \lambda}$$

$(x^- \sim x^- + 2\pi r^-)$

Entropy: $$S = \frac{R_{AdS}^3}{4G_N^5}\lambda(2\pi r^-)\frac{r_0^3}{2}V_2$$

The solution (after performing NMT)

$$\mathrm{d}s^2 = \mathrm{e}^{\frac{3}{2}\Phi} r^2 \Bigg[\left(-1 + \frac{r_0^4}{2r^4}\right) \mathrm{d}x^+ \mathrm{d}x^- + \frac{r_0^4}{4r^4} \left(\lambda^2 (\mathrm{d}x^-)^2 + \lambda^{-2} (\mathrm{d}x^+)^2\right)$$

$$- \sigma^2 r^2 \left(1 - \frac{r_0^4}{r^4}\right) (\mathrm{d}x^+)^2 \Bigg] + \mathrm{e}^{-\frac{\Phi}{2}} r^2 \left[\frac{1}{r^4 - r_0^4} \mathrm{d}r^2 + \mathrm{d}\vec{x}^2\right]$$

$$+ \mathrm{e}^{-\frac{\Phi}{2}} ds^2(CP^2) + \mathrm{e}^{\frac{3}{2}\Phi} \eta^2$$

$$\mathrm{e}^{-2\Phi} = 1 + \sigma^2 \lambda^2 \frac{r_0^4}{r^2} ,$$

$$B^{NS} = \sigma \frac{r^2}{2} \mathrm{e}^{2\Phi} \left[\left(2 - \frac{r_0^4}{r^4}\right) dx^+ - \frac{r_0^4}{2r^4} \lambda^2 dx^-\right] \wedge \eta \qquad \text{+ 5-form}$$

The solution asymptotes to the Schrödinger spacetime when $r \to \infty$ (boundary)

Note:

NMT does not change the entropy.

NMT changes only the asymptotic behavior and the horizon structure is preserved.

2. brane-wave deformation

[Hartnoll-KY]

Our idea

$$ds^2 = r^2[-2dx^+dx^- + (dx^i)^2] + \frac{dr^2}{r^2} - \underline{f(X_5)}r^4(dx^+)^2 + ds^2_{X_5}$$

$$F_5 = 4(1+*)\mathrm{vol}(X_5)$$

Allow the coordinate dependence on the internal manifold X_5

➡ Only the (++)-component of Einstein eq. is modified.

The function f has to satisfy $\boxed{-\nabla^2_{X_5} f = 12 f}$

EX

For $X_5 = S^5$ we know the eigenvalues: $\boxed{-\nabla^2_{S^5} = \ell(\ell+4)}$

Thus the spherical harmonics with $\ell = 2$ gives a Schrödinger inv. sol.

The sol. preserves 8 SUSY (1/4 BPS) ➡ super Schrödinger symm.

The solution with an arbitrary dynamical exponent z

$$ds^2 = r^2[-2dx^+dx^- + (dx^i)^2] + \frac{dr^2}{r^2} - f(X_5)\underline{r^{2z}}(dx^+)^2 + ds^2_{X_5}$$

$$F_5 = 4(1+*)\mathrm{vol}(X_5)$$

Dynamical exponent appears

The differential eq. is $\boxed{-\nabla^2_{X_5} f = 4(z^2-1)f}$

For $X_5 = S^5$ case f = a spherical harmonics with $\boxed{\ell = 2z-2}$

The moduli space of the solution is given by spherical harmonics

Instability in the brane-wave method

(++)-component of the metric may take the positive sign.

(because spherical harmonics have zero at certain points on the internal manifold)

$\Rightarrow$ Instability (Its existence can be shown after a bit complicated argument)

SUSY solutions are unstable. Does it sound curious?

Ans. Supersymmetries preserved by the solutions are kinematical and do not imply the stability.

Kinematical SUSY:	$\{Q,Q\} \sim P^{+} = M$	(light-cone momentum)
Dynamical SUSY:	$\{Q,Q\} \sim P^{-} = H$	(light-cone Hamiltonian)

However, this instability can be removed by adding a *B*-field.

The solution with NS-NS *B*-field

($z = 2$ case)

$$ds^2 = r^2[-2dx^+dx^- + (dx^i)^2] + \frac{dr^2}{r^2} - f(X_5)r^4(dx^+)^2 + ds^2_{\mathrm{KE}} + \eta^2$$

$$F_5 = 4(1+*)\mathrm{vol}(X_5) \qquad \boxed{B_2 = \sigma r^2 dx^+ \wedge \eta} \longrightarrow \text{non-SUSY}$$

The function f has to satisfy the equation

$$\boxed{-\nabla^2_{X_5} f = 12 f - 12\sigma^2}$$

By rewriting f as

$$\boxed{f = \sigma^2 + \tilde{f}} \Rightarrow \boxed{-\nabla^2_{X_5}\tilde{f} = 12\tilde{f}}$$

Here $\tilde{f}$ is still given by a spherical harmonics with $\ell = 2$

f is lifted up due to the presence of *B*-field

Super Schrödinger inv. solutions - after that

1) A.Donos, J.P. Gauntlett, 0901.0818

IIB and 11D SUGRA: solutions with kinematical SUSY and no instability with B-field

But they got the restriction for the value of z:

$z \geq 3$ for IIB $z \geq 4$ for 11D

2) A.Donos, J.P. Gauntlett, 0905.1098

IIB and 11D SUGRA: $z = 2$ is possible

3) A.Donos, J.P. Gauntlett, 0907.1761

IIB SUGRA: solutions with dynamical SUSY, conformal SUSY, kinematical SUSY

super Schrödinger algebras [M. Sakaguchi, K.Y.]

agree!

5. A new approach to AdS/NRCFT

Another direction to AdS/NRCFT

So far we have discussed AdS/NRCFT based on the light-like compactification.

(The embedding law of Schrödinger algebra into rel. conformal one)

Another scenario: Start from the well-known NRCFT and look for its gravity dual.

(possibly using the standard NR limit)

A few examples of NRCFT (1+2 D)

NL Schrödinger, Jackiw-Pi model (NR CSM), Its supersymmetric extensions

Jackiw-Pi model Schrödinger invariant [Jackiw-Pi]

$$\mathcal{L} = \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + i\Phi^* D_t\Phi - \frac{1}{2m}(D_i\Phi)D_i\Phi + \lambda|\Phi|^4$$

Obtained by taking the standard NR limit of a *relativistic* Chern-Simons matter system

NR limit of ABJM theory (*N*=6 Chern-Simons matter system)

[Nakayama-Sakaguchi-KY, 0902.2204] [Lee^3 , 0902.3857]

After taking a mass deformation to the ABJM model, we can take a NR limit

Here we shall discuss the SUSY only.

The original ABJM: 12 SUSY + 12 conformal SUSY, SO(6) = SU(4) R-symmetry

↓ Mass deformation

A massive ABJM: 12 SUSY + 0 conformal SUSY, SU(2) x SU(2) x U(1)

↓ NR limit with all particles (no anti-particle)

[Hosomichi-Lee^3-Park]
[Gomis et.al]

NR ABJM

kinematical SUSY: 10, dynamical SUSY: 2, conformal SUSY: 2

R-symmetry: SU(2) x SU(2) x U(1)

What is the gravity dual for NR ABJM ?

The gravity dual for rel. ABJM = $AdS_4 \times CP^3$

For mass deformed ABJM ⟹ Bena-Warner geometry

But it seems difficult to consider the NR limit of the BW geometry.

Another approach is to consider the gravity sol. dual to the NR fixed point directly.

A candidate of the gravity dual

$\underline{\text{5D Schrödinger spacetime}} \times \underline{M_6}$ [Ooguri-Park, 0905.1954]

Schrödinger symm. ↑ fix from the R-symmetry structure of NR ABJM ↑

SU(2) x SU(2) x U(1)

⟹ Dynamical SUSYs are not preserved.

Improvement ? [Jeong-Kim-Lee- O Colgain-Yavartanoo,0911.5281]

It is an interesting topic to look for the gravity dual for NR ABJM

Summary and Discussion

Summary

No example of AdS/NRCFT in which both sides are clearly understood.

1. If we start from the gravity (with the embedding of Sch. algebra)

   → Difficulty of DLCQ (including interactions)

2. If we start from the concrete NRCFT (with the usual NR limit)

   → What is the gravity solution?

   NR ABJM → gravity dual ??

Other topics:

1. null warped AdS3 in topologically massive gravity (TMG)

   [Anninos-Li-Padi-Song-Strominger, 0807.3040]

2. 3 pt. function in NRCFT ⟷ computation in the gravity side

   [Henkel, hep-th/9310081] [Fueres-Moroz, 0903.1844] [Volovich-Wen, 0903.2455]

3. Schrödinger-Virasoro algebra

   [Henkel, hep-th/9310081] [Compere-de Buyl-Detournay-K.Y., 0908.1402]

4. Embedding of Lifshitz spacetime into string theory

   [Azeyanagi-Li-Takayanagi, 0905.0688] [Li-Nishioka-Takayanagi, 0908.0363]

5. Asymptotically Lifshitz black hole

   [Danielsson-Thorlacius, 0812.5088] [Balasubramanian-McGreevy, 0909.0263 (analytic sol.)]

Thank you!

Correlation functions in NRCFT

2-pt. function can completely be fixed from the symmetry (up to an overall const.)

$$A_2(1,2) = c\,\delta_{\Delta_1,\Delta_2} t_{12}^{-\Delta_1} e^{\frac{iM}{2}\frac{x_{12}^2}{t_{12}}}$$

3-pt. function contains an arbitrary function in comparison to rel. CFT

$$A_3(1,2,3) = \prod_{i<j} t_{ij}^{\Delta/2-(\Delta_i+\Delta_j)} e^{i(\frac{M_1}{2}\frac{x_{13}^2}{t_{13}}+\frac{M_2}{2}\frac{x_{23}^2}{t_{23}})} F(v_{12})$$

Schrödinger invariant variables:

$$v_{ij} = \frac{(\vec{x}_{in}t_{jn} - \vec{x}_{jn}t_{in})^2}{2t_{ij}t_{in}t_{jn}} = \frac{1}{2}\left(\frac{x_{jn}^2}{t_{jn}} - \frac{x_{in}^2}{t_{in}} + \frac{x_{ij}^2}{t_{ij}}\right), \qquad i<j<n$$

Q1. What is the corresponding coset ?

Ans. to Q1. Fix the coset from physical assumptions for the denominator

$$\begin{cases} \text{Assump.1} \quad \text{No translation condition} \qquad L \ \text{doesn't contain} \ P_i \ \text{and} \ H \\ \text{Assump.2} \quad \text{Local Lorentz condition} \qquad L \ \text{contains} \ J_{ij} \ \text{and} \ G_i \end{cases}$$

$\Downarrow$

Schrödinger coset $(\alpha, \beta : \ \text{const.})$

$$G/L = \{H, M, P_i, D\}, \quad L = \{J_{ij}, G_i, \alpha C + \beta M\}$$

Q2. What is the symmetric 2-form ?

Ans. to Q2. NW argument?

It is possible if the coset is reductive

Reductiveness : $[\mathfrak{m}, \mathfrak{l}] \subset \mathfrak{m} \qquad (M \equiv G/L)$

EX pp-wave, Bargmann

However, the Schrödinger coset is NOT reductive.

$\Rightarrow$ Nappi-Witten argument is not applicable directly.

What should we do?

The construction of symm. 2-form for the non-reductive coset [Fels-Renner, 2006]

The condition for the symm. 2-form

$$\Omega_{[m][n]} f_{[k]p}{}^{[m]} + \Omega_{[k][m]} f_{[n]p}{}^{[m]} = 0$$

$[m], \ldots$ are indices for $\mathfrak{m}$, p is for $\mathfrak{l}$

L-invariance of symm. 2-form

A generalization of NW argument

The indices [m],… are defined up to *L*-transformation

The group structure constant is generalized

The Schrödinger coset :

$$G/L = \{H, P_i, M, D\}, \qquad L = \{J, G_i, \alpha C + \beta M + \gamma D\} \qquad (i = 1, 2)$$

Structure constants:

$$[m],\ [n] = \{H, P_1, P_2, M, D\}$$

$$f_{[m]J}{}^{[n]} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad f_{[m]G_1}{}^{[n]} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f_{[m]\alpha C + \beta M + \gamma D}{}^{[n]} = \begin{array}{rl} & \begin{array}{ccccc} & & & M & D \end{array} \\ \begin{array}{c} \\ \\ \\ \\ D \end{array} & \begin{pmatrix} -2\gamma & 0 & 0 & 0 & -\alpha \\ 0 & -\gamma & 0 & 0 & 0 \\ 0 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{2\beta} & \underline{2\gamma} \end{pmatrix} \end{array} \qquad f_{[m]G_2}{}^{[n]} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[D, \alpha C + \beta M + \gamma D] = \alpha[D, C] = -2\alpha C = \overset{\overset{\mathfrak{m}}{\pitchfork}}{2\beta M} + \overset{\overset{\mathfrak{m}}{\pitchfork}}{2\gamma D}$$

$$\sim 2\alpha C + 2\beta M + 2\gamma D \quad \in \mathfrak{l}$$

NR limit of N=2 Chern-Simons matter system

N=2 relativistic Chern-Simons matter system [Lee-Lee-Weinberg]

$$\mathcal{L}_{\text{rel}} = \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{M}} \qquad \mathcal{L}_{\text{CS}} = \kappa A_0 F_{12} + \frac{\kappa}{2c}\epsilon^{ij}\partial_t A_i A_j$$

$$\mathcal{L}_{\text{M}} = -D_\mu \phi^* D^\mu \phi - i\bar{\psi}\gamma^\mu D_\mu \psi - \left(\frac{e^2}{\kappa c^2}\right)^2 |\phi|^2 \left(|\phi|^2 - v^2\right)^2 + \frac{e^2}{\kappa c^2}\left(3|\phi|^2 - v^2\right) i\bar{\psi}\psi$$

complex scalar 2-comp.complex fermion

Expanding the potential → Mass $m^2c^2 = \frac{e^4}{\kappa^2 c^4} v^4$

NR limit

Field expansion:

$$\phi = \frac{1}{\sqrt{2m}}\left[\underbrace{\mathrm{e}^{-imc^2 t}\Phi}_{\text{particle}} + \underbrace{\mathrm{e}^{imc^2 t}\hat{\Phi}^*}_{\text{anti-particle}}\right] \qquad \psi = \sqrt{c}\left[\underbrace{\mathrm{e}^{-imc^2 t}\Psi}_{\text{particle}} + \underbrace{\mathrm{e}^{imc^2 t}C\hat{\Psi}^*}_{\text{anti-particle}}\right]$$

Here we keep particles only $\hat{\Phi} = \hat{\Psi} = 0$ Take $c \to \infty$ limit

Scalar field fluctuations

S⁵ part

$$\nabla^2\phi = m^2\phi \quad \Rightarrow \quad \phi = \Phi(r)\underline{Y_\lambda(X_5)}\mathrm{e}^{-i\omega x^+ + i\mathbf{k}\cdot\mathbf{x} - iMx^-}$$

Laplacian for S^5 :

$$\left[-\nabla^2_{X_5} + M^2 f\right] Y_\lambda = \underline{\lambda} Y_\lambda$$

Eigenvalues for S5

Eq. for the radial direction:

$$\frac{d^2\Phi}{dr^2} + \frac{5}{r}\frac{d\Phi}{dr} + \frac{2M\omega - \mathbf{k}^2}{r^4}\Phi = \frac{m^2+\lambda}{r^2}\Phi$$

The solution:

$$\Phi = \frac{K_\nu(p/r)}{r^2}$$

(modified Bessel function)

$$p = \sqrt{\mathbf{k}^2 - 2M\omega}, \quad \nu = \underline{\sqrt{m^2+\lambda+4}}$$

λ : largely negative ⇒ ν : purely imaginary (instability) (while p is real)

⇒ The scaling dimension of the dual op. becomes complex

NR super Chern-Simons matter systems

N=2 NR Chern-Simons matter system [Leblanc-Lozano-Min, hep-th/9206039]

N=3 NR Chern-Simons matter system [Nakayama-Ryu-Sakaguchi-KY, 0811.2461]

N=6 NR Chern-Simons matter system [Nakayama-Sakaguchi-KY,0902.2204]

(NR ABJM)

Depending on the matter contents in the NR limit, we may get other NR CSM systems.

Note Interacting SUSY singlet is possible [Nakayama-Sakaguchi-KY, 0812.1564]

There is no direct analog of the Coleman-Mandula theorem for NR SUSY.

It is interesting to study NR SUSY itself.

R-symmetry of NR ABJM

SU(4)

$\downarrow$

R-symmetry: SU(2) x SU(2) x U(1)

14 SUSY charges of NR ABJM

= 2 dynamical SUSY + 2 conformal SUSY + 10 kinematical SUSY

II

5 complex SUSY charges

$$\{Q_1^0, Q_1^{0*}\} = 2M, \quad \{Q_1^{m*}, Q_1^n\} = 2M\delta^{mn} + 2mR^{[mn]}$$

singlet under SU(2) x SU(2) (*m, n* = 1,2,3,4) SU(2) x SU(2)

dynamical SUSY & conformal SUSY are also singlet under SU(2) x SU(2)

U(1) R-symmetry:

$$i[R, Q_1^0] = -iQ_1^0, \quad i[R, Q_1^m] = \frac{i}{3}Q_1^m,$$

$$i[R, Q_2] = -iQ_2, \quad i[R, S] = -iS$$

$$\{Q_2^*, S\} = -\frac{1}{2}D - \frac{i}{2}J + \frac{3}{4}iR$$

Memo

Operator insertion corresponding to the NMT

$$\mathcal{O}_\mu^{IJ} = \frac{i}{g_{\mathrm{YM}}^2}\mathrm{tr}\{F_\mu{}^\nu \Phi^{[I} D_\nu \Phi^{J]} + \sum_K (D_\mu \Phi^K)\Phi^{[K}\Phi^I\Phi^{J]}\} + \text{fermions}$$