Higher Dimensional Black Holes

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Purpose of this talk

– give a brief overview of recent progress in understanding basic properties of $D \geq 4$ black holes
Black holes in general relativity

**Definitions:**

Consider a strongly causal, asymptotically flat spacetime \((M, g_{ab})\)

Let \(\mathcal{I}^+\) be a set of *idealized* distant observers (i.e., future *null infinity*)

- **Black hole** \(B := M \ominus \) Chronological past of \(\mathcal{I}^+\)
- **Event horizon of** \(B\) \(\mathcal{H} := \) Boundary of \(B\)
- **Predictable** \(B\) \(\Leftrightarrow\) Outside \(B\) (including \(\mathcal{H}\)) is globally hyperbolic

**Remarks:**

- \(\mathcal{H}\) is, by definition, a *null hypersurface*
- \(\mathcal{H}\) is a *global notion*; it has no distinguished local significance
Area Theorem: (Hawking 71)

Consider a predictable black hole that is a solution to Einstein’s equation with the null energy conditions.

The surface area $A$ of horizon cross-sections of $\mathcal{H}$ can never decrease with time.

The null energy conditions: $T_{ab}k^a k^b \geq 0$ for any null vectors $k^a$.

Remark:

A resemblance to 2nd-law of thermodynamics:
(Entropy $S$ never decreases: $\delta S \geq 0$) (Bekenstein 73)
Towards local characterization of black holes

- Gravitating source bends the spacetime geometry

\[ ds^2 = -(1 - 2\Phi)dt^2 + \frac{dr^2}{1 - 2\Phi} + r^2d\Omega^2 : -\Phi : \text{Newton potential} \]

Static observers (along \( t^a = (\partial/\partial t)^a \)) are no longer geodeisc but are accelerated: i.e.,

\[ t^c \nabla_c t^a = \kappa(r)(\partial/\partial r)^a \]

- A distinguished null hypersurface \( \mathcal{N} \) in the limit \( \Phi \to 1/2 \)

since \( g^{ab}(dr)_a(dr)_b \to 0, t^a t_a \to 0 \) as \( \Phi \to 1/2 \) \( (r \to r_H) \)

\[ t^c \nabla_c t^a = \kappa t^a \text{ on } \mathcal{N} \]

- \( t^a \) is tangent and normal to \( \mathcal{N} \)

- \( \kappa = \kappa(r_H) \): redshifted proper acceleration of observer \( t^a \) on \( \mathcal{N} \)
Killing Horizons

Definitions:

Killing horizon $\mathcal{N}$ :⇔ A null hypersurface whose null generators coincide with the orbits of a one-parameter group of isometries (so that there is a Killing field $K^a$ normal to $\mathcal{N}$)

Surface gravity $\kappa$ of $\mathcal{N}$ :⇔ A function on $\mathcal{N}$ that satisfies

$$\nabla^a (K^b K_b) = -2\kappa K^a \cdots \cdots (\ast)$$

Remarks:

− eq. (\ast) is rewritten as $K^b \nabla_b K^a = \kappa K^a$

− The notion of a Killing horizon is independent of the notion of event horizon

− Surface gravity $\kappa$ must be constant along each null generator of $\mathcal{N}$, but can, in general, vary from generator to generator
In a wide variety of circumstances, the event horizon $\mathcal{H}$ of a stationary black hole is also a Killing horizon.

e.g., Kerr metric (rotating black hole)

\[
\begin{align*}
\text{d}s^2 &= \rho^2 \left( \frac{\text{d}r^2}{\Delta} + \text{d}\theta^2 \right) + (r^2 + a^2) \sin^2 \theta \text{d}\phi^2 \\
&\quad - \text{d}t^2 + \frac{2Mr}{\rho^2} (a \sin^2 \theta \text{d}\phi - \text{d}t)^2
\end{align*}
\]

where $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$ and

\[
K^a = t^a + \frac{a}{r_H^2 + a^2} \phi^a
\]

Although both $t^a$ and $\phi^a$ are spacelike near $\mathcal{H}$, $K^a K_a \to 0$ as $r \to r_H$.
Carter’s Rigidity Theorem: (Carter 73)

Consider a stationary-axisymmetric black hole with $t^a$ and $\varphi^a$ which satisfy $t \wedge dt \wedge \varphi = 0 = \varphi \wedge d\varphi \wedge t$

There exists a Killing field $K^a$ of the form

$$K^a = t^a + \Omega_H \varphi^a$$

which is normal to $\mathcal{H}$. The constant $\Omega_H$ is called the angular velocity of $\mathcal{H}$. Furthermore, the surface gravity $\kappa$ must be constant over $\mathcal{H}$

Remarks:

– purely geometrical result: no use of Einstein’s field equations

– leaves open the possibility that there could exist stationary BHs whose event horizons are not Killing horizons
BH Mechanics and thermodynamics

Mechanics of stationary black holes (Bardeen, Carter & Hawking 73)

$$\kappa = \text{const.}, \quad \delta M = \frac{1}{8\pi} \kappa \delta A + \Omega_H \delta J$$

$M$: Mass, $\kappa$: Surface gravity, $\Omega_H$: Horizon angular velocity, $J$: Angular momentum

The dominant energy conditions, $T_{ab} V^a W^a \geq 0$ for any causal vectors $V^a, W^a$

⇒ Mathematical analogue of 0th & 1st laws of equilibrium thermodynamics

$$T = \text{const.}, \quad \delta E = T \delta S + \text{work term}$$

Quantum effects ⇒ $T = \kappa / 2\pi$ (Hawking 75)
If a **stationary** black hole corresponds to an **equilibrium** thermodynamic system, then such a stationary BH should be described by merely a **small numbers of parameters**
Towards BH uniqueness theorems

Topology Theorem: (Hawking 73)

Let $(M, g)$ be a stationary predictable black hole spacetime that satisfies the dominant energy conditions.

Spatial cross-section, $\Sigma$, of each connected component of the event horizon $\mathcal{H}$ is homeomorphic to a 2-sphere.
Towards BH uniqueness theorems

**Rigidity Theorem:** (Hawking 73)

Let \((M, g)\) be an asymptotically flat, regular stationary, predictable black hole spacetime that is a vacuum or electro-vacuum solution to Einstein’s equations. Assume further that \((M, g)\) be analytic.

The event horizon \(\mathcal{H}\) must be a **Killing horizon**

If rotating, the BH spacetime must be **axisymmetric**

**Remarks:**

- *makes no assumptions of symmetries beyond stationarity*
- *makes use of Einstein’s field equations*
- *use the result of Topology theorem: \(\Sigma \approx S^2\)*
Uniqueness Theorems:  
(Israel-Carter-Robinson-Mazur-Bunting-Chrusciel)

Let $\mathcal{M} = (M, g)$ be a regular, stationary predictable BH solution of a vacuum or electro-vacuum Einstein’s equations. Furthermore, assume $\mathcal{M} = (M, g)$ be analytic and $\mathcal{H}$ be connected.

The BH is uniquely specified by its mass, angular momentum, and charges.

Remarks:

- vacuum rotating black hole spacetime $\Rightarrow$ Kerr-metric
- based on the results of Topology and Rigidity theorems
Uniqueness theorems

- If weak cosmic censorship (Penrose) holds, gravitational collapse always forms a black hole
  \[
  \iff \text{strong support from e.g., BH thermodynamics}
  \]

- The Kerr-metric is stable (Press-Teukolsky 73, Whiting 89)
  \[
  \Rightarrow \text{describes a possible final state of dynamics}
  \]
  \[
  \Rightarrow \text{describes astrophysical black holes–formed via gravitational collapse–in our universe}
  \]

- A proof in smooth (not real-analytic) setup
  \[
  \text{(Ionescu-Klainerman 07)}
  \]
Summary: Black holes in $4D$ general relativity

Asymptotically flat stationary BHs in 4-dimensions

- **Exact solutions** .............. e.g., Kerr metric
- **Stability** .................. Stable $\Rightarrow$ final state of dynamics
- **Topology** ................. Shape of the horizon $\approx$ 2-sphere
- **Symmetry** ................. Static or axisymmetric
- **Uniqueness** ............... Vacuum $\Rightarrow$ Kerr-metric
- **BH Thermodynamics** .... Quantum aspects

Which properties of $4D$ BHs are extended to $D > 4$?
Exact Solutions — much larger variety

- Rotating Holes (Myers & Perry 86)
- Rotating Rings (Emparan & Reall 02)
Exact Solutions — much larger variety

Rotating Holes (Myers & Perry 86) Rotating Rings (Emparan & Reall 02)

Stability — not fully studied yet

Static vacuum ⇒ stable (AI & Kodama 03)
Rotating holes ⇒ partial results:

Special case (e.g., Kunduri-Lucietti-Reall 06, Murata-Soda 07)
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  Some restrictions, (Galloway & Schoen 05)
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Symmetry — rigid
at least, one rotational symmetry:
(Hollands, AI & Wald 07)
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- **Symmetry** — rigid
  - at least, one rotational symmetry: (Hollands, AI & Wald 07)
- **Uniqueness** — non-unique
  - Hole and Rings w/ the same $(J, M)$
    - Static holes: e.g., (Gibbons, Ida & Shiromizu 02)
    - Uniqueness in $5D$ rotating holes/rings (Morisawa-Ida 04, Hollands & Yazadjiev 07)
    - (Morisawa, Tomizawa & Yasui 07)
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  (Morisawa, Tomizawa & Yasui 07)

• **BH Thermodynamics** — generalize to $D > 4$ e.g., (Rogatko 07)
Exact solutions in $D > 4$

- Static spherical holes in $\forall D > 4$  \quad \text{(Tangherlini 63)}

- Stationary rotating black holes in $\forall D > 4$ \quad \text{(Myers-Perry 82)}
  
  - Topology of horizon cross-sections $\approx S^{D-2}$
  
  - $[(D+1)/2]$ commuting Killing fields $\Rightarrow [(D-1)/2]$ spins
  
  - for $D = 4, 5$, $\exists$ Kerr upper-bound on angular momentum $J$
  
  - for $D \geq 6$, \text{No upper-bound on $J$} $\Rightarrow$ ultra-spinning

$$\exists \text{ horizon} \iff 0 = g^{rr} = \Pi_i \left(1 + \frac{(J_i/M)^2}{r^2}\right) - \frac{GM}{r^{D-3}}$$

as the last term dominates for small $r$ when $D \geq 6$

Sufficient conditions for no-bound:

- two $J_i = 0$ for $D(\text{odd}) \geq 7$, one $J_i = 0$ for $D(\text{even}) \geq 6$
Exact solutions in $D > 4$

Stationary, rotating black-rings in $D = 5$ \hspace{1cm} (Emparan-Reall 02)

- Topology of the horizon $\approx S^1 \times S^2$
- 3-commuting Killing fields $\text{Isom}: \mathbb{R} \times SO(2) \times SO(2)$
- not uniquely specified by global charges $(M, J_1, J_2)$

Two ring-solutions with the same $(M, J_1, J_2 = 0)$

$\Rightarrow$ In $D = 5$, Uniqueness Theorem no longer holds as it stands
Exact solutions in $D > 4$

- Solutions akin to Emparan-Reall’s ring ($M, J_1 \neq 0, J_2 = 0$)
  - Black-ring w/ ($M, J_1 = 0, J_2 \neq 0$) (Mishima & Iguchi 05)
  - Black-ring w/ two angular momenta ($M, J_1 \neq 0, J_2 \neq 0$) (Pomeransky & Sen’kov 06)
    - Uniqueness proof (Morisawa-Tomizawa & Yasui 07)
Multi-black vacuum solutions:

- Black di-rings ("ring" + "ring") (Iguchi & Mishima 07)
- Black-Saturn ("hole" + "ring") (Elvang & Figueras 07)
- Orthogonal-di-/Bicycling-Rings ("ring" + "ring") (Izumi 07 Elvang & Rodriguez 07)
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- **MP in AdS or dS**  
  e.g., (Gibbons, Lu, Page, Pope 04)

- **on Gibbons-Hawking**  
  e.g. (BMPV 97, Gauntlett-Gutowski-Hull-Pakis-Reall 03)

- **on 4-Euclid space**  
  e.g. (Elvang-Emparan-Mateos-REall 04)

- **on Kaluza-Klein**

- **on Eguchi-Hanson**  
  e.g., (Ishihara, Kimura, Matsuno, Nakagawa, Tomizawa 06-08)

- **on Taub-NUT**  
  e.g., (Bena-Kraus-Warner 05)

- **Black-p-branes**

- **Black holes on Braneworld**
Studies of $D > 4$ black holes

- $4D$ Black holes:
  - “Special” in many respects

- $D > 4$ Black holes: **Much larger variety**
  - Classify (or get phase space diagram for) them!
    - need study
      - Dynamical stability
      - Possible horizon topology
      - Symmetry properties
Stability of static black holes

Gravitational perturbations of $\forall D > 4$ static black holes

– 3 types: tensor-, vector-, scalar-type w.r.t. $(D-2)$-base space
  – get a single *master equation* for each type of perturbations
    (Kodama & AI 03)
  ⇒ make complete stability analysis possible
  ⇒ Stable for vacuum case (AI & Kodama 03)

– Einstein-$\Lambda$-Maxwell case: not completed yet

– New ingredient in $D \geqslant 5$
  Tensor-mode w.r.t. $(D-2)$-horizon manifold $\Sigma$

  *c.f.* if $\Sigma$ is a *highly clumpy* Einstein-manifold,
  ⇒ tensor-mode instability (Gibbons & Hartnoll 02)
Static background $D = m + n$ (warped product type) metric:

$$ds_{(D)}^2 = (m)g_{AB}dy^A dy^B + r(y)^2 d\sigma_{(n)}^2$$

e.g., when $m = 2$

$$(^2)g_{AB}dy^A dy^B = -f(r)dt^2 + \frac{1}{f(r)} dr^2 , \quad d\sigma_{(D-2)}^2 = n\text{-Einstein}$$

Master equations for each tensor/vector/scalar-type of perturbations:

$$\frac{\partial^2}{\partial t^2} \Phi = -A \Phi = \left( \frac{\partial^2}{\partial r_*^2} - U(r) \right) \Phi$$

– looks just like a Schroedinger equation: If $A \geq 0 \Rightarrow$ stable
Stability of rotating holes: Some partial analysis

Perturbations on cohomogeneity-1 MP holes:

\[ D = \text{odd}, \ J_1 = J_2 = \cdots J_{(D-1)/2} \]
\[ \Rightarrow \text{enhanced symmetry: } \mathbb{R} \times U((D-1)/2) \Rightarrow \text{depends only on } r \]
\[ \Rightarrow \text{Perturbation equations reduce to ODEs} \]

- \( D(\text{odd}) \geq 7: \Rightarrow \text{Stable w.r.t. a subclass of tensor perturbations} \)
  \( \text{(tensor-modes w.r.t. } (D-3)\text{-base space)} \)
  \( \text{(Kunduri-Lucietti-Reall 06)} \)

- \( D = 5: \ \text{Decoupled master equations for zero-modes of vector and tensor (gravity) fields} \)
  \[ \Rightarrow \text{Towards complete stability analysis of (coho-1) MP holes} \]
  \( \text{(Murata & Soda 07)} \)
Stability of rotating holes: Some partial analysis

Perturbations on cohomogeneity-2 MP holes:

A single rotation: symmetry enhance $U(1)^N \Rightarrow U(1) \times SO(D - 3)$

$$ds_D^2 = ds_{(4)}^2 \text{(looks like } D = 4 \text{ Kerr metric)} + r^2 \cos^2 \theta \cdot d\Omega_{(D-4)}^2$$

For $D \geq 7$, decoupled master equation for tensor-type perturbations w.r.t. $(D - 4)$-base space $\Leftrightarrow$ Massless Klein-Gordon equation

- Conserved energy integral $\Rightarrow$ Stable
- AdS case $\Rightarrow$ superradiant instability for $|\Lambda| > a^2/r_H^4$ (?) (Kodama 07)

- observed also in coho-mo-1 AdS-MP-holes (Kunduri-Lucietti-Reall 06)

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\[ E := - \int_S dS n^a \chi^b T_{ab} \quad \chi^a : \text{co-rotate Killing vector} \]  \hspace{1cm} (2)

Note: \( \chi^a \) can be non-spacelike if \( |\Lambda| \leq a^2/r_H^4 \Rightarrow E \geq 0 \)

Kerr-AdS
Stability of rotating holes: Some partial analysis

Indication of instability of cohomogeneity-2 MP holes (heuristic argument)

- \( D \geq 6 \Rightarrow \text{no upper-bound on } J: \)
  - ultra-spinning hole looks like “pancake”
  - looks like black-p-brane near the rotation axis
- \( \Rightarrow \) expected to be unstable due to Gregory-Laflamme modes (Emparan & Myers 03)
Myers-Perry solution:

\[
    ds^2 = -dt^2 + \frac{M}{\rho^2 r^{D-5}} \left( dt + a \sin^2 \theta d\phi \right)^2 + \frac{\rho^2}{\Delta} dr^2 \\
    + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\Omega^2_{(D-4)}
\]

where

\[
    \rho^2 = r^2 + a^2 \cos^2 \theta \quad \Delta = r^2 + a^2 - \frac{M}{r^{D-5}}
\]

In the ultra-spinning limit: \( a \to \infty \) with \( \mu = \frac{M}{a^2} \) kept finite, near the pole \( \theta = 0 \) (\( \sigma := a \sin \theta \)) the metric becomes

\[
    ds^2 = - \left( 1 - \frac{\mu}{r^{D-5}} \right) dt^2 + \left( 1 - \frac{\mu}{r^{D-5}} \right)^{-1} dr^2 + r^2 d\Omega^2_{(D-4)} + d\sigma^2 + \sigma^2 d\phi^2
\]

⇒ Black-membrane metric ⇒ Gregory-Laflamme instability
– Similar heuristic arguments apply also to thin black-rings, other multi-rings, Saturns, etc.

i.e., – They are expected to suffer from GL-instability
 BHs in \( D = 4 \) GR

BHs in \( D > 4 \)

Symmetry properties

Remarks

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Topology of event horizon

- **Method 1: global analysis**  \( \text{(Chrusciel & Wald 94)} \)
  - Combine **Topological Censorship** and **Cobordism** of spacelike hypersurface \( S \) with boundaries at horizon and infinity

  **Topological Censorship** \( \Rightarrow \) \( S \) is simply connected

  **\( \Sigma = \partial S \) is cobordant to \( S^{D-2} \) via \( S \)**

  In \( 4D \) \( \Rightarrow \) \( \partial S \) must be \( S^2 \)

  - powerful method in \( 4D \) but turns out to be not so in \( D \geq 6 \)
    e.g., \( \text{(Helfgott-Oz-Yanay 05)} \)
Method 2: local analysis  (Hawking 72)

– Combine variational analysis $\delta \theta / \delta \lambda$ and fact that outer-trapped surface must be inside BH, to show

$$\int_{\Sigma} R > 0$$

w/ $\Sigma$ being a horizon cross-section and $R$ scalar curvature of $\Sigma$

⇒ in $4D$, $\Sigma \approx S^2$ via Gauss-Bonnet Theorem

– generalizes to $D > 4$  (Galloway & Schoen 05)
BHs in $D = 4$ GR

Symmetry properties

Remark

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**Topology Theorem:** (Galloway & Schoen 05  Galloway 07)

Consider a $\forall D \geq 4$ (stationary) black hole spacetime satisfying the dominant energy conditions

The topology of (event) horizon cross-section $\Sigma$ must be such that $\Sigma$ admits metrics of positive scalar curvature

Remarks:

- $\Sigma$ can be topologically e.g., $S^{D-2}$, $S^m \times \cdots \times S^n$ 
  $\Rightarrow$ much larger variety in $D > 4$ than $4D$
- $5D \Rightarrow S^3$ **Black holes and $S^1 \times S^2$ Black-rings**
- **What if $\Lambda < 0$ and $D \geq 6$? $\Rightarrow$ more variety?**
Assertion:

(1) The event horizon of a stationary, electro-vacuum BH is a Killing horizon

(2) If rotating, the BH spacetime must be axisymmetric

* Event Horizon: a boundary of causal past of distant observers
* Killing Horizon: a null hypersurface with a Killing symmetry vector field being normal to it

The event horizon is rigidly rotating with respect to infinity

\[ \cdot \cdot \cdot \quad \text{Black Hole Rigidity} \]
Why “rigidity” interesting?

- relates “global” (even horizon) to “local” (Killing horizon)
Why “rigidity” interesting?

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- foundation of BH Thermodynamics
  (Constancy of surface gravity $\Rightarrow$ Oth Law of Thermodynamics)
### Why “rigidity” interesting?

- relates **“global” (even horizon)** to **“local” (Killing horizon)**
- foundation of BH Thermodynamics
  - *(Constancy of surface gravity ⇒ Oth Law of Thermodynamics)*
- rotating hole ⇒ **extra-(axial) symmetry**
Why “rigidity” interesting?

- relates “global” (even horizon) to “local” (Killing horizon)
- foundation of BH Thermodynamics
  (Constancy of surface gravity $\Rightarrow$ Oth Law of Thermodynamics)
- rotating hole $\Rightarrow$ extra-(axial) symmetry
- a critical step toward proof of “Uniqueness” in $4D$ case
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- In $D > 4$, Uniqueness no longer holds as it stands, and there seems to be a much larger variety of exact BH solutions
Why “rigidity” interesting?

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- foundation of BH Thermodynamics
  (Constancy of surface gravity ⇒ Oth Law of Thermodynamics)
- rotating hole ⇒ extra-(axial) symmetry
- a critical step toward proof of “Uniqueness” in 4D case
- In $D > 4$, Uniqueness no longer holds as it stands, and there seems to be a much larger variety of exact BH solutions

⇒ “Rigidity”—if holds also in $D > 4$—places important restrictions on possible exact BH solutions
An important question:

– Does there exist a $D > 4$ BH solution with only two commuting Killing fields (i.e., w/ isom. $\mathbb{R} \times U(1)$)?

(Reall 03)

– all known $D > 4$ BH solutions have multiple rotational symmetries

⇒ Hunt (less-symmetric) black objects!

– need to show the existence of, at least, one $U(1)$-symmetry
However Hawking’s proof for $4D$ case relies on the fact that event horizon cross-section $\Sigma$ is topologically $2$-sphere. This makes the Generalization to $D > 4$ not at all obvious.

**Goal:** Prove BH Rigidity Theorem in $D \geq 4$

No Assumption on Topology of Event Horizon
Let \((M, g)\) be a \(D \geq 4\), analytic, asymptotically flat, stationary vacuum BH solution to Einstein’s equation. Assume event horizon \(\mathcal{H}\) is analytic, non-degenerate, and topologically \(\mathbb{R} \times \Sigma\) with cross-sections \(\Sigma\) being compact, connected.

**Theorem 1:** There exists a Killing field \(K^a\) in the entire exterior of the BH such that \(K^a\) is normal to \(\mathcal{H}\) and commutes with the stationary Killing vector field \(t^a\) \(\Rightarrow\) “Killing horizon”

**Theorem 2:** If \(t^a\) is not normal to \(\mathcal{H}\), i.e., \(t^a \neq K^a\), then there exist mutually commuting Killing vector fields \(\varphi^a_{(1)}, \ldots, \varphi^a_{(j)}\) \((j \geq 1)\) with period \(2\pi\) and \(t^a = K^a + \Omega_{(1)}\varphi^a_{(1)} + \cdots + \Omega_{(j)}\varphi^a_{(j)}\), where \(\Omega_{(j)}\)'s constants. \(\Rightarrow\) “Axisymmetry”
Brief sketch of proof of Theorem 1

1. Find Killing Vector on the horizon $\mathcal{H}^+$

2. Extend $K^a$ to outside the BH
“Trial foliation” $\Sigma$ & “candidate” vector $K^a$

**Step 1**
Construct a “candidate” Killing field $K^a$ on $\mathcal{H}$ which satisfies

- $K^a K_a = 0$ and $\mathcal{L}_t K^a = 0$ on $\mathcal{H}$
- $\mathcal{L}_K g_{ab} = 0$ (Killing eqn.) on $\mathcal{H}$
- $\alpha = const.$ $(K^c \nabla_c K^a = \alpha K^a)$ on $\mathcal{H}$

Try this one! $K^a = t^a - s^a$

**Step 2**

- Show Taylor expansion $\partial^m (\mathcal{L}_K g_{ab}) / \partial \lambda^m = 0$ at $\mathcal{H}$
- Extend $K^a$ to the entire spacetime by invoking analyticity
However, there is no reason why $\alpha$ need be constant.

— wish to find “correct” $\tilde{K}^a$ with $\tilde{\alpha} = \text{const.} =: \kappa$ on $\mathcal{H}$ by choosing a new “correct” foliation $\tilde{\Sigma}$

Both $K^a$ and $\tilde{K}^a$ are null

$$\tilde{K}^a = f(x) K^a$$

**Task:** Find a solution to equation for coordinate transformation from trial $\Sigma$ to correct $\tilde{\Sigma}$:

$$-\mathcal{L}_s f(x) + \alpha(x) f(x) = \kappa$$

When one solves this equation, the spacetime dimensionality comes to play a role.

**K^a + s^a = t^a = \tilde{K}^a + \tilde{s}^a**
Find correct foliation $\tilde{\Sigma}$: $4D$ case  
Hawking 73

In $4D$, horizon cross-section $\Sigma$ is 2-sphere, and therefore the orbits of $s^a$ must be closed.

There is a discrete isometry “$\Gamma$” which maps each null generator into itself.

Discrete isometry, $\Gamma$, helps to

- define the surface gravity as
  \[ \kappa \equiv P^{-1} \int_0^P \alpha[\phi_s(x)] ds \]
- find correct foliation $\tilde{\Sigma}$
- show **Step 2**
Find correct foliation $\tilde{\Sigma}$: $D > 4$ case

No reason that the isometry $s^a$ need have closed orbits on $\Sigma$. $
Rightarrow$ in general, there is No discrete isometry $\Gamma$.

e.g., 5D Myers-Perry BH w/ 2-rotations $\Omega(1), \Omega(2)$:

$$\Sigma \approx S^3, \quad t^a = K^a + s^a$$

$$s^a = \Omega(1)\varphi^a_{(1)} + \Omega(2)\varphi^a_{(2)}$$

Each rotation Killing vector $\varphi^a$ has closed orbits but $s^a$ does not if $\Omega(1)$ and $\Omega(2)$ are incommensurable.

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Talk at Onomichi 12 Feb. ’08
Solution to $D > 4$ case:

(i) When $s^a$ has closed orbits on $\Sigma \Rightarrow$ we are done!

\[
\kappa = \frac{1}{P} \int_0^P \alpha[\phi_s(x)] \, ds \quad P : \text{period} \quad \phi_s : \text{isom. on } \Sigma \text{ by } s^a
\]

(ii) When $s^a$ has No closed orbits $\Rightarrow$ Use Ergodic Theorem!

\[
\kappa = \lim_{T \to \infty} \frac{1}{T} \int_0^T \alpha[\phi_s(x)] \, ds = \frac{1}{\text{Area}(\Sigma)} \int_\Sigma \alpha(x) \, d\Sigma
\]

"time-average" "space-average"

- can show that the limit "$\kappa$" exists and is constant
- can find well-behaved transformation $\Sigma \to \tilde{\Sigma}$
Solution to $D > 4$ case:

— wish to solve equation, $\alpha(x)f(x) - \mathcal{L}_s f(x) = \kappa$

to find the “correct” horizon Killing field, $\tilde{K}^a = f(x) K^a$

Solution:

\[ f(x) = \kappa \int_0^\infty P(x, T) dT, \quad P(x, T) = \exp \left( - \int_0^T \alpha[\phi_s(x)] ds \right) \]

— since $\forall \epsilon > 0$, $P(x, T) < e^{(\epsilon - \kappa)T}$, for sufficiently large $T$,

$f(x)$ above is well-defined
— wish to show $t^a = K^a + \Omega_{(1)} \varphi^{(1)} + \cdots + \Omega_{(j)} \varphi^{(j)}$
Brief sketch of proof of Theorem 2

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• Get horizon Killing vector field $K^a$ by Theorem 1

⇒ Then $S^a \equiv t^a - K^a$ generates Abelian group, $\mathcal{G}$, of isometries on horizon cross-sections $\Sigma$
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— Extend $U(1)^N$ into the entire spacetime by analyticity
Immediate generalizations:

- can apply to **Einstein-Λ-Maxwell** system
e.g., **charged-AdS-BHs**
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- can apply to Einstein-$\Lambda$-Maxwell system
e.g., charged-AdS-BHs

- combined together with Staticity Theorems

$$d = 4 \quad \text{Sudarsky & Wald (92)} \quad d > 4 \quad \text{Rogatko (05)}$$

⇒ The assertion is rephrased as

Stationary, non-extremal BHs in $D \geq 4$ Einstein-Maxwell system are either static or axisymmetric
Remarks

— can apply to any “horizon” defined as the “boundary” of causal past of a complete timelike orbit $\gamma$ of $t^a$
eq 4

Cosmological horizon

e.g., cosmological horizon
Remarks

- can apply to any “horizon” defined as the “boundary” of causal past of a complete timelike orbit $\gamma$ of $t^a$
eq 4

- can remove analyticity assumption for the BH interior

by using initial value formulation w/ initial data for $K^a$ on the bifurcate horizon

Cosmological horizon

\[ \gamma \]
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  Horizon Killing field $K^a$ may **not** have
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- **Non-trivial topology** at infinity / BH exterior
  - Horizon Killing field $K^a$ may **not** have a single-valued analytic extension

- **Extremal BHs** (i.e., BHs w/ degenerate horizon $\kappa = 0$)