高次元数値相対論

吉野裕高 (アルバータ大)

柴田大 (京大基研)

PRD80, 084025 (2009) [arXiv:0907.2760 [gr-qc]]

arXiv:0912.3606 [gr-qc]

中尾憲一, 阿部博之 (大阪市立大)

PRD80, 084028 (2009) [arXiv:0908.0799 [gr-qc]]

2009年12月24日

研究会「高次元 Black Hole 研究最前線」（京大基研）

Contents

- Introduction
- BSSN formalism
- Cartoon method
- Codes
- (In)stability of a 5D MP black hole
- Summary

Contents

Numerical relativity

Numerical relativity is a method of solving Einstein equations by fully numerical simulations.

- **4D numerical relativity**
  - **has long history motivated by gravitational wave observations.**

- **Higher-dimensional numerical relativity**
  - **Very new topic**
  - **Evolution of Gregory-Laflamme instability**

    (Their formulation is specific to the setup)

Choptuik, Lehner, Olabarrieta, Petryk, Pretorius and Villegas, PRD68, 044001 (2003).

Our target physics

- **(In)stability of higher-dimensional Kerr black holes**

  Shibata and Yoshino, arXiv:0912.3606 [gr-qc]

- **BH formation in high-energy particle collisions**

  by Okawa, Yoshino, Shibata ….

- **Evolution of BHs in RSII braneworld scenarios**

  by Tanahashi, Tanaka, ….

Our target physics

- **(In)stability of higher-dimensional rotating black holes**

  Shibata and Yoshino, arXiv:0912.3606 [gr-qc]

- **BH formation in high-energy particle collisions**

  by Okawa, Yoshino, Shibata ....

- **Evolution of BHs in RSII braneworld scenarios**

  by Tanahashi, Tanaka, ...

Our target physics

- **(In)stability of higher-dimensional rotating black holes**

  Shibata and Yoshino, arXiv:0912.3606 [gr-qc]

- **BH formation in high-energy particle collisions**

  by Okawa, Yoshino, Shibata ....

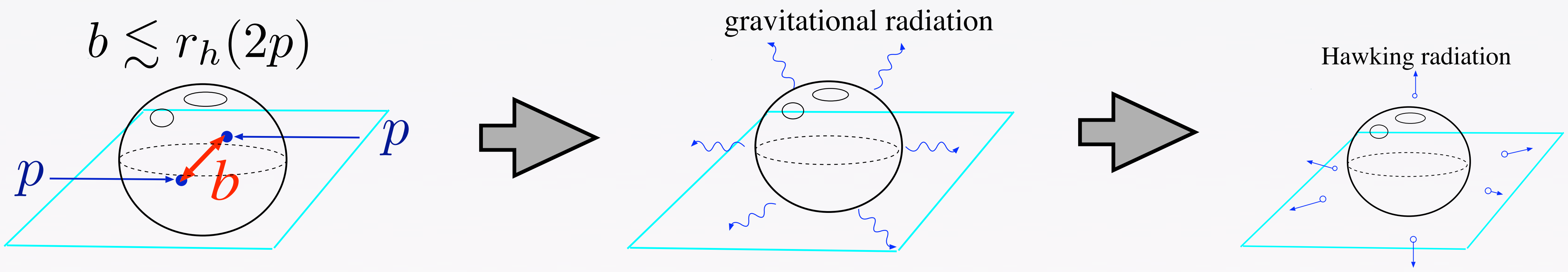
- **Evolution of BHs in RSII braneworld scenarios**

  by Tanahashi, Tanaka, ...

BH formation in high-energy particle collisions

- **BH production at the LHC?**
  - **ADD scenario, RS I scenario**
  - **Planck energy = O(TeV)**

BH production **Balding** **Evaporation**



  - **BH production rate (cross section)**
  - **M and J after the balding (radiation efficiency)**

- **AH studies (instant of collision)**

Yoshino and Nambu, PRD 67, 024009 (2003).
Yoshino and Rychkov, PRD 71, 104028 (2005).

D	4	5	6	7	8	9	10	11
$\sigma_{\rm AH}/\pi[r_h(2p)]^2$	0.76	1.54	2.15	2.52	2.77	2.95	3.09	3.20

Our target physics

- **(In)stability of higher-dimensional rotating black holes**

  Shibata and Yoshino, arXiv:0912.3606 [gr-qc]

- **BH formation in high-energy particle collisions**

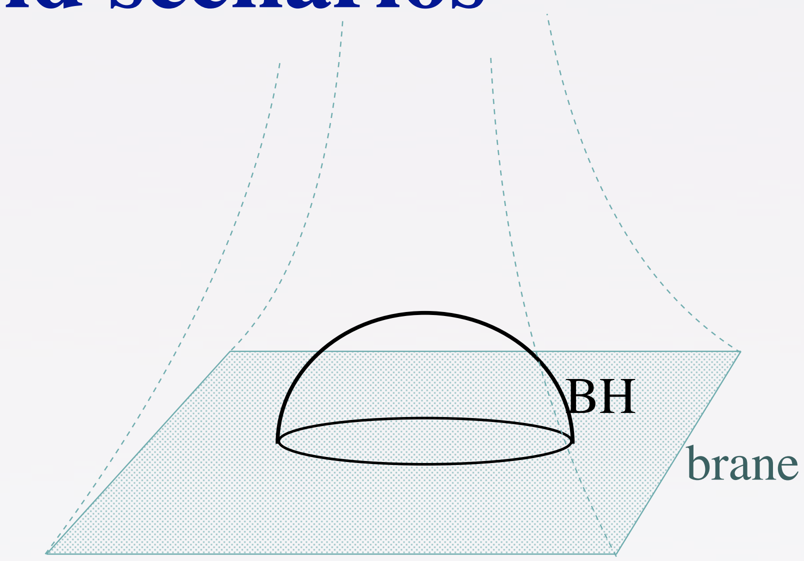
  by Okawa, Yoshino, Shibata ....

- **Evolution of BHs in RSII braneworld scenarios**

  by Tanahashi, Tanaka, ...

Evolution of BHs in RS II braneworld scenarios

- No one has discovered an exact solution of the braneworld BH exists (in 5D case).



- AdS/CFT correspondence may imply the non-existence of such solutions.

Tanaka, Prog. Theor. Phys. Suppl. 148, 307-316 (2002).
Emparan, Fabbri and Kaloper, JHEP08, 043 (2002).

A 4D BH with quantum fields ⟷ A 5D classical BH

↑ ↑

evaporates by Hawking rad. ⟷ escapes into the bulk (?)

- My recent numerical calculation also supports the non-existence of a static BH solution.

Yoshino, JHEP 0901, 268 (2009) [arXiv:0812.0465].

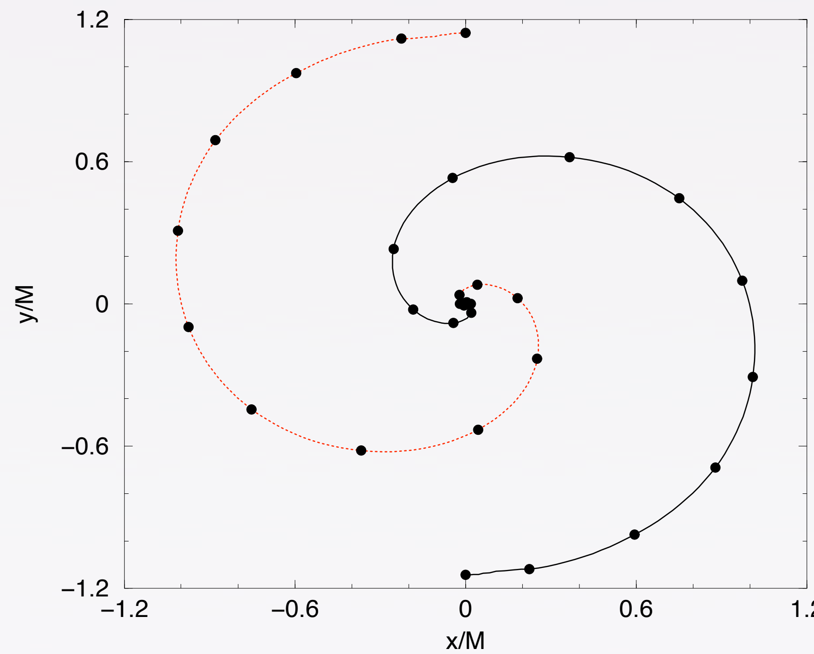
4D numerical relativity (BH systems)

- **Simulations of binary BHs (2005)**

  Pretorius, PRL95, 121101 (2005).

  Campanelli et al., PRL96, 111101 (2005) [gr-qc/011048].

  Baker et al., PRL96, 111102 (2006) [gr-qc/0511103].

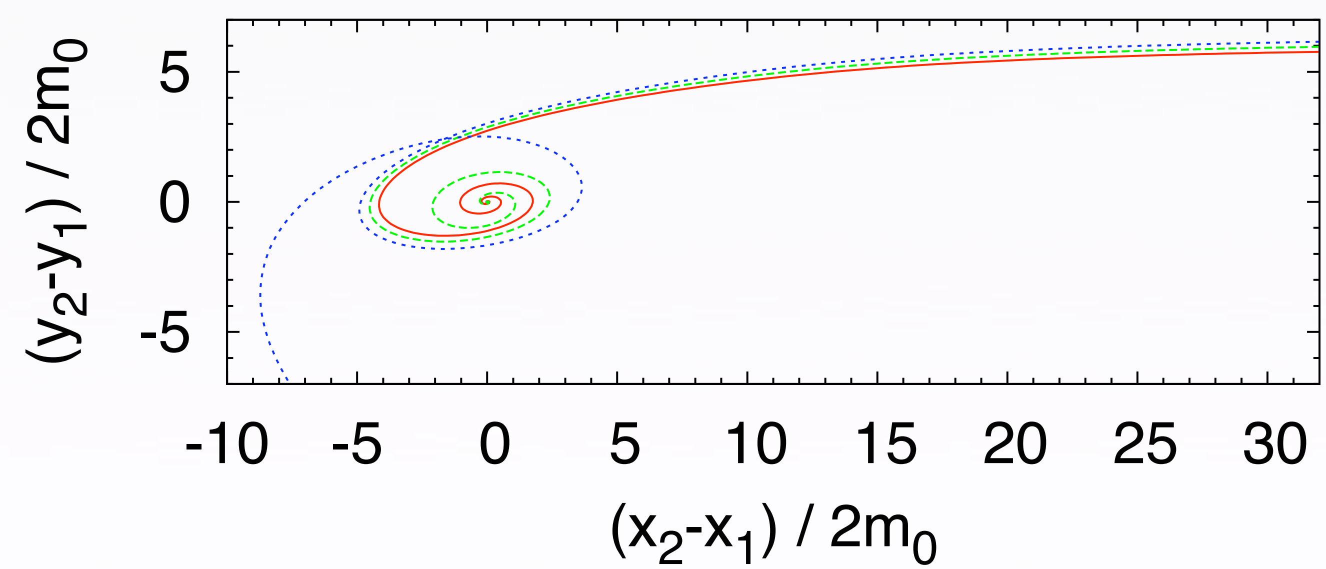


- **Simulations of high-velocity collisions of black holes (2008)**

  Shibata, Okawa and Yamamoto, PRD78, 101501(R) (2008).

  Sperhake, Cardoso, Pretorius, Hinderer and Yunes, arXiv:0907.1252.

SACRA code



4D numerical relativity (techniques)

- Formulation
  - BSSN
  - hyperbolic formulation
  - harmonic formulation
- Appropriate gauge conditions
- Handling BH interior
  - BH excision
  - Puncture method
- Simulating spacetimes with symmetry effectively
- Gravitational wave detection
- AH finder
- AMR

4D numerical relativity (techniques)

- Formulation
  - BSSN
  - hyperbolic formulation
  - harmonic formulation
- Appropriate gauge conditions
- Handling BH interior
  - BH excision
  - Puncture method
- Simulating spacetimes with symmetry effectively
- Gravitational wave detection
- AH finder
- AMR

Plan of talk

In this talk, I would like to explain our formulation and codes of higher-dimensional numerical relativity and its application.

- **BSSN formalism** ← (tools)
- **Cartoon method** ← (tools)
- **Codes** ← **(tests)**
- **(In)stability of a 5D MP black hole** ← **(1st application)**
- **Summary**

Contents

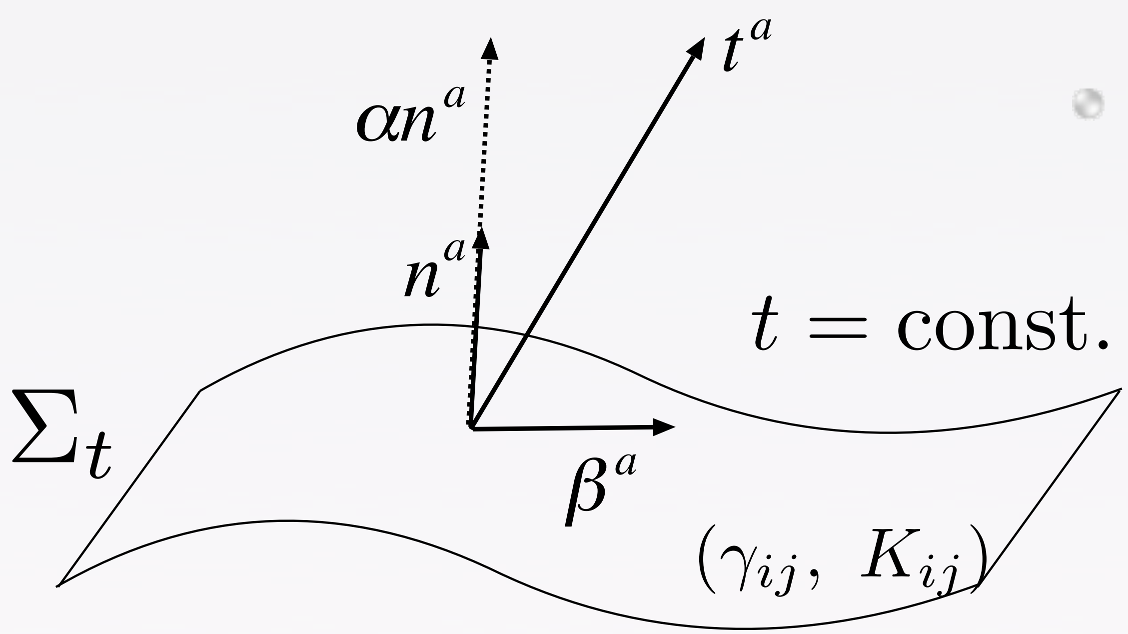
- Introduction
- BSSN formalism
- Cartoon method
- Codes
- (In)stability of a 5D MP black hole
- Summary

ADM formalism

- **Einstein equation** $${}^{(D)}G_{ab} = 8\pi T_{ab}$$

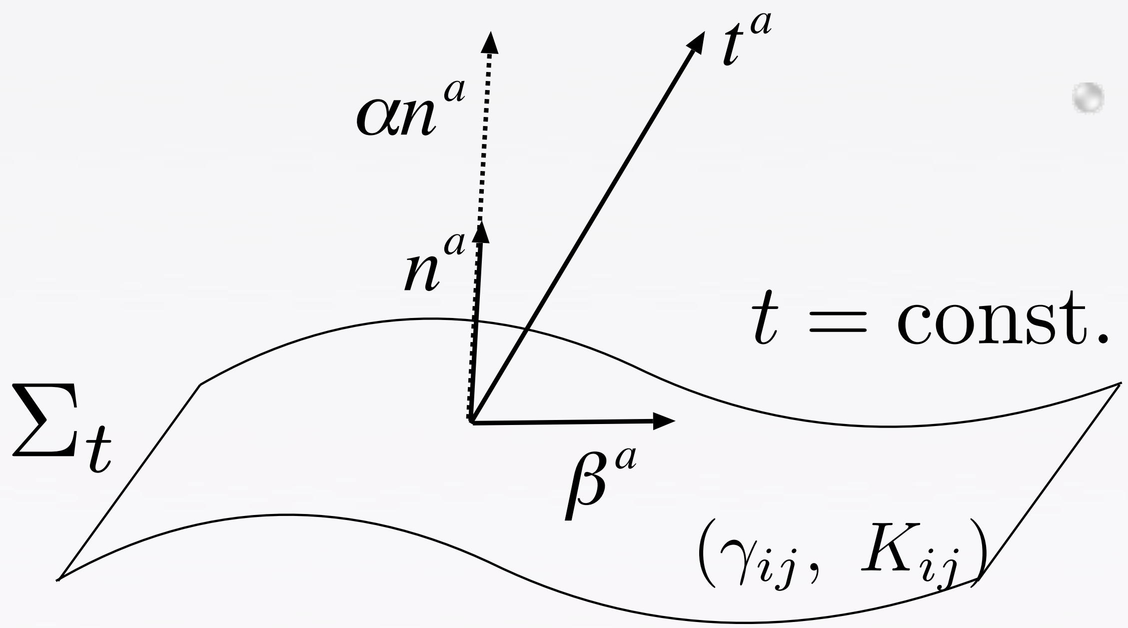
ADM formalism

- **Einstein equation** $${}^{(D)}G_{ab} = 8\pi T_{ab}$$
- **(D-1)+1 splitting**
  - **induced metric** γ_{ij}
  - **extrinsic curvature** $K_{ij} = -(1/2)\mathcal{L}_n \gamma_{ij}$



ADM formalism

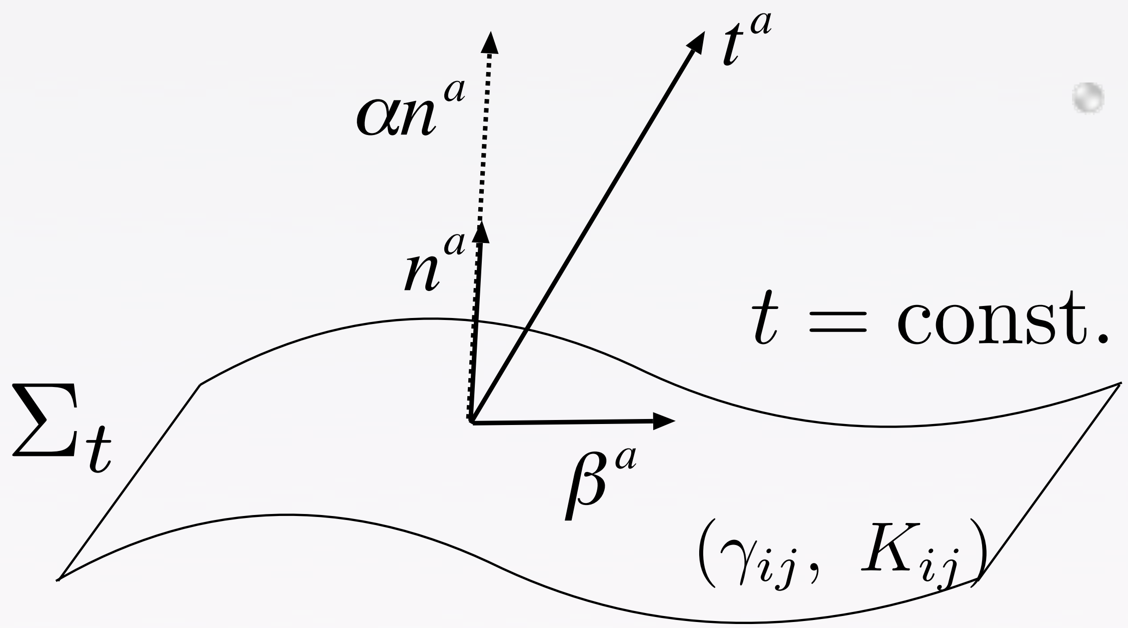
- **Einstein equation** $\quad {}^{(D)}G_{ab} = 8\pi T_{ab}$
- **(D-1)+1 splitting**
  - **induced metric** $\quad \gamma_{ij}$
  - **extrinsic curvature** $\quad K_{ij} = -(1/2)\mathcal{L}_n \gamma_{ij}$



- $G_{ab}n^a n^b = 8\pi T_{ab} n^a n^b$
- $G_{ab}n^a \gamma^b{}_i = 8\pi T_{ab} n^a \gamma^b{}_i$
- $G_{ab}\gamma^a{}_i \gamma^b{}_j = 8\pi T_{ab} \gamma^a{}_i \gamma^b{}_j$

ADM formalism

- **Einstein equation** $${}^{(D)}G_{ab} = 8\pi T_{ab}$$
- **(D-1)+1 splitting**
  - **induced metric** γ_{ij}
  - **extrinsic curvature** $K_{ij} = -(1/2)\mathcal{L}_n \gamma_{ij}$

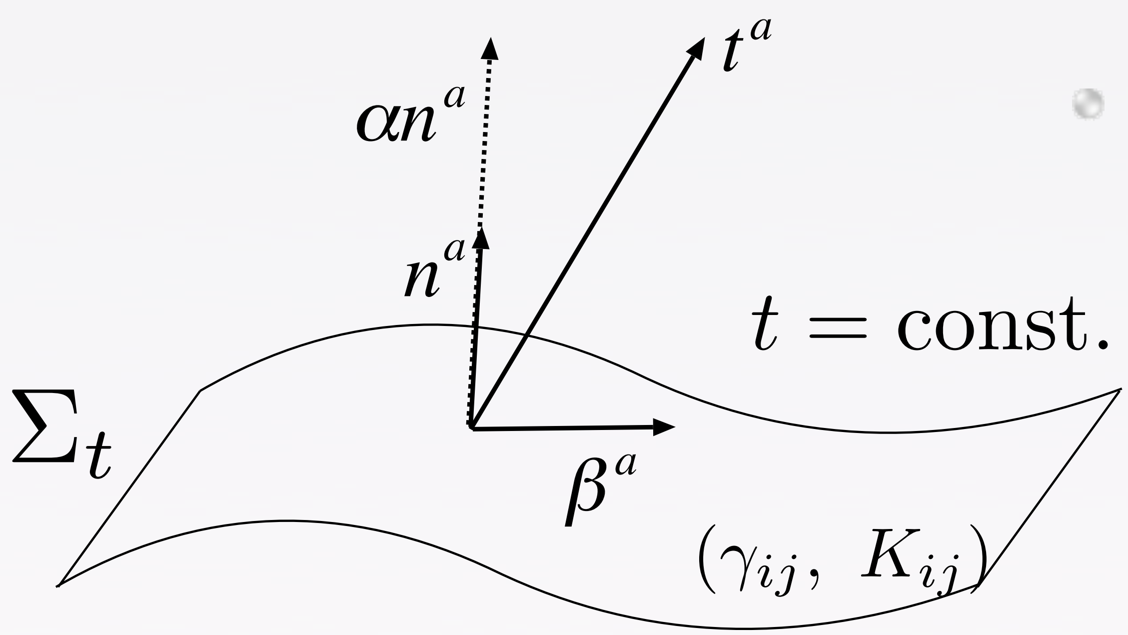


- $G_{ab}n^a n^b = 8\pi T_{ab} n^a n^b \Rightarrow R + K^2 - K_{ij}K^{ij} = 16\pi\rho$
- $G_{ab}n^a \gamma^b{}_i = 8\pi T_{ab} n^a \gamma^b{}_i \Rightarrow D_j K^j_i - D_i K = 8\pi j_i$
- $G_{ab}\gamma^a{}_i \gamma^b{}_j = 8\pi T_{ab}\gamma^a{}_i \gamma^b{}_j$

$$\Rightarrow \quad \partial_t K_{ij} = -D_i D_j \alpha + \alpha\left(R_{ij} - 2K_{ik}K^k{}_j + K_{ik}K\right)$$
$$-8\pi\alpha\left[S_{ij} + \frac{1}{D-2}\gamma_{ij}(\rho - S^k{}_k)\right]$$

ADM formalism

- **Einstein equation** $$^{(D)}G_{ab} = 8\pi T_{ab}$$
- **(D-1)+1 splitting**
  - **induced metric** γ_{ij}
  - **extrinsic curvature** $K_{ij} = -(1/2)\mathcal{L}_n \gamma_{ij}$



- $G_{ab}n^a n^b = 8\pi T_{ab} n^a n^b$ ⇨ $R + K^2 - K_{ij}K^{ij} = 16\pi\rho$ **(Hamiltonian constraint)**
- $G_{ab}n^a \gamma^b{}_i = 8\pi T_{ab} n^a \gamma^b{}_i$ ⇨ $D_j K^j_i - D_i K = 8\pi j_i$ **(Momentum constraint)**
- $G_{ab}\gamma^a{}_i \gamma^b{}_j = 8\pi T_{ab} \gamma^a{}_i \gamma^b{}_j$

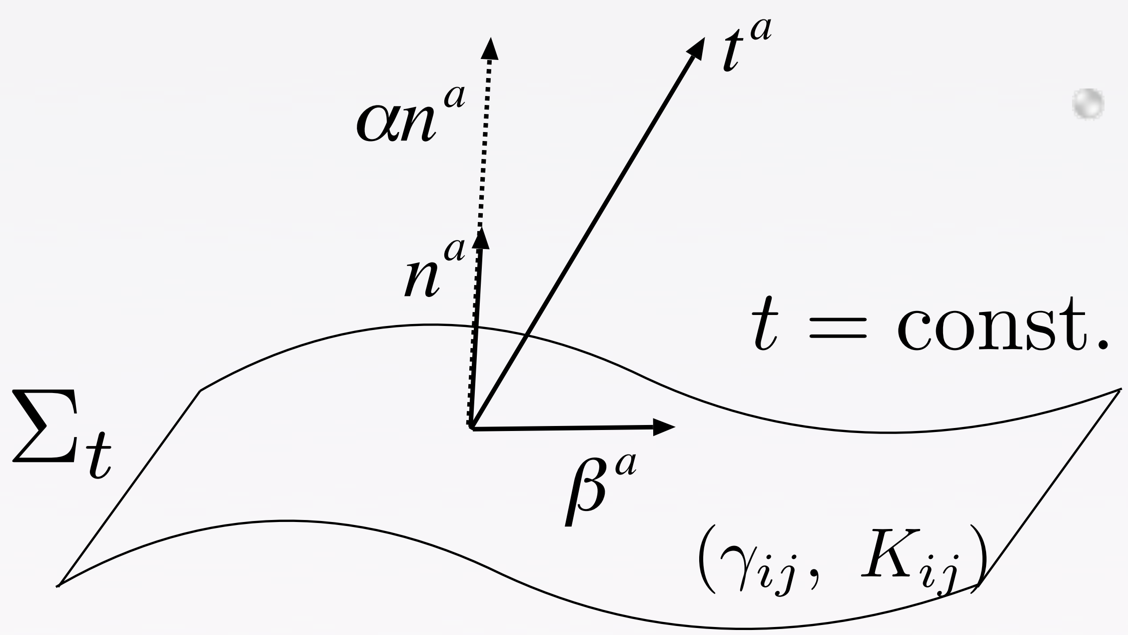
⇨ **Evolution equation (extrinsic curvature)**

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha\left(R_{ij} - 2K_{ik}K^k{}_j + K_{ik}K\right) - 8\pi\alpha\left[S_{ij} + \frac{1}{D-2}\gamma_{ij}(\rho - S^k{}_k)\right]$$

ADM formalism

- **Einstein equation** $${}^{(D)}G_{ab} = 8\pi T_{ab}$$

- **(D-1)+1 splitting**



- **induced metric** γ_{ij}
- **extrinsic curvature** $K_{ij} = -(1/2)\mathcal{L}_n \gamma_{ij}$

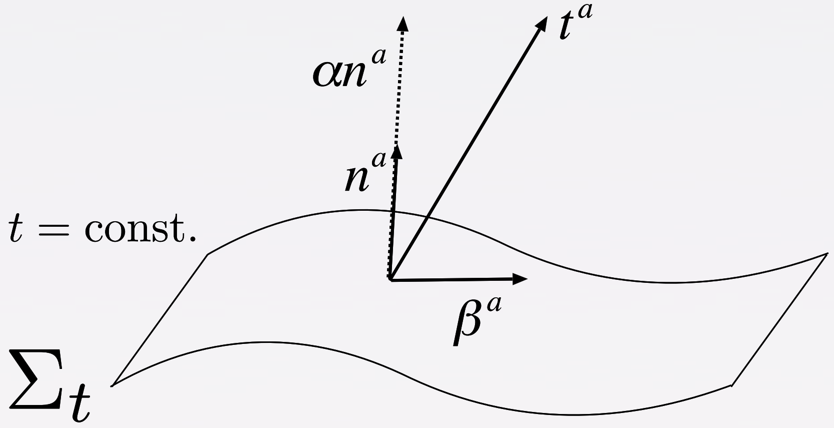
Evolution equation (metric)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

- $G_{ab} n^a n^b = 8\pi T_{ab} n^a n^b \Rightarrow$ $$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$ **(Hamiltonian constraint)**
- $G_{ab} n^a \gamma^b{}_i = 8\pi T_{ab} n^a \gamma^b{}_i \Rightarrow$ $$D_j K^j_i - D_i K = 8\pi j_i$$ **(Momentum constraint)**
- $G_{ab} \gamma^a{}_i \gamma^b{}_j = 8\pi T_{ab} \gamma^a{}_i \gamma^b{}_j \Rightarrow$

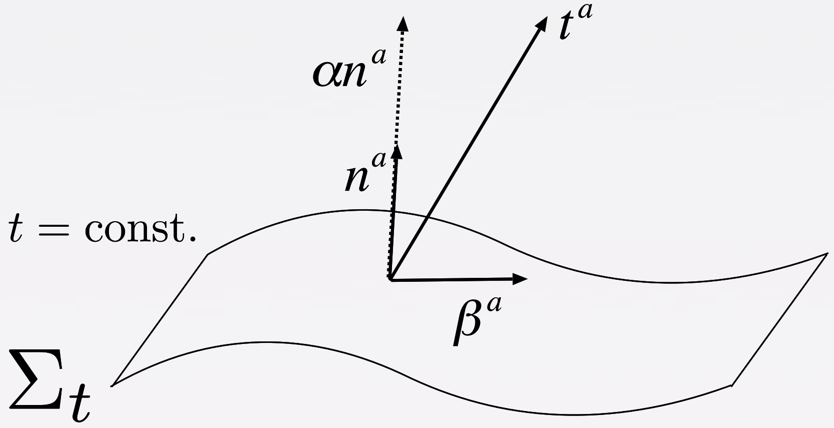
Evolution equation (extrinsic curvature)

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha \left(R_{ij} - 2K_{ik}K^k{}_j + K_{ik}K\right)$$
$$-8\pi\alpha \left[S_{ij} + \frac{1}{D-2}\gamma_{ij}(\rho - S^k{}_k)\right]$$



BSSN formalism

- $\gamma_{ij} = (1/\chi)\tilde{\gamma}_{ij}$ where $\tilde{\gamma} = 1$
- $K_{ij} = \frac{1}{\chi}\left[\tilde{A}_{ij} + \frac{K}{D-1}\tilde{\gamma}_{ij}\right]$ where $\tilde{A}^i{}_i = 0$



BSSN formalism

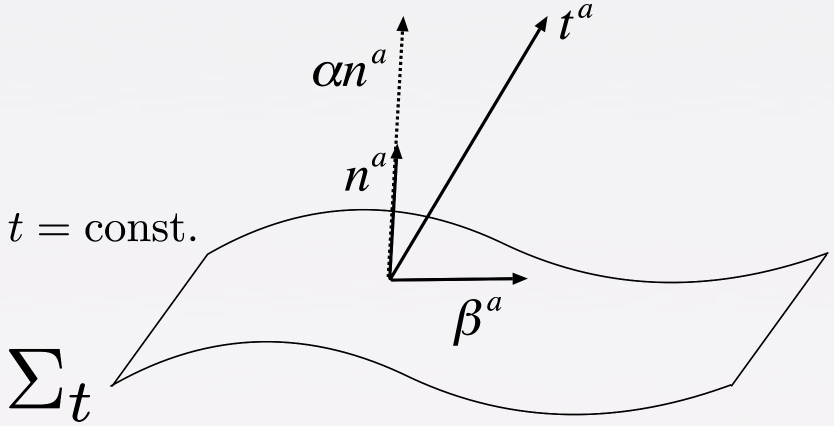
- $\gamma_{ij} = (1/\chi)\tilde{\gamma}_{ij}$ where $\tilde{\gamma} = 1$
- $K_{ij} = \frac{1}{\chi}\left[\tilde{A}_{ij} + \frac{K}{D-1}\tilde{\gamma}_{ij}\right]$ where $\tilde{A}^i{}_i = 0$

evolution equation (metric)

- $\partial_t \chi = \frac{2}{(D-1)}\chi\left(\alpha K - \partial_i \beta^i\right) + \beta^i \partial_i \chi$
- $\partial_t \tilde{\gamma}_{ij} = 2\alpha \tilde{A}_{ij} + \beta^l \partial_l \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta_k + \tilde{\gamma}_{ij}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{\gamma}_{ij}$

evolution equation (extrinsic curvature)

- $\partial_t K = -D^2\alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{K^2}{D-1}\right) + \frac{8\pi\alpha}{D-2}\left[(D-3)\rho + S\right] + \beta^i \partial_i K.$
- $$\partial_t \tilde{A}_{ij} = \chi\left[-(D_i D_j \alpha)^{\mathrm{TF}} + \alpha\left(R_{ij}^{\mathrm{TF}} - 8\pi S_{ij}^{\mathrm{TF}}\right)\right] + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) + \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{A}_{ij}$$



BSSN formalism

- $\gamma_{ij} = (1/\chi)\tilde{\gamma}_{ij}$ where $\tilde{\gamma} = 1$
- $K_{ij} = \dfrac{1}{\chi}\left[\tilde{A}_{ij} + \dfrac{K}{D-1}\tilde{\gamma}_{ij}\right]$ where $\tilde{A}^i{}_i = 0$

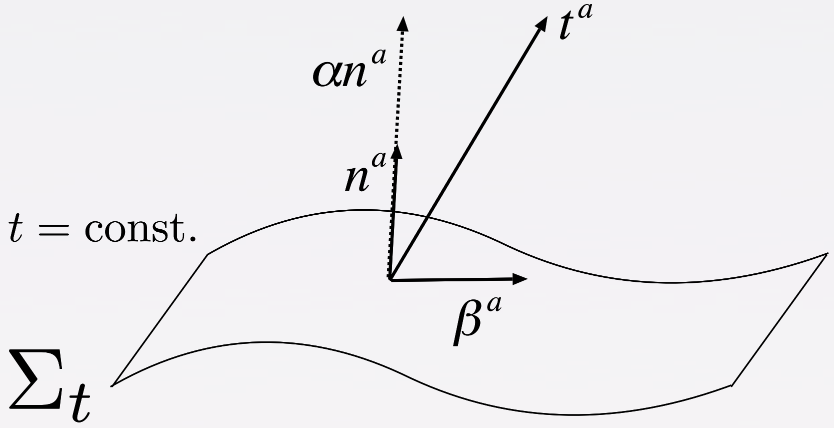
evolution equation (metric)

- $$\partial_t \chi = \frac{2}{(D-1)}\chi\left(\alpha K - \partial_i \beta^i\right) + \beta^i \partial_i \chi$$
- $$\partial_t \tilde{\gamma}_{ij} = 2\alpha \tilde{A}_{ij} + \beta^l \partial_l \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta_k + \tilde{\gamma}_{ij}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{\gamma}_{ij}$$

evolution equation (extrinsic curvature)

- $$\partial_t K = -D^2\alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{K^2}{D-1}\right) + \frac{8\pi\alpha}{D-2}\left[(D-3)\rho + S\right] + \beta^i \partial_i K.$$
- $$\partial_t \tilde{A}_{ij} = \chi\left[-(D_i D_j \alpha)^{\mathrm{TF}} + \alpha\left(R_{ij}^{\mathrm{TF}} - 8\pi S_{ij}^{\mathrm{TF}}\right)\right]$$
$$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) + \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{A}_{ij}$$

- $$R_{ij} = -\frac{1}{2}\tilde{\gamma}^{kl}\left(\tilde{\gamma}_{ij,kl} + \tilde{\gamma}_{kl,ij} - \tilde{\gamma}_{kj,il} - \tilde{\gamma}_{il,kj}\right) + \cdots$$



BSSN formalism

- $\gamma_{ij} = (1/\chi)\tilde{\gamma}_{ij}$ where $\tilde{\gamma} = 1$
- $K_{ij} = \dfrac{1}{\chi}\left[\tilde{A}_{ij} + \dfrac{K}{D-1}\tilde{\gamma}_{ij}\right]$ where $\tilde{A}^i{}_i = 0$
- $\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$

(auxiliary variables)

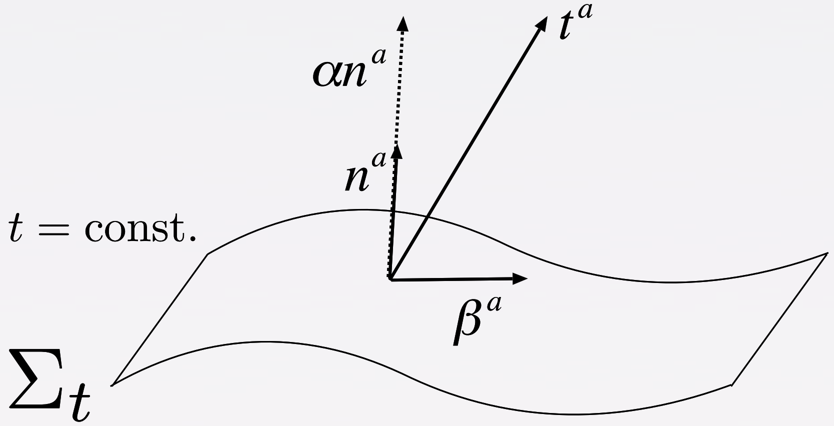
evolution equation (metric)

- $$\partial_t \chi = \frac{2}{(D-1)}\chi\left(\alpha K - \partial_i\beta^i\right) + \beta^i\partial_i\chi$$
- $$\partial_t\tilde{\gamma}_{ij} = 2\alpha\tilde{A}_{ij} + \beta^l\partial_l\tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta_k + \tilde{\gamma}_{ij}\partial_i\beta^k - \frac{2}{D-1}\partial_k\beta^k\tilde{\gamma}_{ij}$$

evolution equation (extrinsic curvature)

- $$\partial_t K = -D^2\alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{K^2}{D-1}\right) + \frac{8\pi\alpha}{D-2}\left[(D-3)\rho + S\right] + \beta^i\partial_i K.$$
- $$\partial_t\tilde{A}_{ij} = \chi\left[-(D_iD_j\alpha)^{\rm TF} + \alpha\left(R^{\rm TF}_{ij} - 8\pi S^{\rm TF}_{ij}\right)\right]$$
  $$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) + \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{D-1}\partial_k\beta^k\tilde{A}_{ij}$$

- $$R_{ij} = -\frac{1}{2}\tilde{\gamma}^{kl}\left(\tilde{\gamma}_{ij,kl} + \tilde{\gamma}_{kl,ij} - \tilde{\gamma}_{kj,il} - \tilde{\gamma}_{il,kj}\right) + \cdots$$



BSSN formalism

- $\gamma_{ij} = (1/\chi)\tilde{\gamma}_{ij}$ where $\tilde{\gamma} = 1$
- $K_{ij} = \dfrac{1}{\chi}\left[\tilde{A}_{ij} + \dfrac{K}{D-1}\tilde{\gamma}_{ij}\right]$ where $\tilde{A}^i{}_i = 0$
- $\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$

(auxiliary variables)

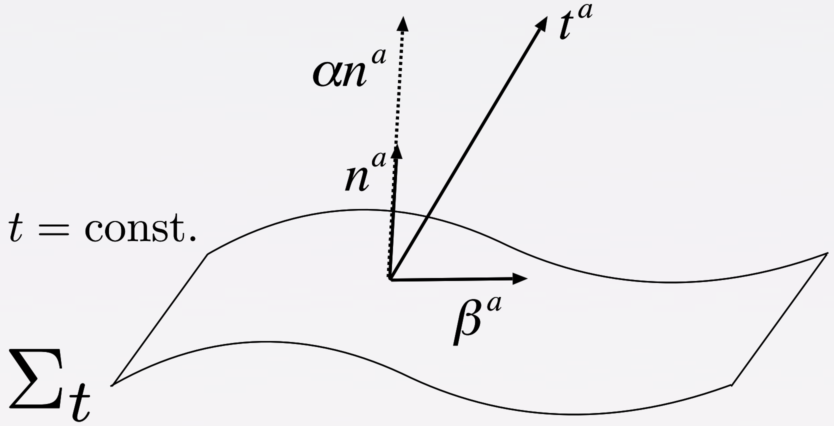
evolution equation (metric)

- $$\partial_t \chi = \frac{2}{(D-1)}\chi\left(\alpha K - \partial_i \beta^i\right) + \beta^i \partial_i \chi$$
- $$\partial_t \tilde{\gamma}_{ij} = 2\alpha \tilde{A}_{ij} + \beta^l \partial_l \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta_k + \tilde{\gamma}_{ij}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{\gamma}_{ij}$$

evolution equation (extrinsic curvature)

- $$\partial_t K = -D^2\alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{K^2}{D-1}\right) + \frac{8\pi\alpha}{D-2}\left[(D-3)\rho + S\right] + \beta^i \partial_i K.$$
- $$\partial_t \tilde{A}_{ij} = \chi\left[-(D_i D_j \alpha)^{\mathrm{TF}} + \alpha\left(R^{\mathrm{TF}}_{ij} - 8\pi S^{\mathrm{TF}}_{ij}\right)\right]$$
$$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) + \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{A}_{ij}$$

- $$R_{ij} = -\frac{1}{2}\tilde{\gamma}^{kl}\tilde{\gamma}_{ij,kl} + \frac{1}{2}\left(\tilde{\gamma}_{ki}\partial_j \tilde{\Gamma}^k + \tilde{\gamma}_{kj}\partial_i \tilde{\Gamma}^k\right) + \cdots$$



BSSN formalism

- $\gamma_{ij} = (1/\chi)\tilde{\gamma}_{ij}$ where $\tilde{\gamma} = 1$
- $K_{ij} = \dfrac{1}{\chi}\left[\tilde{A}_{ij} + \dfrac{K}{D-1}\tilde{\gamma}_{ij}\right]$ where $\tilde{A}^i{}_i = 0$
- $\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$

(auxiliary variables)

evolution equation (metric)

- $$\partial_t \chi = \frac{2}{(D-1)}\chi\left(\alpha K - \partial_i \beta^i\right) + \beta^i \partial_i \chi$$
- $$\partial_t \tilde{\gamma}_{ij} = 2\alpha \tilde{A}_{ij} + \beta^l \partial_l \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta_k + \tilde{\gamma}_{ij}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{\gamma}_{ij}$$

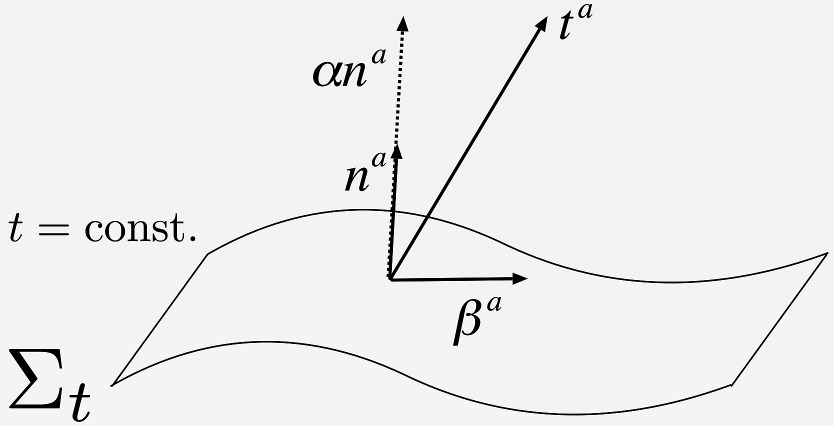
evolution equation (extrinsic curvature)

- $$\partial_t K = -D^2\alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{K^2}{D-1}\right) + \frac{8\pi\alpha}{D-2}\left[(D-3)\rho + S\right] + \beta^i \partial_i K.$$
- $$\partial_t \tilde{A}_{ij} = \chi\left[-(D_i D_j \alpha)^{\mathrm{TF}} + \alpha\left(R^{\mathrm{TF}}_{ij} - 8\pi S^{\mathrm{TF}}_{ij}\right)\right]$$
$$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) + \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{A}_{ij}$$

- $$R_{ij} = -\frac{1}{2}\tilde{\gamma}^{kl}\tilde{\gamma}_{ij,kl} + \frac{1}{2}\left(\tilde{\gamma}_{ki}\partial_j \tilde{\Gamma}^k + \tilde{\gamma}_{kj}\partial_i \tilde{\Gamma}^k\right) + \cdots$$

evolution equation (auxiliary variables)

- $$\partial_t \bar{\Gamma}^i = -2\tilde{A}^{ij}\partial_j \alpha + 2\alpha\left[\bar{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{D-2}{D-1}\bar{\gamma}^{ij}K_{,j} - 8\pi\bar{\gamma}^{ij}j_j - \frac{(D-1)}{2}\frac{\chi_{,j}}{\chi}\tilde{A}^{ij}\right]$$
$$+\beta^j \partial_j \bar{\Gamma}^i - \bar{\Gamma}^j \partial_j \beta^i + \frac{2}{D-1}\bar{\Gamma}^i \partial_j \beta^j + \frac{D-3}{D-1}\bar{\gamma}^{ik}\beta^j_{,jk} + \bar{\gamma}^{jk}\beta^i_{,jk}$$



BSSN formalism

- $\gamma_{ij} = (1/\chi)\tilde{\gamma}_{ij}$ where
- $K_{ij} = \dfrac{1}{\chi}\left[\tilde{A}_{ij} + \dfrac{K}{D-1}\tilde{\gamma}_{ij}\right]$ where

(additional constraints)

$$\tilde{\gamma} = 1$$

$$\tilde{A}^i{}_i = 0$$

- $\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$

(auxiliary variables)

evolution equation (metric)

- $\partial_t\chi = \dfrac{2}{(D-1)}\chi\left(\alpha K - \partial_i\beta^i\right) + \beta^i\partial_i\chi$
- $\partial_t\tilde{\gamma}_{ij} = 2\alpha\tilde{A}_{ij} + \beta^l\partial_l\tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta_k + \tilde{\gamma}_{ij}\partial_i\beta^k - \dfrac{2}{D-1}\partial_k\beta^k\tilde{\gamma}_{ij}$

evolution equation (extrinsic curvature)

- $\partial_t K = -D^2\alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \dfrac{K^2}{D-1}\right) + \dfrac{8\pi\alpha}{D-2}\left[(D-3)\rho + S\right] + \beta^i\partial_i K.$
- $$\partial_t\tilde{A}_{ij} = \chi\left[-(D_iD_j\alpha)^{\mathrm{TF}} + \alpha\left(R^{\mathrm{TF}}_{ij} - 8\pi S^{\mathrm{TF}}_{ij}\right)\right]$$
  $$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) + \beta^k\partial_k\tilde{A}_{ij} + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{kj}\partial_i\beta^k - \frac{2}{D-1}\partial_k\beta^k\tilde{A}_{ij}$$

- $R_{ij} = -\dfrac{1}{2}\tilde{\gamma}^{kl}\tilde{\gamma}_{ij,kl} + \dfrac{1}{2}\left(\tilde{\gamma}_{ki}\partial_j\tilde{\Gamma}^k + \tilde{\gamma}_{kj}\partial_i\tilde{\Gamma}^k\right) + \cdots$

evolution equation (auxiliary variables)

- $$\partial_t\bar{\Gamma}^i = -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left[\bar{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{D-2}{D-1}\bar{\gamma}^{ij}K_{,j} - 8\pi\bar{\gamma}^{ij}j_j - \frac{(D-1)}{2}\frac{\chi_{,j}}{\chi}\tilde{A}^{ij}\right]$$
  $$+\beta^j\partial_j\bar{\Gamma}^i - \bar{\Gamma}^j\partial_j\beta^i + \frac{2}{D-1}\bar{\Gamma}^i\partial_j\beta^j + \frac{D-3}{D-1}\bar{\gamma}^{ik}\beta^j_{,jk} + \bar{\gamma}^{jk}\beta^i_{,jk}$$

Contents

- Introduction
- BSSN formalism
- Cartoon method
- Codes
- (In)stability of a 5D MP black hole
- Summary

Cartoon method

Cartoon method is a very effective method for simulating spacetimes with symmetries (using Cartesian coordinates).

- Cartoon method (1)
  - 4D
  - 5D
- Cartoon method (2)

Cartoon method

Cartoon method is a very effective method for simulating spacetimes with symmetries (using Cartesian coordinates).

- **Cartoon method (1)**
  - **4D**
  - **5D**
- **Cartoon method (2)**

Cartoon method (4D)

Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

- **Axisymmetric system (x=y, z)**

Cartoon method (4D)

Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

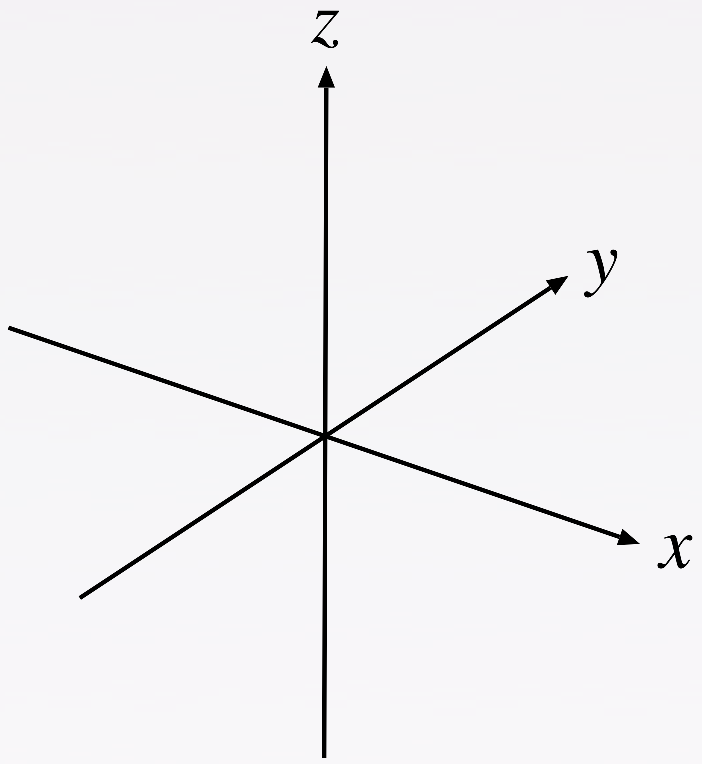
- **Axisymmetric system (x=y, z)**
  - **cylindrical coordinates** (ρ, ϕ, z)

$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

Cartoon method (4D)

Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

- **Cartesian coordinates** (x, y, z)



- **Axisymmetric system (x=y, z)**
  - **cylindrical coordinates** (ρ, ϕ, z)

$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

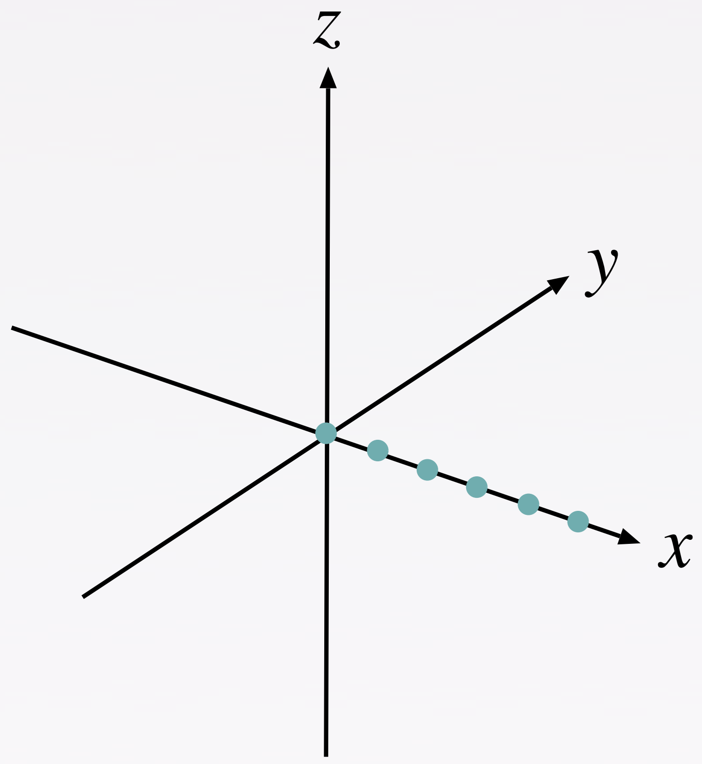
Cartoon method (4D)

Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

- **Axisymmetric system (x=y, z)**
  - **cylindrical coordinates** (ρ, ϕ, z)

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

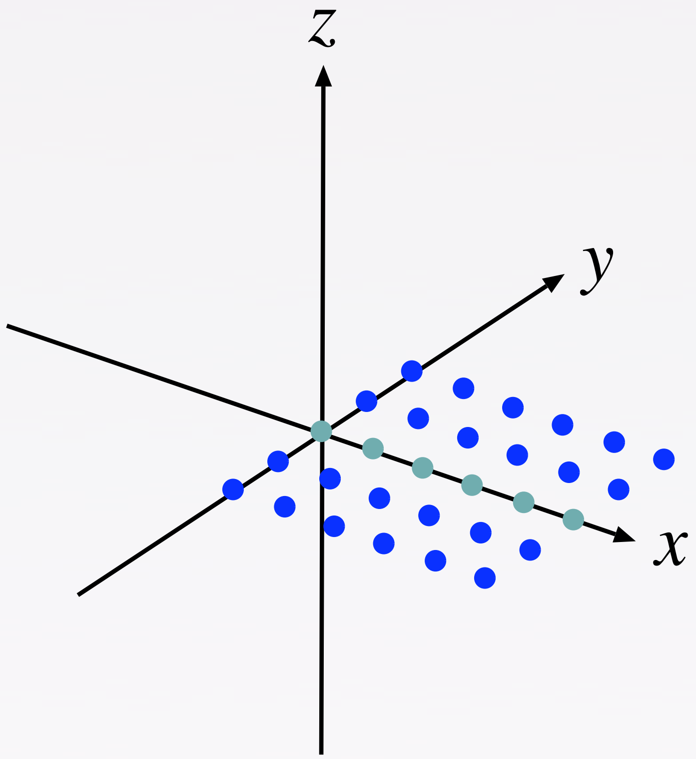
- **Cartesian coordinates** (x, y, z)



Cartoon method (4D)

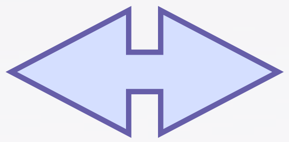
Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

- **Cartesian coordinates** (x, y, z)



- **Axisymmetric system (x=y, z)**
  - **cylindrical coordinates** (ρ, ϕ, z)

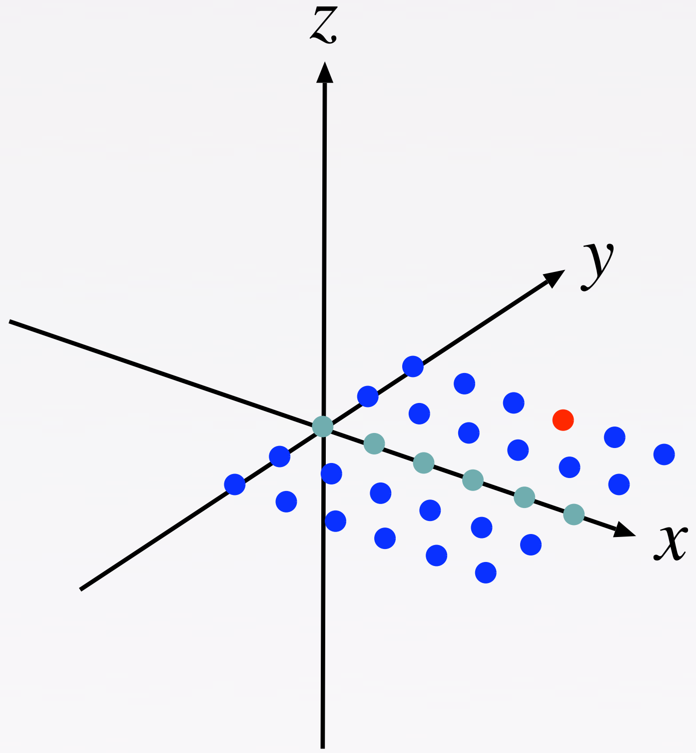
$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$



Cartoon method (4D)

Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

- **Cartesian coordinates** (x, y, z)



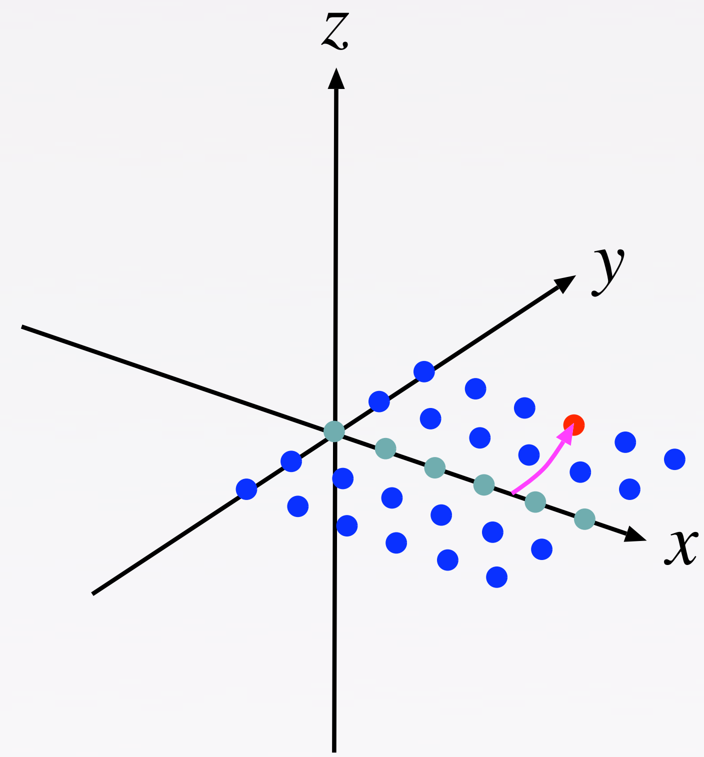
- **Axisymmetric system (x=y, z)**
  - **cylindrical coordinates** (ρ, ϕ, z)

$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

Cartoon method (4D)

Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

- **Cartesian coordinates** (x, y, z)



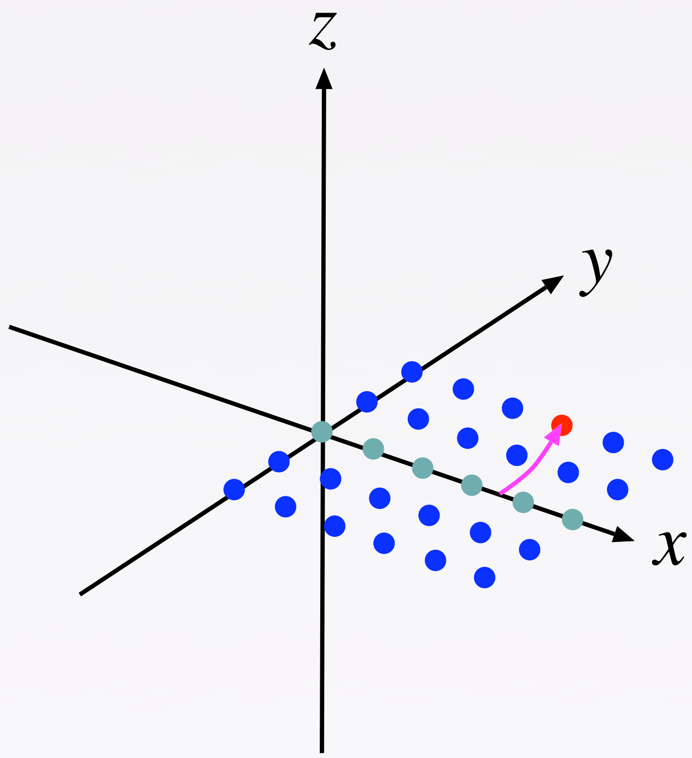
- **Axisymmetric system (x=y, z)**
  - **cylindrical coordinates** (ρ, ϕ, z)

$$
\begin{aligned}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z
\end{aligned}
$$

Cartoon method (4D)

Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

- **Cartesian coordinates** (x, y, z)



- **Axisymmetric system (x=y, z)**
  - **cylindrical coordinates** (ρ, ϕ, z)

$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$
$$z = z$$

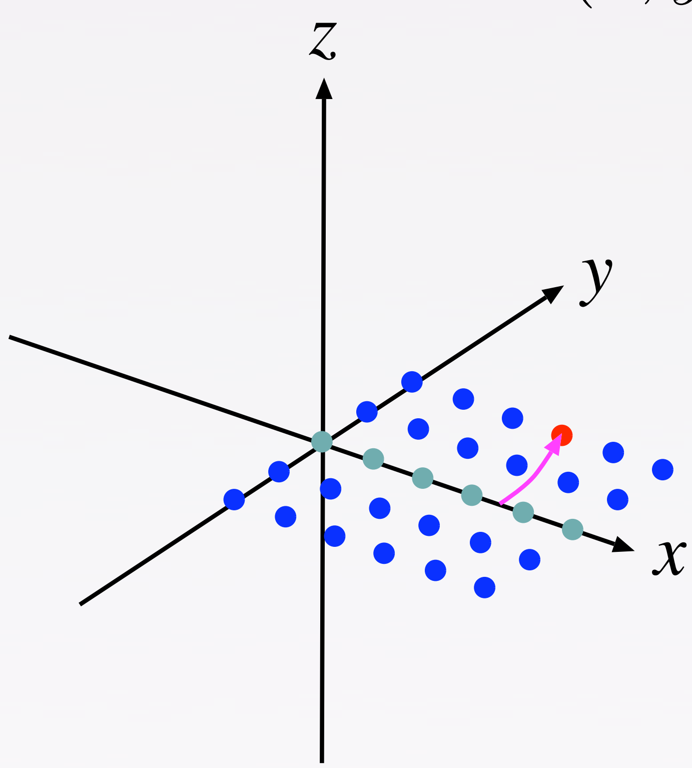
Scalar

$$\Psi(x, y, z) = \Psi(\rho, 0, z)$$

Cartoon method (4D)

Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

- **Cartesian coordinates** (x, y, z)



- **Axisymmetric system (x=y, z)**
  - **cylindrical coordinates** (ρ, ϕ, z)

$$
\begin{aligned}
x &= \rho \cos\phi \\
y &= \rho \sin\phi \\
z &= z
\end{aligned}
$$

Vector

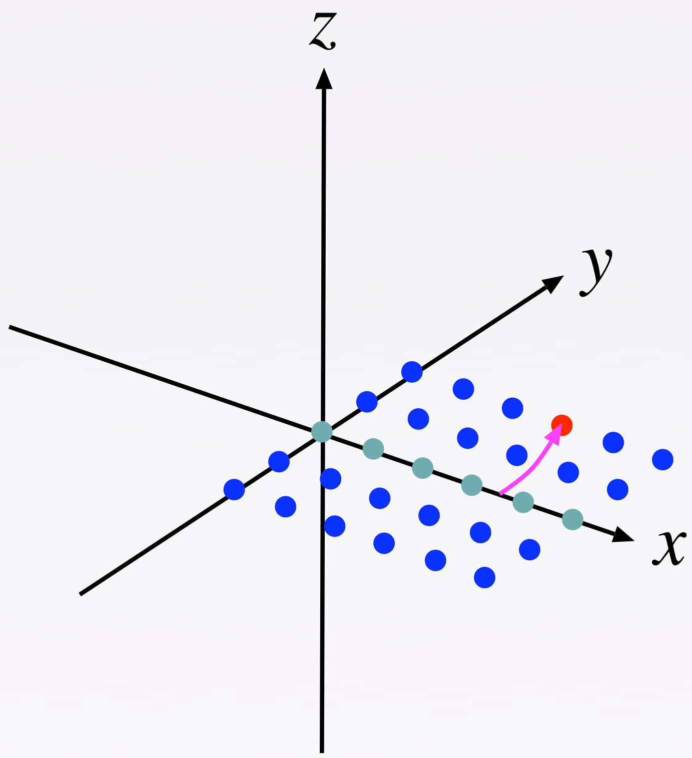
$$T^z(x, y, z) = T^z(\rho, 0, z)$$

$$
\begin{aligned}
T^x(x, y, z) &= (x/\rho) T^x(\rho, 0, z) - (y/\rho) T^y(\rho, 0, z) \\
T^y(x, y, z) &= (y/\rho) T^x(\rho, 0, z) + (x/\rho) T^y(\rho, 0, z)
\end{aligned}
$$

Cartoon method (4D)

Alcubierre, Brandt, Brugmann, Holz, Seidel, Takahashi, and Thornburg, gr-qc/9908012.

- **Cartesian coordinates** (x, y, z)



- **Axisymmetric system (x=y, z)**
  - **cylindrical coordinates** (ρ, ϕ, z)

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

2-rank symmetric tensor

$$S^{zz}(x,y,z) = S^{zz}(\rho,0,z)$$

$$S^{zx}(x,y,z) = (x/\rho)S^{zx}(\rho,0,z) - (y/\rho)S^{zy}(\rho,0,z)$$
$$S^{zy}(x,y,z) = (y/\rho)S^{zx}(\rho,0,z) + (x/\rho)S^{zy}(\rho,0,z)$$

$$S^{xx}(x,y,z) = (x/\rho)^2 S^{xx}(\rho,0,z) + (y/\rho)^2 S^{yy}(\rho,0,z) - (2xy/\rho^2)S^{xy}(\rho,0,z)$$
$$S^{yy}(x,y,z) = (y/\rho)^2 S^{xx}(\rho,0,z) + (x/\rho)^2 S^{yy}(\rho,0,z) + (2xy/\rho^2)S^{xy}(\rho,0,z)$$
$$S^{xy}(x,y,z) = (xy/\rho)[S^{xx}(\rho,0,z) - S^{yy}(\rho,0,z)] + [(x^2-y^2)/\rho^2]S^{xy}(\rho,0,z)$$

Cartoon method

Cartoon method is a very effective method for simulating spacetimes with symmetries (using Cartesian coordinates).

- Cartoon method (1)
  - 4D
  - 5D
- Cartoon method (2)

Cartoon method (5D)

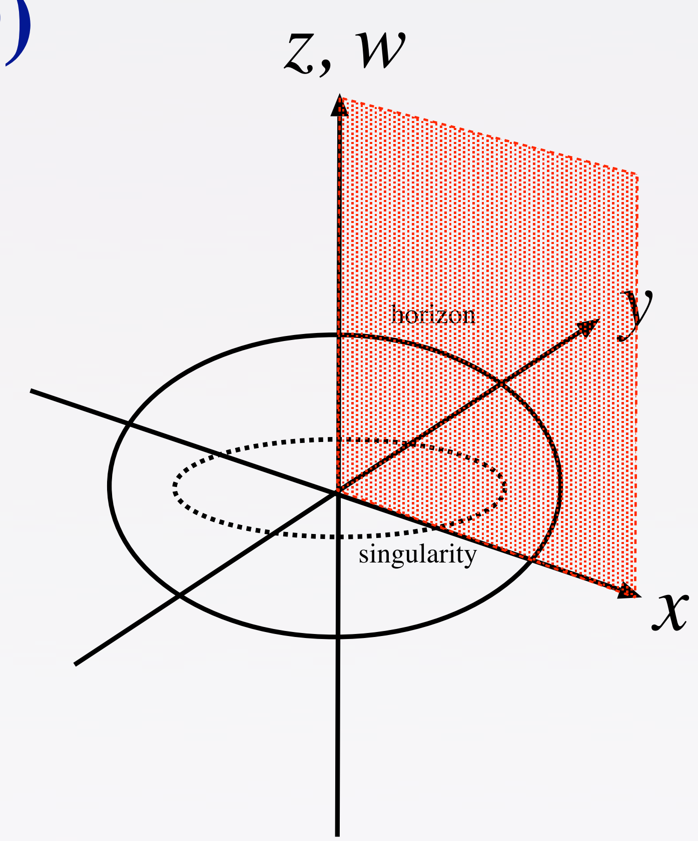
- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)

Cartoon method (5D)

- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)

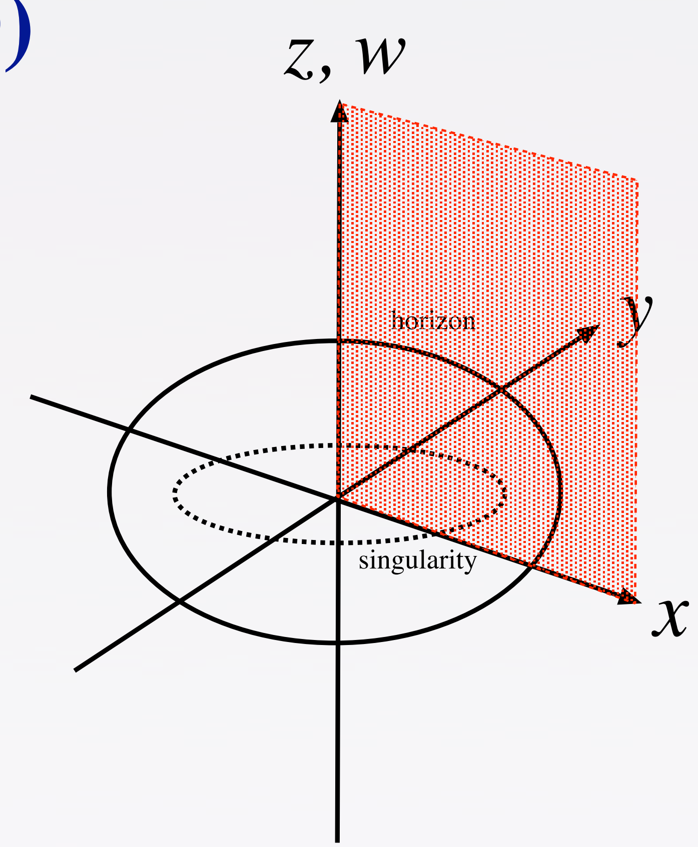
Cartoon method (5D)

- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)



Cartoon method (5D)

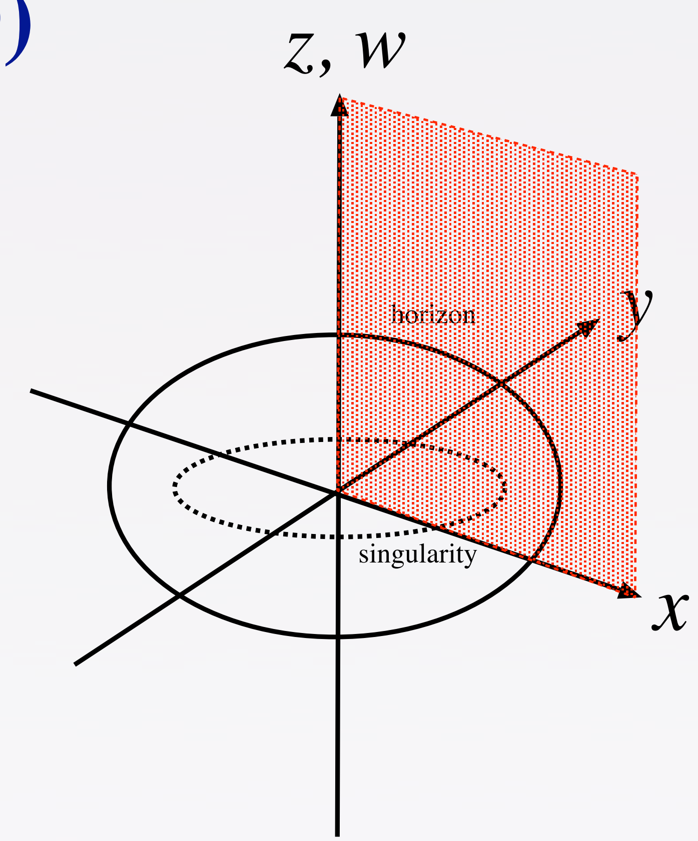
- **Space is 4D (x, y, z, w)**
  - **U(1)×U(1) symmetry (x=y, z=w)**
  - **SO(3) symmetry (x=y=z, w)**



Data of (x, 0, z, 0)

Cartoon method (5D)

- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)



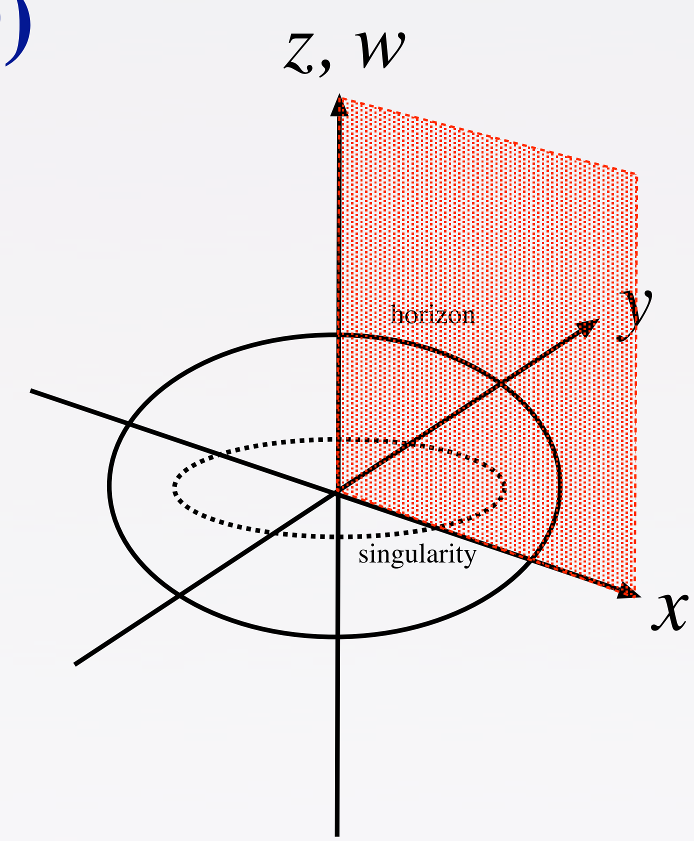
Data of (x, 0, z, 0) ⇨ Data of (x, y, z, 0)

⇧

1st cartoon

Cartoon method (5D)

- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)

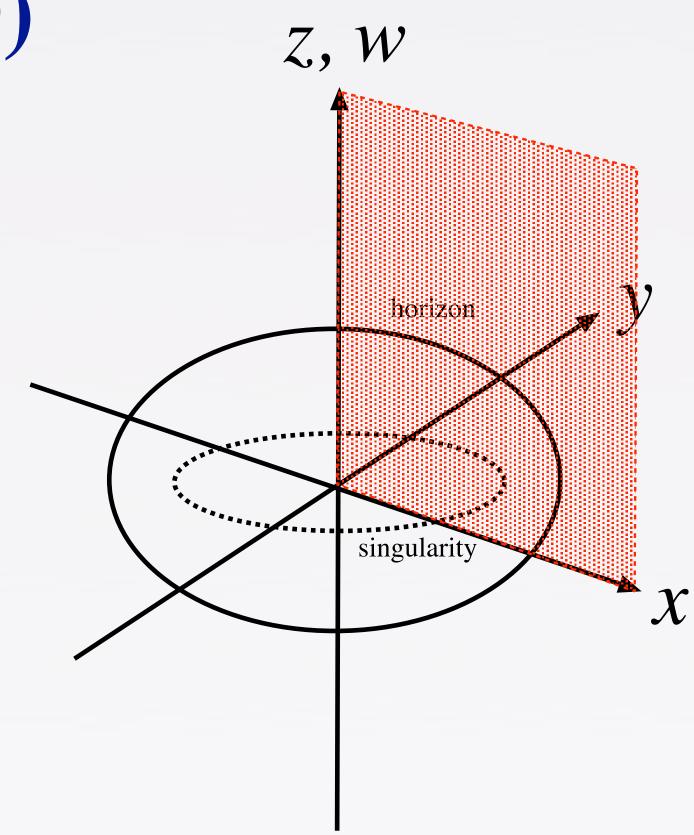


Data of (x, 0, z, 0) ➔ Data of (x, y, z, 0) ➔ Data of (x, y, z, w)

1st cartoon

2nd cartoon

Cartoon method (5D)



- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)

Data of (x, 0, z, 0) → Data of (x, y, z, 0) → Data of (x, y, z, w)

1st cartoon

2nd cartoon

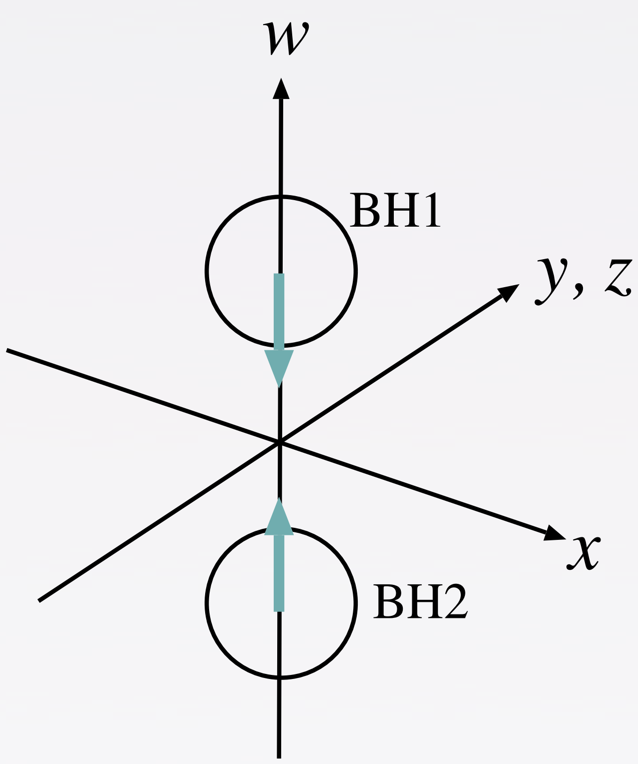
Double Cartoon method

Cartoon method (5D)

- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)

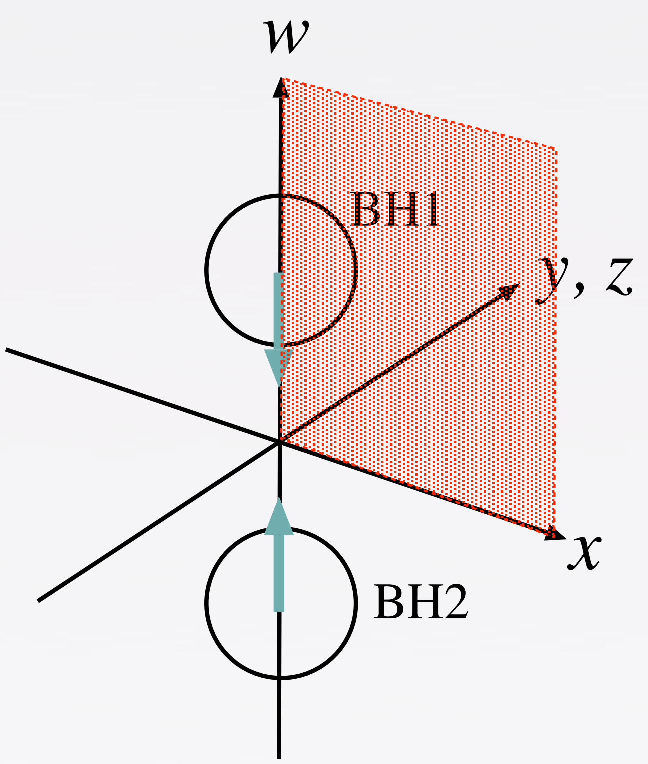
Cartoon method (5D)

- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)



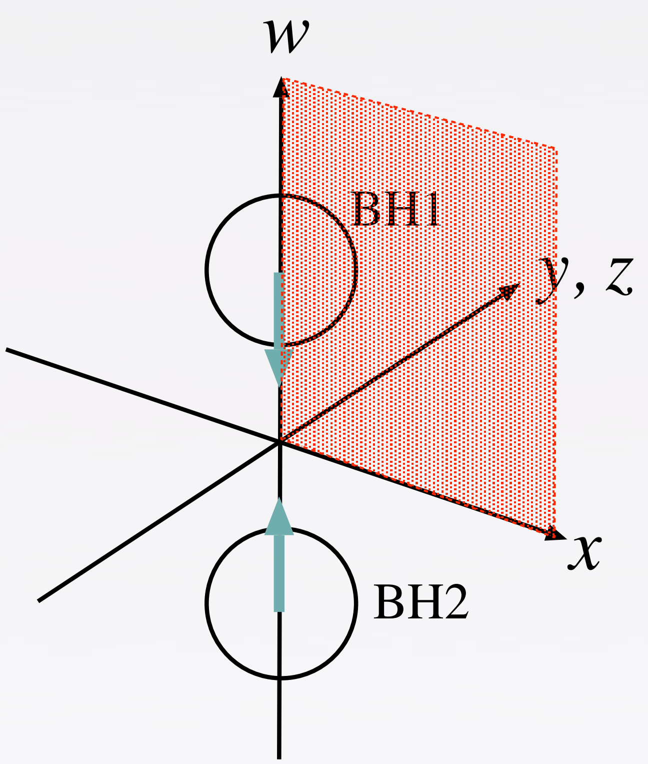
Cartoon method (5D)

- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)



Cartoon method (5D)

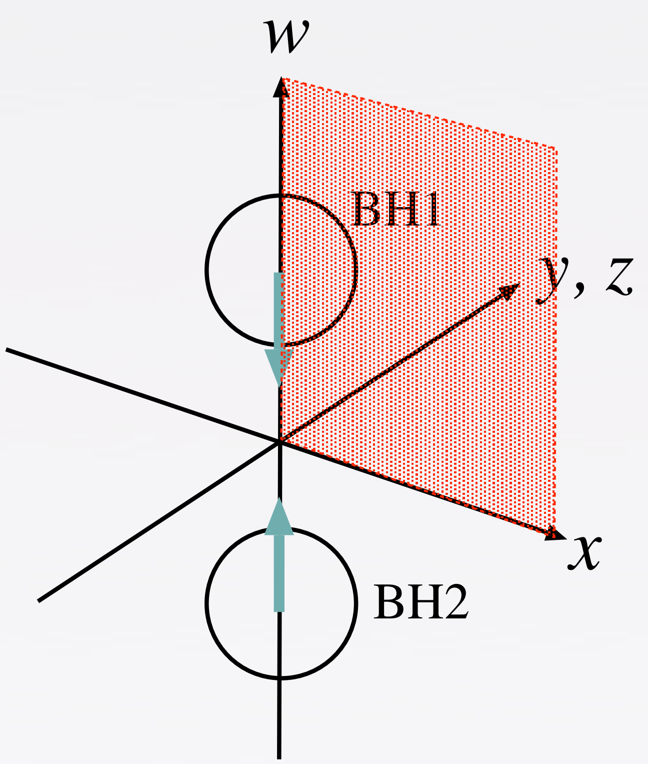
- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)



Scalar

$$\Psi(x, y, z, w) = \Psi(r, 0, 0, w)$$

Cartoon method (5D)



- Space is 4D (x, y, z, w)
  - U(1)×U(1) symmetry (x=y, z=w)
  - SO(3) symmetry (x=y=z, w)

Vector

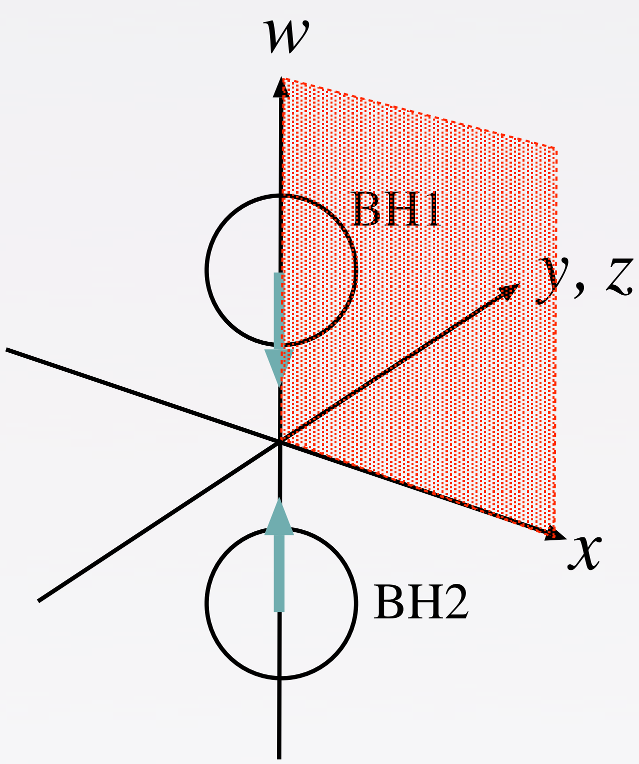
$$T^x(x,y,z,w) = (x/r)T^x(r,0,0,w)$$

$$T^y(x,y,z,w) = (y/r)T^x(r,0,0,w)$$

$$T^z(x,y,z,w) = (z/r)T^x(r,0,0,w)$$

$$T^w(x,y,z,w) = T^w(r,0,0,w)$$

Cartoon method (5D)



- **Space is 4D (x, y, z, w)**
  - **U(1)×U(1) symmetry (x=y, z=w)**
  - **SO(3) symmetry (x=y=z, w)**

2-rank symmetric tensor

$$S^{ww}(x,y,z,w) = S^{ww}(r,0,0,w)$$

$$S^{xw}(x,y,z,w) = (x/r)S^{xw}(r,0,0,w)$$
$$S^{yw}(x,y,z,w) = (y/r)S^{xw}(r,0,0,w)$$
$$S^{zw}(x,y,z,w) = (z/r)S^{xw}(r,0,0,w)$$

$$S^{xx}(x,y,z,w) = (x^2/r^2)S^{xx}(r,0,0,w) + \left(1 - x^2/r^2\right) S^{yy}(r,0,0,w)$$
$$S^{yy}(x,y,z,w) = (y^2/r^2)S^{xx}(r,0,0,w) + \left(1 - y^2/r^2\right) S^{yy}(r,0,0,w)$$
$$S^{zz}(x,y,z,w) = (z^2/r^2)S^{xx}(r,0,0,w) + \left(1 - z^2/r^2\right) S^{yy}(r,0,0,w)$$

$$S^{yz}(x,y,z,w) = (yz/r^2)[S^{xx} - S^{yy}](r,0,0,w)$$
$$S^{zx}(x,y,z,w) = (zx/r^2)[S^{xx} - S^{yy}](r,0,0,w)$$
$$S^{xy}(x,y,z,w) = (xy/r^2)[S^{xx} - S^{yy}](r,0,0,w)$$

Cartoon method

Cartoon method is a very effective method for simulating spacetimes with symmetries (using Cartesian coordinates).

- **Cartoon method (1)**
  - **4D**
  - **5D**
- **Cartoon method (2)**

Cartoon method

Cartoon method is a very effective method for simulating spacetimes with symmetries (using Cartesian coordinates).

- Cartoon method (1)
  - 4D
  - 5D
  - Once a code is generated, it can be adopted for various spacetimes with symmetries by just changing subroutines for cartoon.
  - But spacetime dimensionality is fixed.
- Cartoon method (2)

Cartoon method

Cartoon method is a very effective method for simulating spacetimes with symmetries (using Cartesian coordinates).

- Cartoon method (1)
  - 4D
  - 5D
  - Once a code is generated, it can be adopted for various spacetimes with symmetries by just changing subroutines for cartoon.
  - But spacetime dimensionality is fixed.
- Cartoon method (2)

Modified cartoon method

- **Consider D-dimensional spacetime with SO(D-3) symmetry**

  (x, y, $z = w_1 = \ldots = w_{D-3}$)

- **We only prepare grids of (x, y, z) plane.**

- **Derivatives with respect to w_i (e.g., of a scalar function) can be evaluated by**

$$
\begin{aligned}
\alpha_{,w_i} &= 0, \\
\alpha_{,xw_i} = \alpha_{,yw_i} = \alpha_{,zw_i} &= 0, \\
\alpha_{,w_iw_j} &= \frac{\alpha_{,z}}{z}\delta_{ij}
\end{aligned}
$$

Cartoon method

Cartoon method is a very effective method for simulating spacetimes with symmetries (using Cartesian coordinates).

- Cartoon method (1)
  - 4D
  - 5D
  - Once a code is generated, it can be adopted for various spacetimes with symmetries by just changing subroutines for cartoon.
  - But spacetime dimensionality is fixed.
- Cartoon method (2)
  - One code can be applied for various spacetime dimensionality.
  - The type of symmetry is fixed.

Contents

- Introduction
- BSSN formalism
- Cartoon method
- Codes
- (In)stability of a 5D MP black hole
- Summary

Our codes

- Yoshino's Codes
  - Codes for 5D spacetimes
  - Time direction: 4th-order Runge-Kutta
  - Space direction: 4th-order finite differencing (uniform grids)
  - Courant number: $C = \Delta t/\Delta x = 0.5$
  - Cartoon method (1)
    - SO(4) symmetry (x=y=z=w) [1D code]
    - U(1)×U(1) symmetry (x=y, z=w) [2D code]
    - SO(3) symmetry (x=y=z, w) [2D code]
    - U(1) symmetry (x=y=z, w) [3D code]

Our codes

- Shibata's codes
  - SACRA-5D
    - U(1) symmetry (x, y, z=w) [3D code]
    - Cartoon method (2)
    - Adoptive Mesh Refinement
  - SACRA-ND
    - SO(D−3) symmetry (x, y, $z=w_1=\ldots=w_{D-4}$) [3D code]
    - Cartoon method (2)
    - Adoptive Mesh Refinement

Code tests

- Linear gravitational waves in a flat spacetime
  - Comparison with analytic solutions
  - Energy extraction by the Landau-Lifshitz pseudo tensor

- Schwarzschild spacetime
  - Schwarzschild spacetime in geodesic slices
  - Long term evolutions in the 1+log slicing
  - Evolution starting from a limit surface

Code tests

- Linear gravitational waves in a flat spacetime
  - Comparison with analytic solutions
  - Energy extraction by the Landau-Lifshitz pseudo tensor
- Schwarzschild spacetime
  - Schwarzschild spacetime in geodesic slices
  - Long term evolutions in the 1+log slicing
  - Evolution starting from a limit surface

Linear gravitational wave in a flat spacetime

- **Linear gravitational waves in a flat spacetime**
  - **scalar mode**
  - **vector mode**
  - **tensor mode**

(c.f. Kodama and Ishibashi, 2003)

Linear gravitational wave in a flat spacetime

- Linear gravitational waves in a flat spacetime
  - scalar mode
  - vector mode
  - $\boxed{\text{tensor mode}}$

(c.f. Kodama and Ishibashi, 2003)

- U(1)×U(1)-symmetry (x=y, z=w)

Linear gravitational wave in a flat spacetime

- **Linear gravitational waves in a flat spacetime**
  - **scalar mode**
  - **vector mode**
  - **tensor mode**

(c.f. Kodama and Ishibashi, 2003)

- **U(1)×U(1)-symmetry (x=y, z=w)**
  - **hyper-spherical coordinate** (r, θ, ϕ, χ)

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \cos\chi \\ w &= r \cos\theta \sin\chi \end{aligned}$$

$$h_{ij} = H(t,r) r^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin^2\theta(1-3\sin^2\theta) & 0 \\ 0 & 0 & 0 & \cos^2\theta(3\sin^2\theta - 2) \end{pmatrix}$$

Linear gravitational wave in a flat spacetime

- **Linear gravitational waves in a flat spacetime**
  - **scalar mode**
  - **vector mode**
  - **tensor mode**

(c.f. Kodama and Ishibashi, 2003)

- **U(1)×U(1)-symmetry (x=y, z=w)**
  - **hyper-spherical coordinate** (r, θ, ϕ, χ)

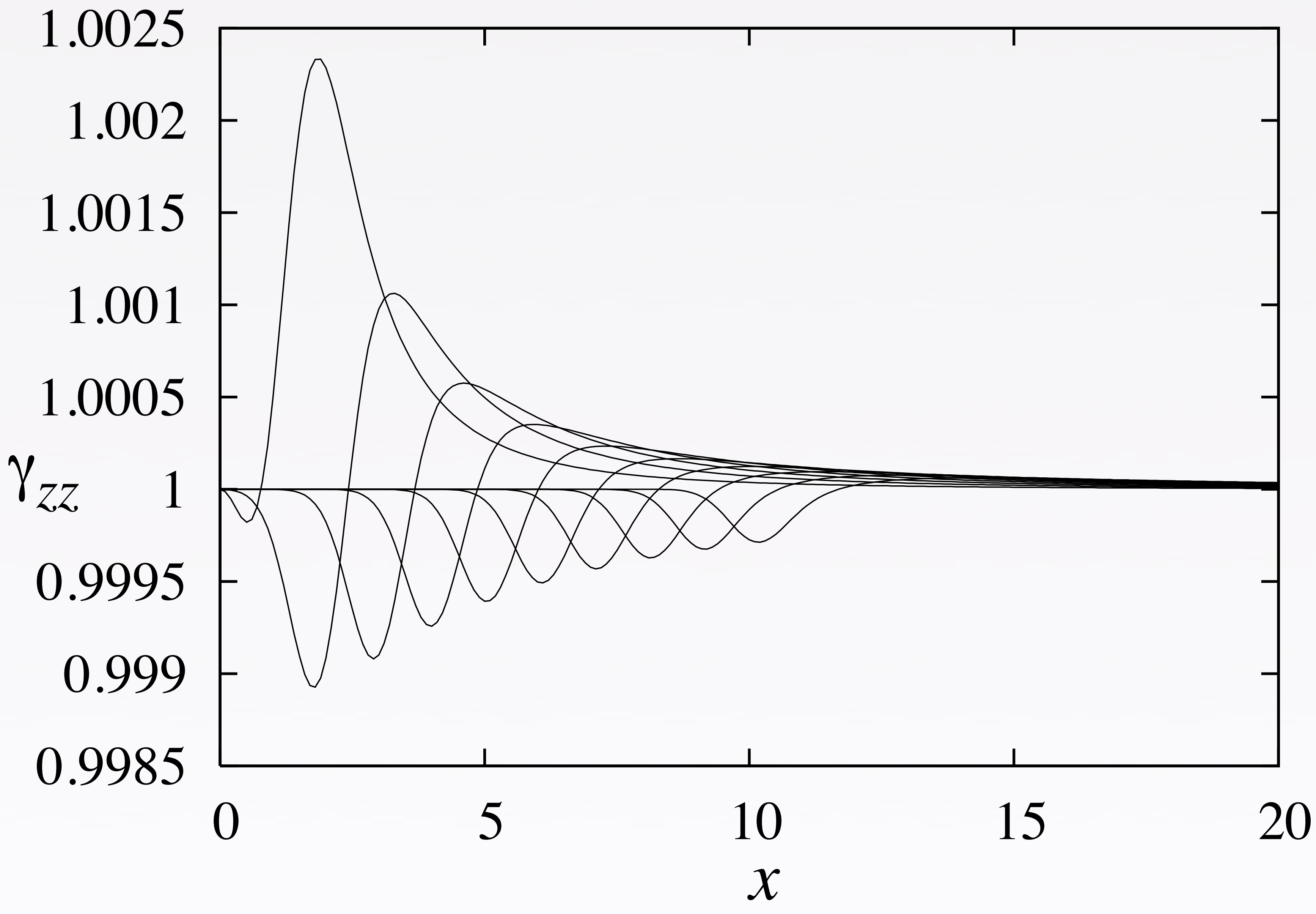
$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta\cos\chi \\ w &= r\cos\theta\sin\chi \end{aligned}$$

$$h_{ij} = H(t,r)r^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin^2\theta(1-3\sin^2\theta) & 0 \\ 0 & 0 & 0 & \cos^2\theta(3\sin^2\theta-2) \end{pmatrix}$$

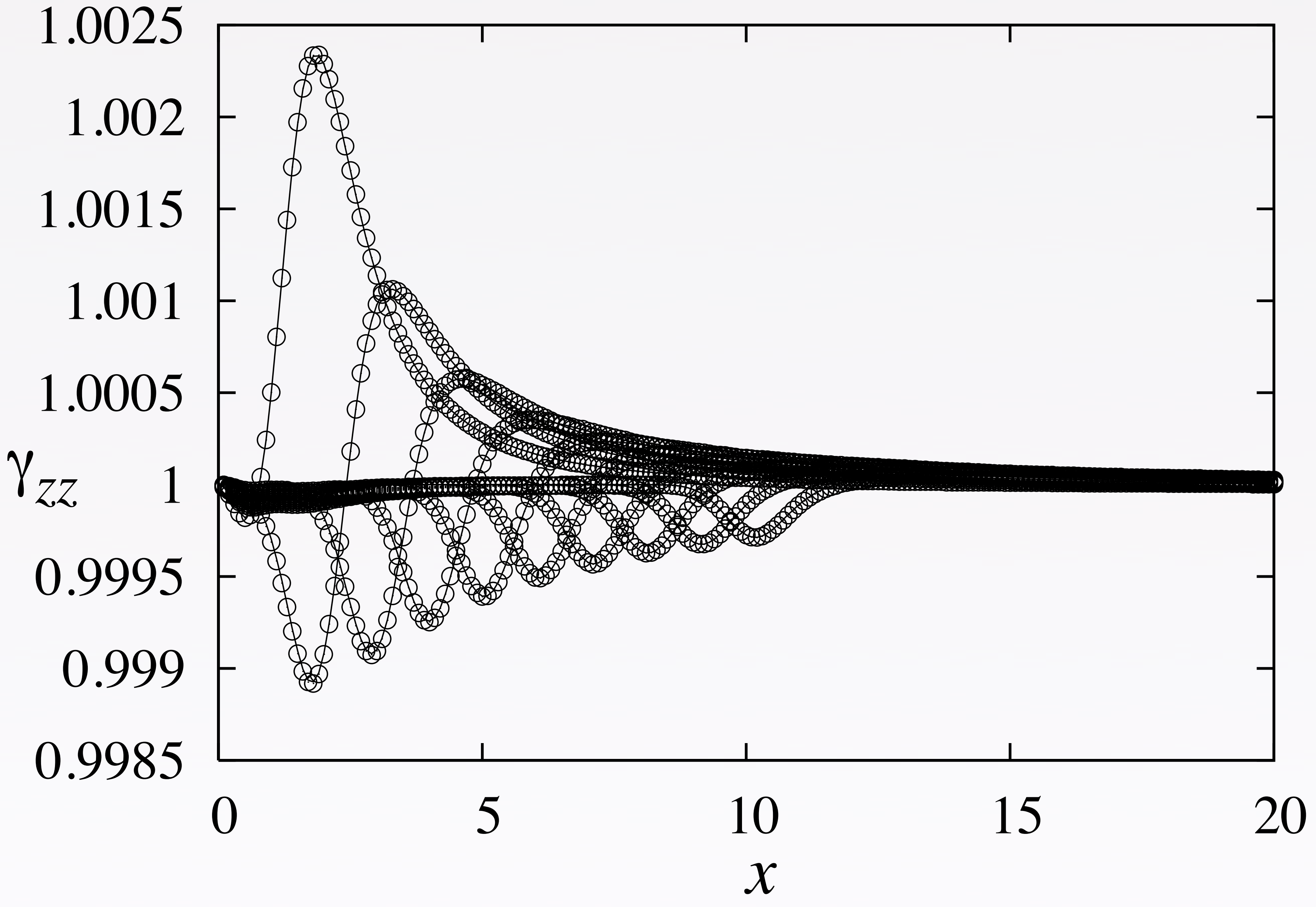
(special solution)

$$H(t,r) = A_0\omega_0 \int_0^{2\pi} d\theta \sin(3\theta) e^{-\omega_0^2(t-r\sin\theta)^2/2}$$

Linear gravitational wave in a flat spacetime



Linear gravitational wave in a flat spacetime



Code tests

- Linear gravitational waves in a flat spacetime
  - Comparison with analytic solutions
  - Energy extraction by the Landau-Lifshitz pseudo tensor
- Schwarzschild spacetime
  - Schwarzschild spacetime in geodesic slices
  - Long term evolutions in the 1+log slicing
  - Evolution starting from a limit surface

Energy extraction by the Landau-Lifshitz pseudo tensor

- **Super-potential** $H^{\mu\alpha\nu\beta} = \tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} - \tilde{g}^{\alpha\nu}\tilde{g}^{\mu\beta}$ $\quad \tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu},$
- **Landau-Lifshitz pseudo tensor** $16\pi G t^{\mu\nu}_{\rm LL} = (-g)^{-1} H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} - (2R^{\mu\nu} - g^{\mu\nu}R)\,.$

$$
\begin{aligned}
16\pi G(-g)t^{\mu\nu}_{\rm LL} &= \tilde{g}^{\mu\nu}{}_{,\alpha}\tilde{g}^{\alpha\beta}{}_{,\beta} - \tilde{g}^{\mu\alpha}{}_{,\alpha}\tilde{g}^{\nu\beta}{}_{,\beta} + \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\tilde{g}^{\alpha\rho}{}_{,\sigma}\tilde{g}^{\sigma\beta}{}_{,\rho} \\
&\quad - \left(g^{\mu\alpha}g_{\beta\rho}\tilde{g}^{\nu\rho}{}_{,\sigma}\tilde{g}^{\beta\sigma}{}_{,\alpha} + g^{\nu\alpha}g_{\beta\rho}\tilde{g}^{\mu\rho}{}_{,\sigma}\tilde{g}^{\beta\sigma}{}_{,\alpha}\right) + g_{\alpha\beta}g^{\rho\sigma}\tilde{g}^{\mu\alpha}{}_{,\rho}\tilde{g}^{\nu\beta}{}_{,\sigma} \\
&\quad + \frac{1}{4(D-2)}\left(2g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}\right)\left[(D-2)g_{\rho\sigma}g_{\gamma\delta} - g_{\sigma\gamma}g_{\rho\delta}\right]\tilde{g}^{\rho\delta}{}_{,\alpha}\tilde{g}^{\sigma\gamma}{}_{,\beta}
\end{aligned}
$$

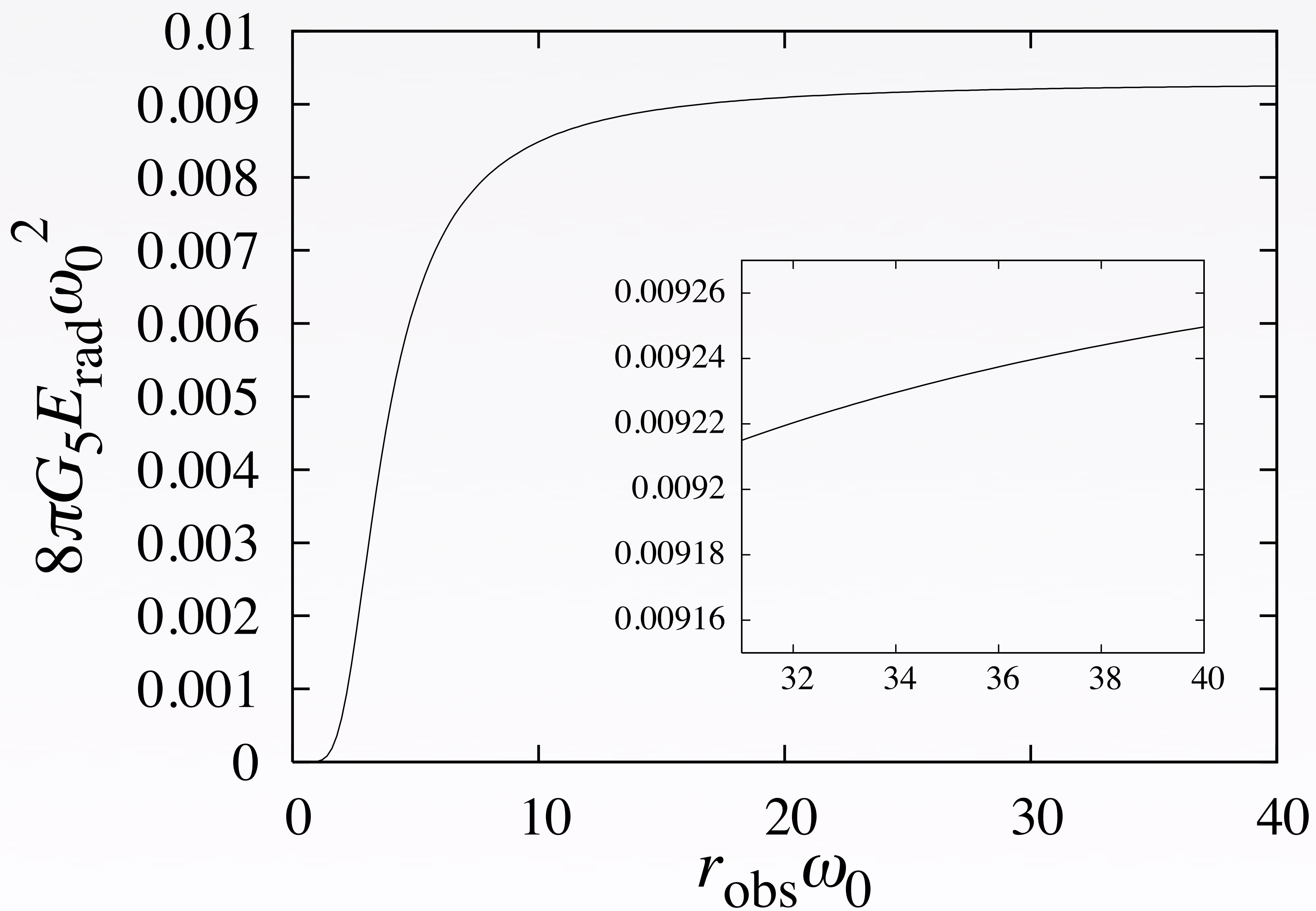
- **Conservation:** $\left[(-g)\left(T^{\mu\nu} + t^{\mu\nu}_{\rm LL}\right)\right]_{,\nu} = 0$
- **For a perturbation of a flat spacetime,** $\quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\quad \hat{h}_{\mu\nu} := h_{\mu\nu} - (1/2)h\eta_{\mu\nu}$

$$
\boxed{
\begin{aligned}
16\pi G t^{\mu\nu}_{\rm LL} &= \hat{h}^{\mu\nu}{}_{,\alpha}\hat{h}^{\alpha\beta}{}_{,\beta} - \hat{h}^{\mu\alpha}{}_{,\alpha}\hat{h}^{\nu\beta}{}_{,\beta} + \frac{1}{2}\eta^{\mu\nu}\hat{h}^{\alpha\rho}{}_{,\sigma}\hat{h}^{\sigma}{}_{\alpha,\rho} \\
&\quad - \left(\hat{h}^{\mu\rho}{}_{,\sigma}\hat{h}_{\rho}{}^{\sigma,\nu} + \hat{h}^{\nu\rho}{}_{,\sigma}\hat{h}_{\rho}{}^{\sigma,\mu}\right) + \hat{h}^{\mu\alpha,\rho}\hat{h}^{\nu}{}_{\alpha,\rho} \\
&\quad + \frac{1}{2}\hat{h}^{\rho\sigma,\mu}\hat{h}_{\rho\sigma}{}^{,\nu} - \frac{1}{4}\eta^{\mu\nu}\hat{h}^{\rho\sigma,\alpha}\hat{h}_{\rho\sigma,\alpha} - \frac{1}{4(D-2)}\left(2\hat{h}^{,\mu}\hat{h}^{,\nu} - \eta^{\mu\nu}\hat{h}^{,\alpha}\hat{h}_{,\alpha}\right).
\end{aligned}
}
$$

- **Total radiated energy:** $\boxed{E_{\rm rad} = \int t^{0i}\hat{n}_i dS dt}$

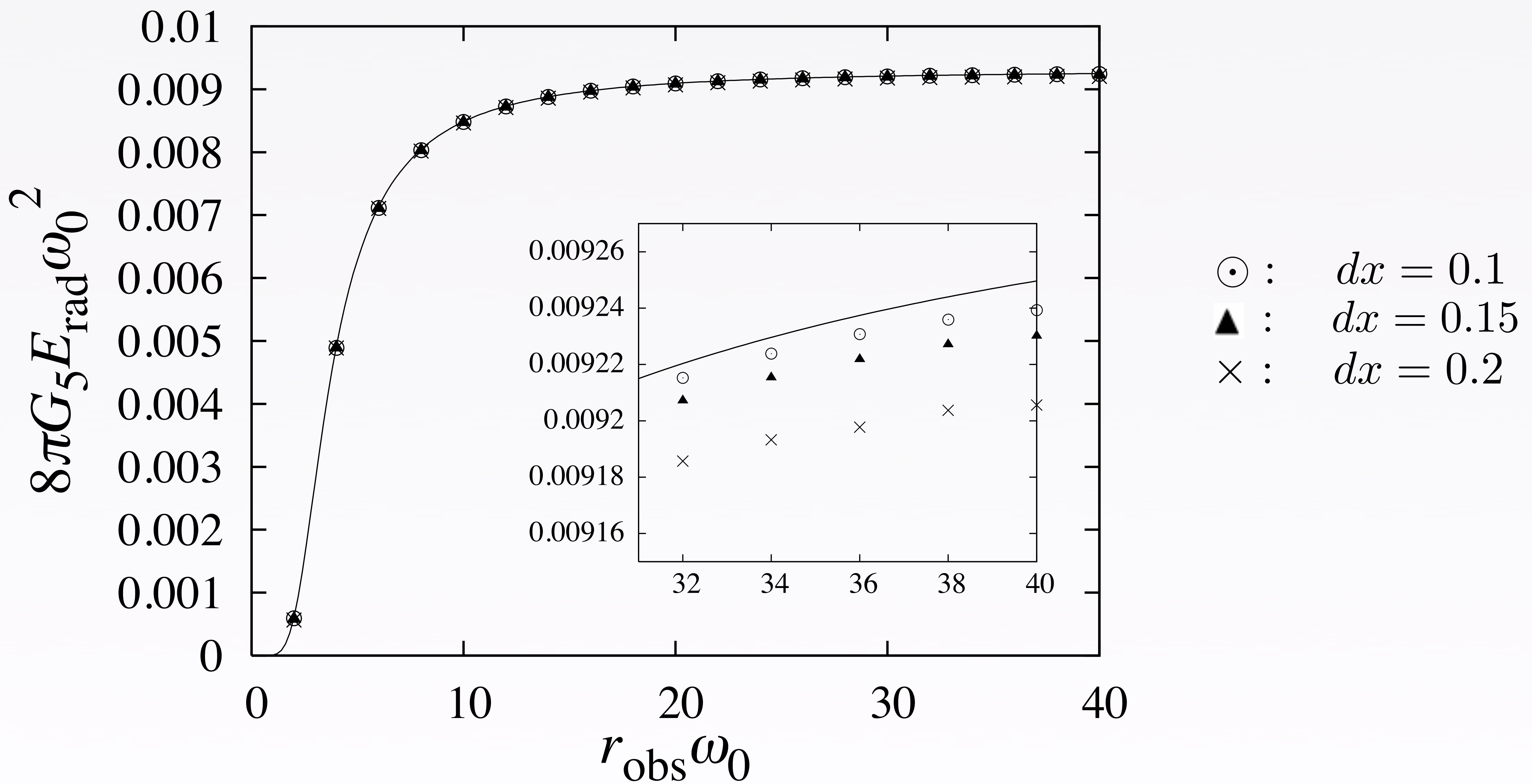
Energy extraction by the Landau-Lifshitz pseudo tensor

$$E_{\rm rad}(r_{\rm obs}) = \int t^{0i}_{\rm LL} n_i dS dt$$



Energy extraction by the Landau-Lifshitz pseudo tensor

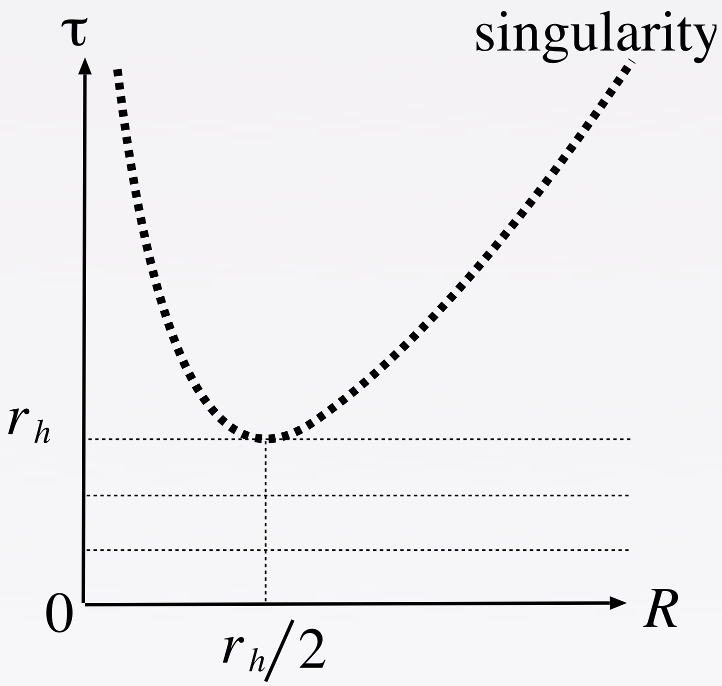
$$E_{\rm rad}(r_{\rm obs}) = \int t^{0i}_{\rm LL} n_i dS dt$$



Code tests

- Linear gravitational waves in a flat spacetime
  - Comparison with analytic solutions
  - Energy extraction by the Landau-Lifshitz pseudo tensor
- Schwarzschild spacetime
  - Schwarzschild spacetime in geodesic slices
  - Long term evolutions in the 1+log slicing
  - Evolution starting from a limit surface

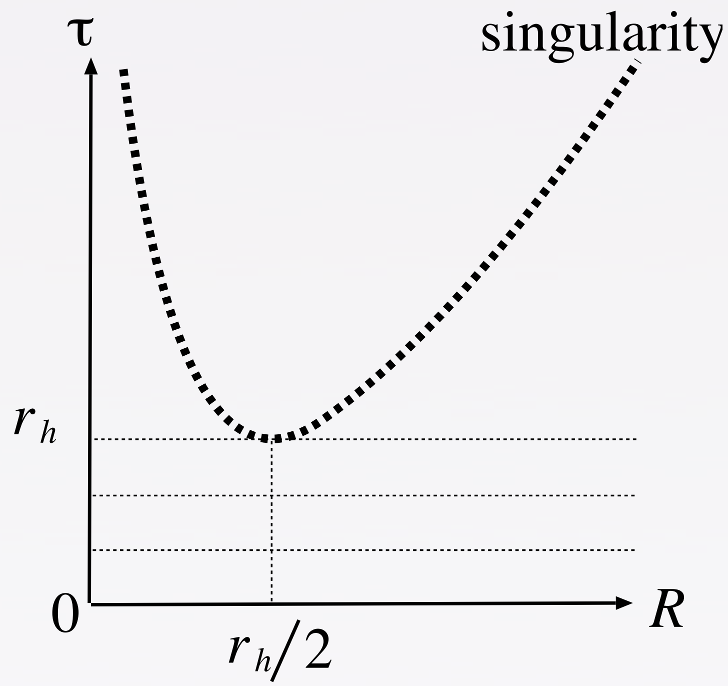
5D Schwarzschild spacetime in geodesic slices



$$ds^2 = -d\tau^2 + \frac{\left[r_0^2 + (r_h/r_0)^2\tau^2\right]^2}{\left[r_0^2 - (r_h/r_0)^2\tau^2\right]}\frac{dR^2}{R^2} + \left[r_0^2 - \left(\frac{r_h}{r_0}\right)^2\tau^2\right]d\Omega_3^2$$

$$r_0^2 = R^2\left(1 + \frac{r_h^2}{4R^2}\right)^2$$

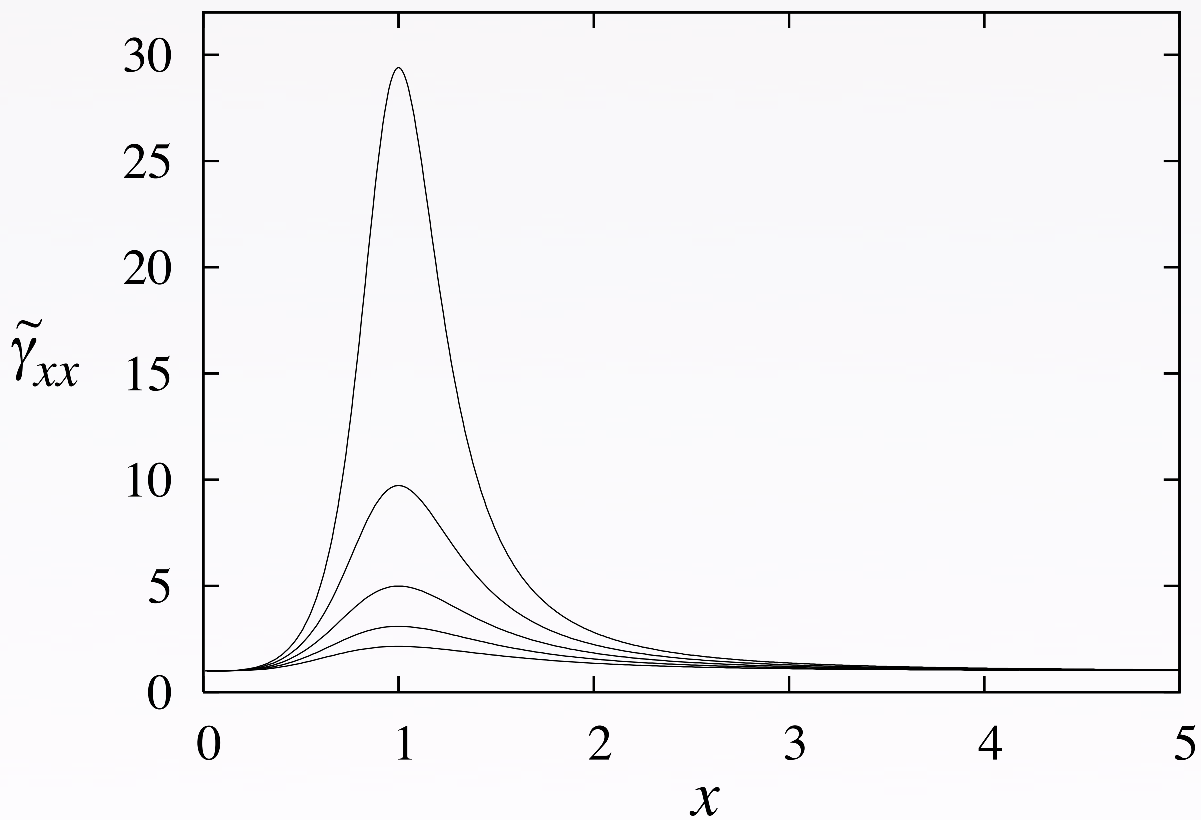
5D Schwarzschild spacetime in geodesic slices



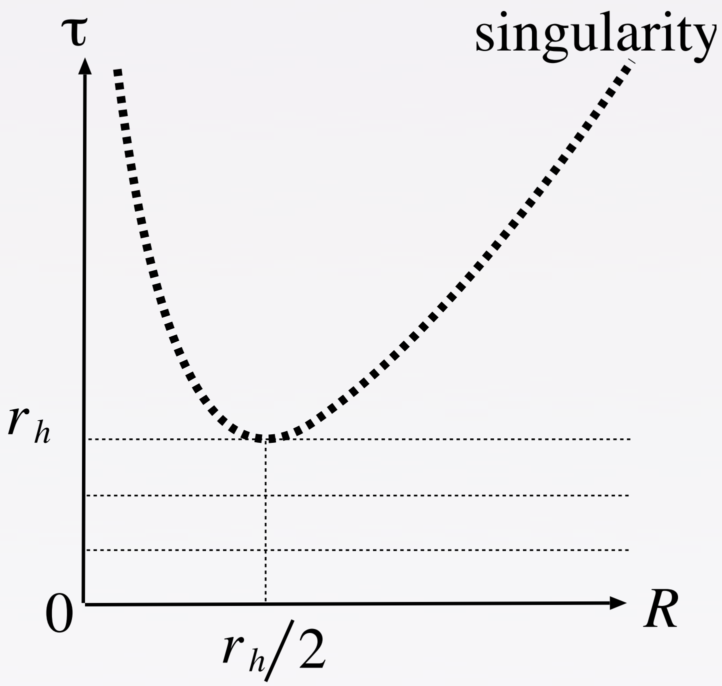
$$ds^2 = -d\tau^2 + \frac{\left[r_0^2 + (r_h/r_0)^2\tau^2\right]^2}{\left[r_0^2 - (r_h/r_0)^2\tau^2\right]}\frac{dR^2}{R^2} + \left[r_0^2 - \left(\frac{r_h}{r_0}\right)^2\tau^2\right]d\Omega_3^2$$

$$r_0^2 = R^2\left(1 + \frac{r_h^2}{4R^2}\right)^2$$

- **snapshots for** $\tau/r_h = 0.5, ..., 0.9$



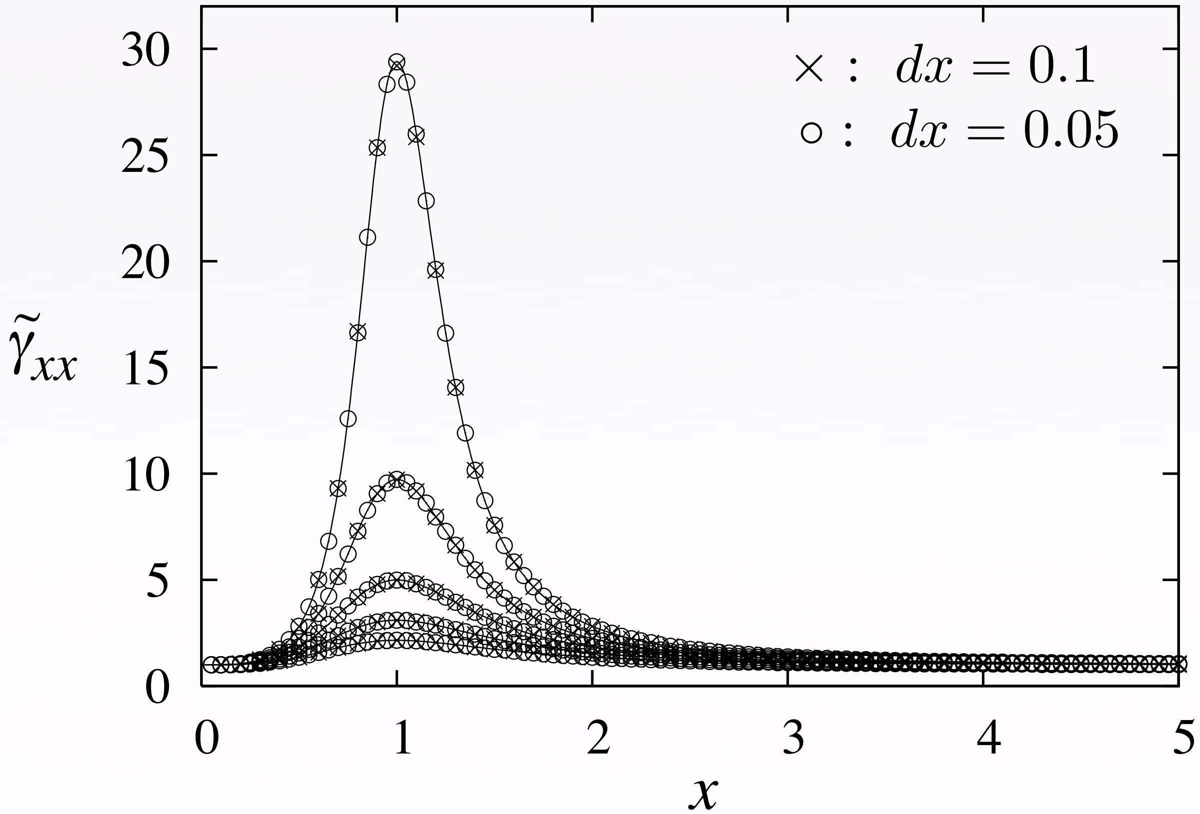
5D Schwarzschild spacetime in geodesic slices



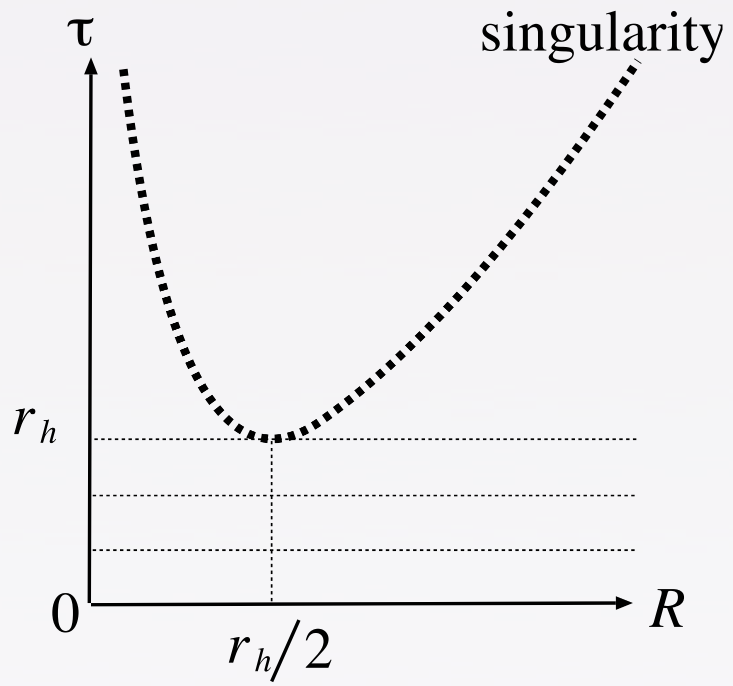
$$ds^2 = -d\tau^2 + \frac{\left[r_0^2 + (r_h/r_0)^2\tau^2\right]^2}{\left[r_0^2 - (r_h/r_0)^2\tau^2\right]}\frac{dR^2}{R^2} + \left[r_0^2 - \left(\frac{r_h}{r_0}\right)^2\tau^2\right] d\Omega_3^2$$

$$r_0^2 = R^2\left(1 + \frac{r_h^2}{4R^2}\right)^2$$

- **snapshots for** $\tau/r_h = 0.5, ..., 0.9$



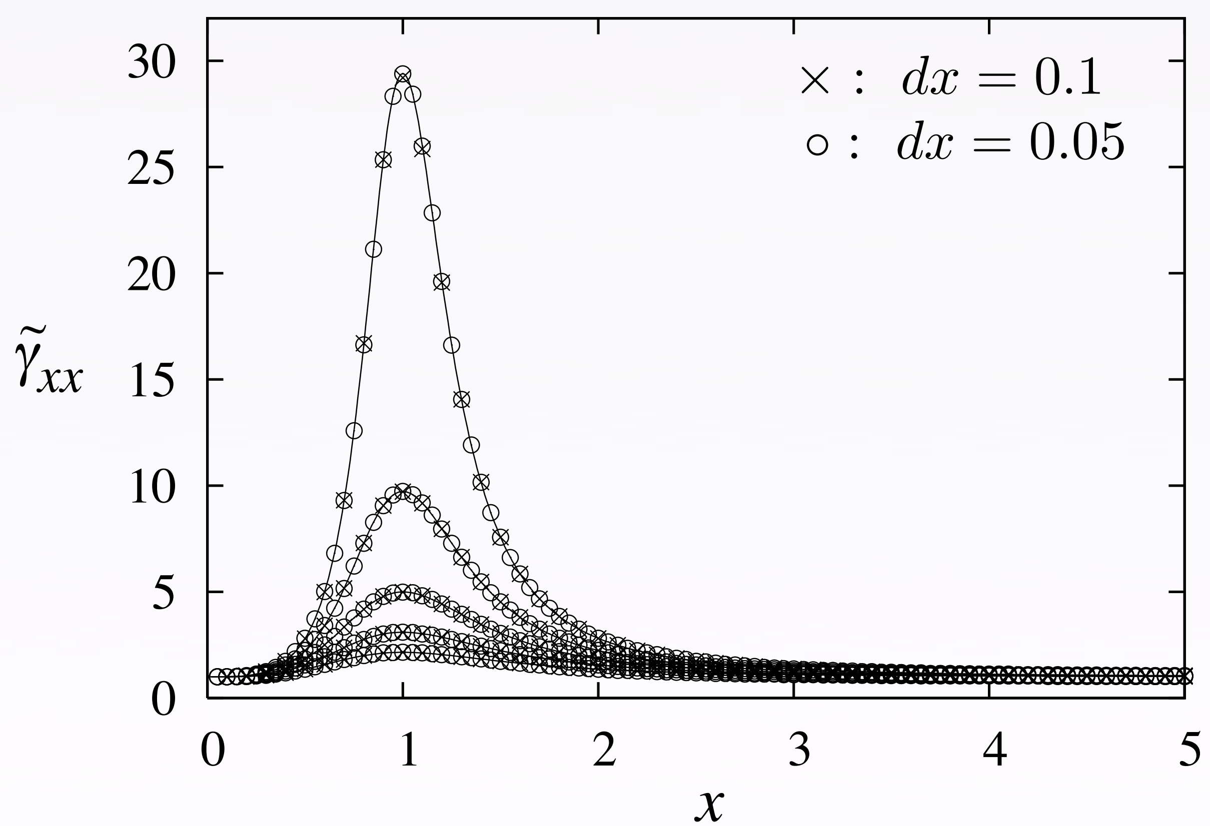
5D Schwarzschild spacetime in geodesic slices

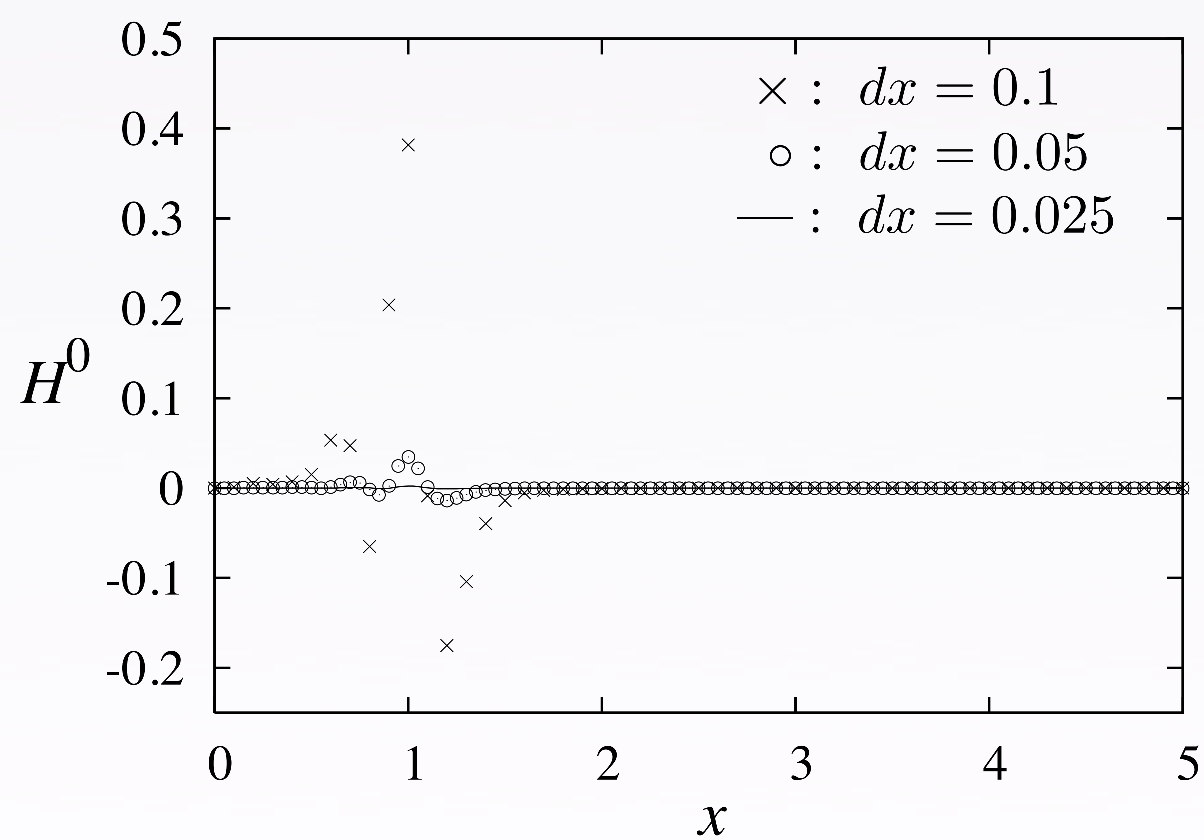


$$ds^2 = -d\tau^2 + \frac{\left[r_0^2 + (r_h/r_0)^2\tau^2\right]^2}{\left[r_0^2 - (r_h/r_0)^2\tau^2\right]}\frac{dR^2}{R^2} + \left[r_0^2 - \left(\frac{r_h}{r_0}\right)^2\tau^2\right]d\Omega_3^2$$

$$r_0^2 = R^2\left(1 + \frac{r_h^2}{4R^2}\right)^2$$

- **snapshots for** $\tau/r_h = 0.5, ..., 0.9$
- **constraint violation at** $\tau = 0.9r_h$





Code tests

- Linear gravitational waves in a flat spacetime
  - Comparison with analytic solutions
  - Energy extraction by the Landau-Lifshitz pseudo tensor
- Schwarzschild spacetime
  - Schwarzschild spacetime in geodesic slices
  - Long term evolutions in the 1+log slicing
  - Evolution starting from a limit surface

Evolution starting from the limit surface

Limit surface

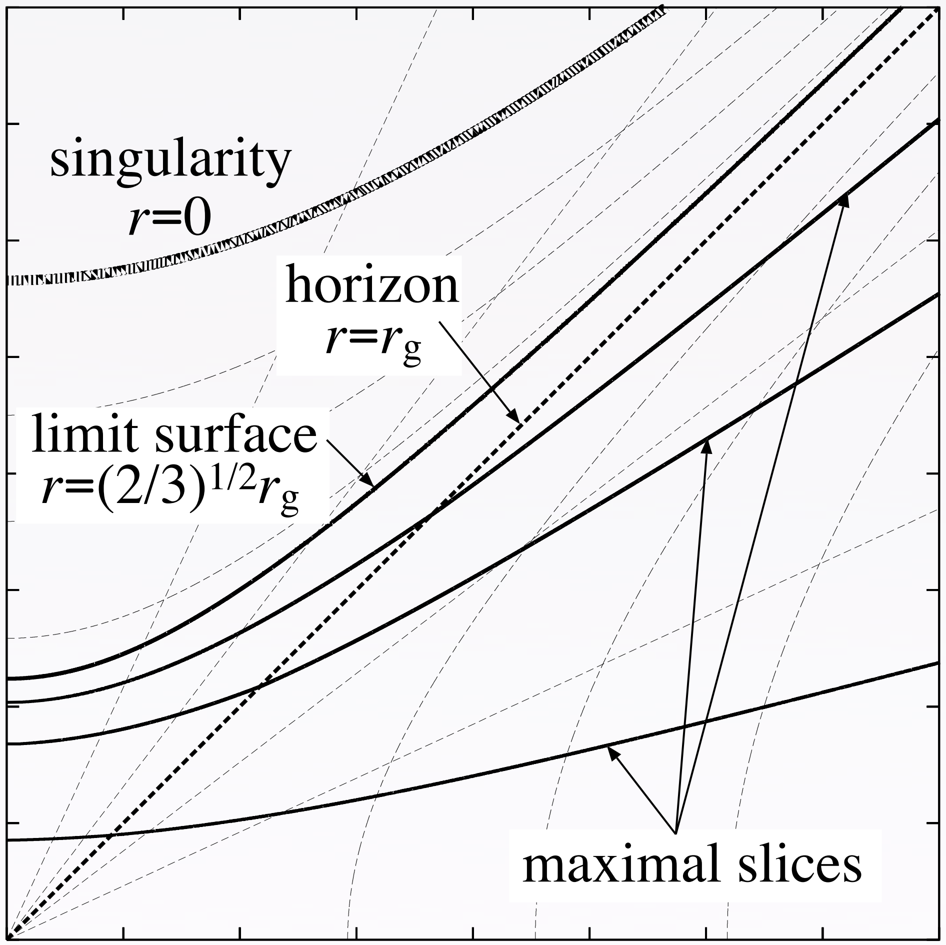
Final state of temporal evolution keeping $K = 0$.

- **(4D)**

Estabrook, Wahlquist, Christensen, DeWitt, Smarr and Tsiang (1973).
Baumgarte and Naculich, PRD75, 067502 (2007).

- **(higher dimensions)**

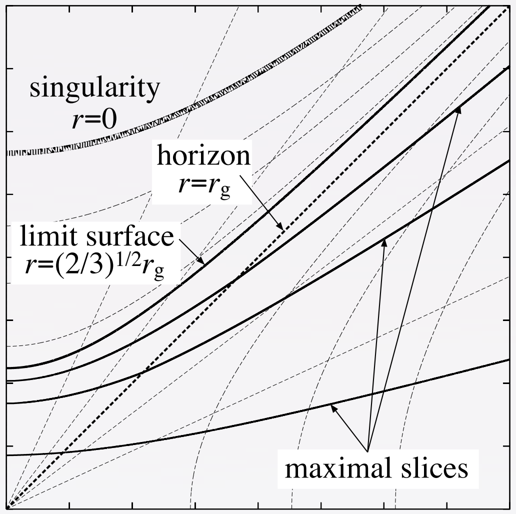
Nakao, Abe, Yoshino and Shibata,
PRD80, 084028 (2009) [arXiv:0908.0799 [gr-qc]]



- **Numerical evolution of a limit surface should be unchanged under:**
  - **1+log slicing condition** $\partial_t \alpha = -2\alpha K$
  - **Γ-driver condition** $\partial_t \beta^i = \dfrac{2}{3} B^i, \quad \partial_t B^i = \partial_t \tilde{\Gamma}^i - \eta B^i$

Evolution starting from the limit surface

- **BSSN variables of limit surfaces**



$$K = 0, \quad \tilde{\gamma}_{ij} = \delta_{ij}, \quad \tilde{\Gamma}^i = 0,$$

$$R = \frac{r}{6}\left(3 + \sqrt{3\left[(r_{\rm g}/r)^2 + 3\right]}\right)\left(\frac{(5+2\sqrt{6})\left[3 - 2(r_{\rm g}/r)^2\right]}{2(r_{\rm g}/r)^2 + 15 + 6\sqrt{2\left[(r_{\rm g}/r)^2 + 3\right]}}\right)^{1/\sqrt{6}}$$

$$\alpha = \sqrt{1 - \left(\frac{r_{\rm g}}{r}\right)^2 + \frac{4}{27}\left(\frac{r_{\rm g}}{r}\right)^6}$$

$$\beta^R = \frac{2}{3\sqrt{3}}\frac{r_{\rm g}^3 R}{r^4}$$

$$\tilde{A}_{RR} = -\frac{2}{\sqrt{3}}\frac{r_{\rm g}^3}{r^4}$$

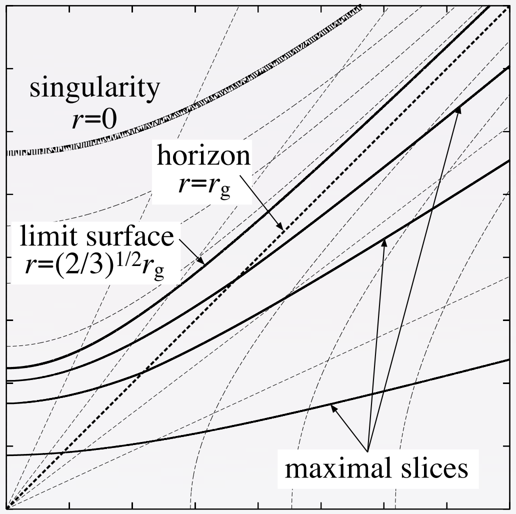
$$\chi = (R/r)^2$$

Evolution starting from the limit surface

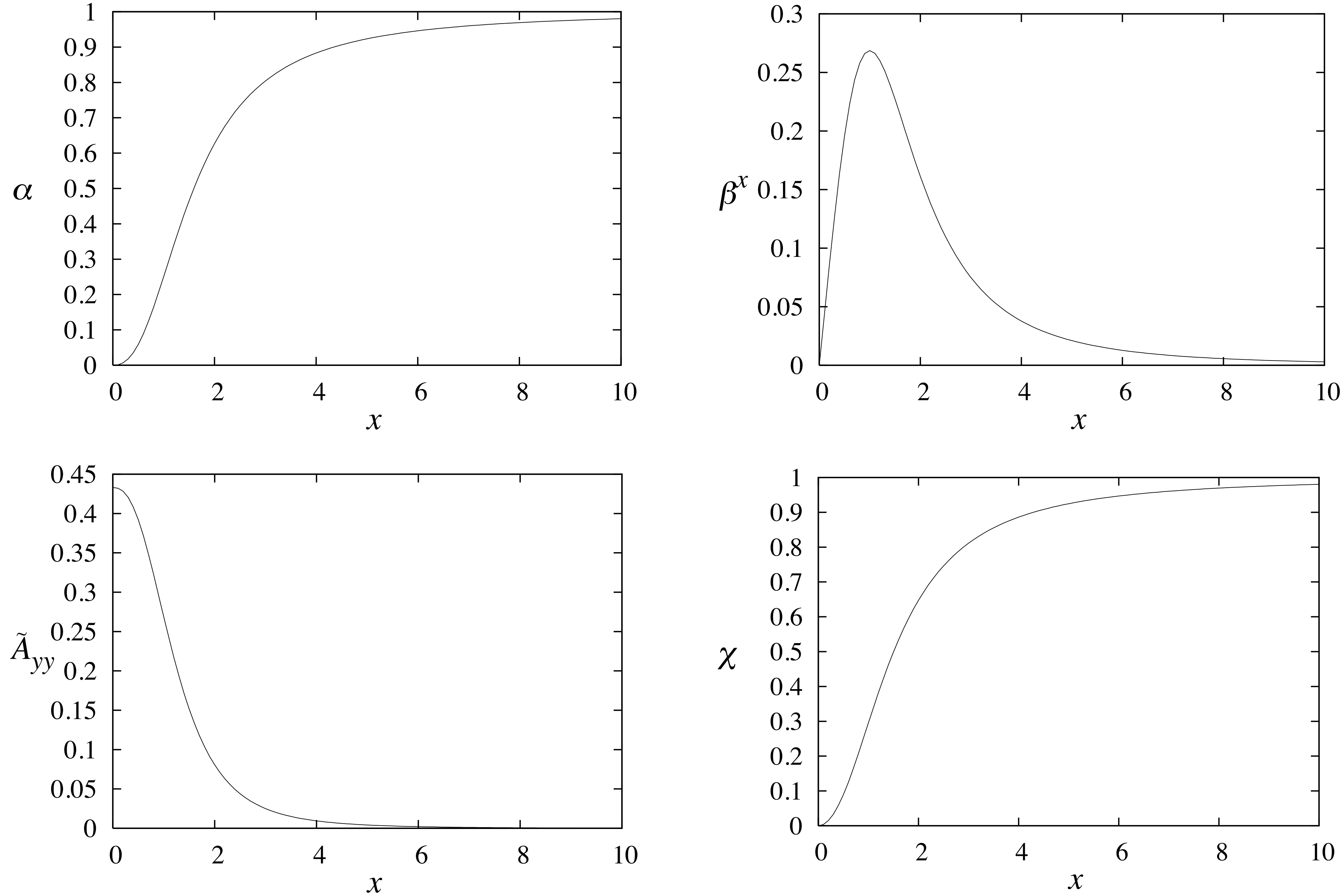
- **BSSN variables of limit surfaces**

$$K = 0, \quad \tilde{\gamma}_{ij} = \delta_{ij}, \quad \tilde{\Gamma}^i = 0,$$

$$R = \frac{r}{6}\left(3 + \sqrt{3\left[(r_{\mathrm{g}}/r)^2 + 3\right]}\right)\left(\frac{(5+2\sqrt{6})\left[3 - 2(r_{\mathrm{g}}/r)^2\right]}{2(r_{\mathrm{g}}/r)^2 + 15 + 6\sqrt{2\left[(r_{\mathrm{g}}/r)^2 + 3\right]}}\right)^{1/\sqrt{6}}$$

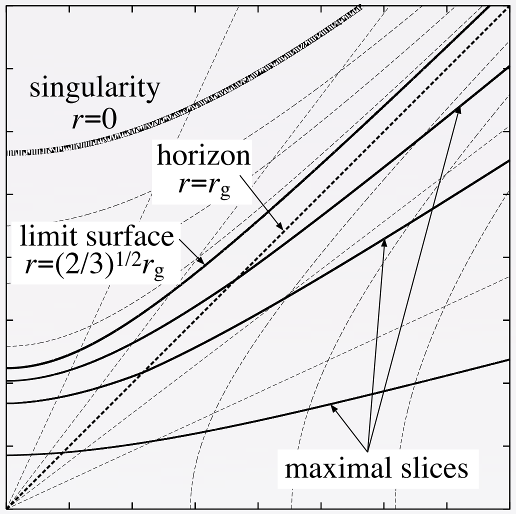


$t = 0$



Evolution starting from the limit surface

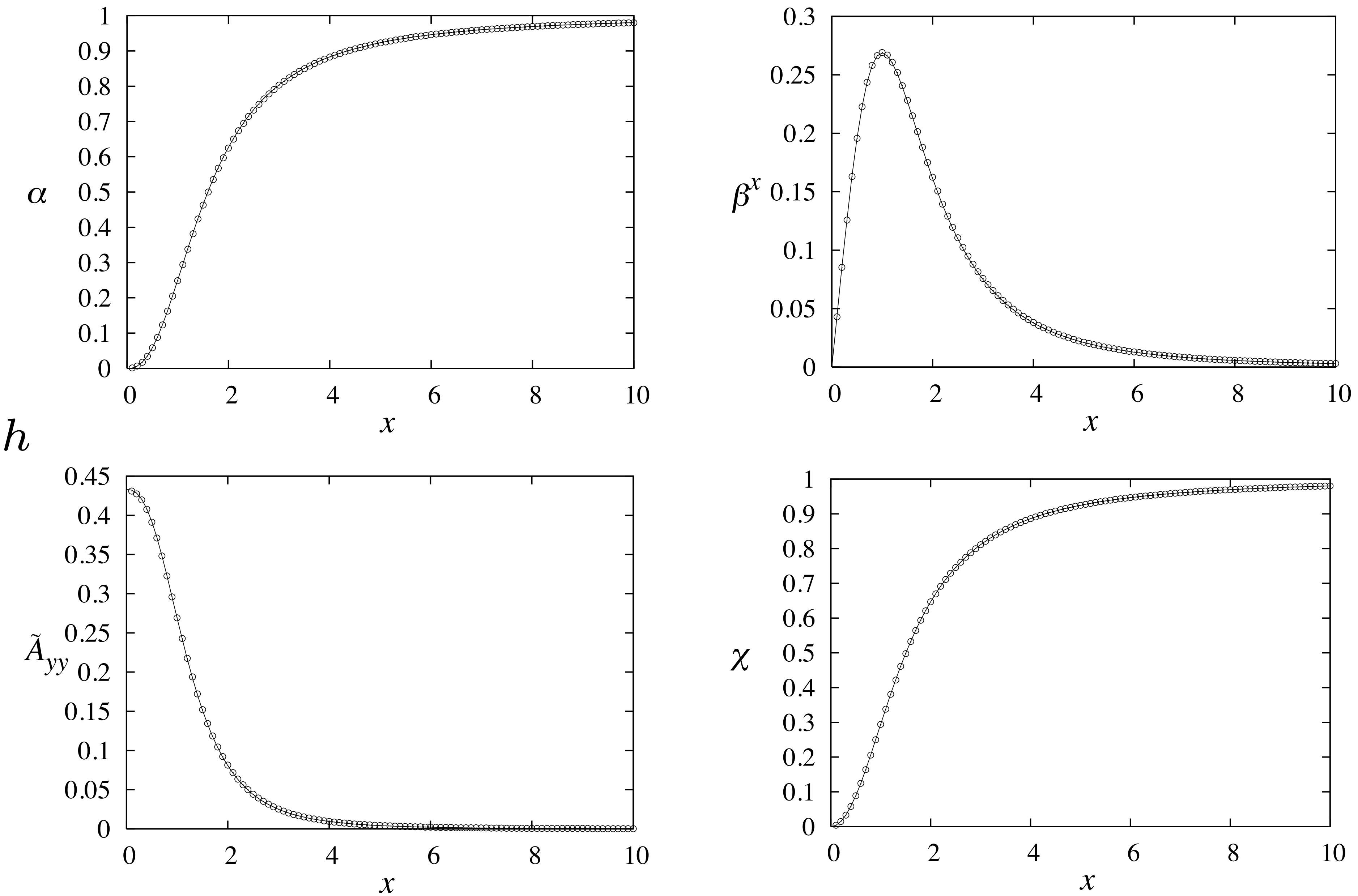
- **BSSN variables of limit surfaces**



$$K = 0, \quad \tilde{\gamma}_{ij} = \delta_{ij}, \quad \tilde{\Gamma}^i = 0,$$

$$R = \frac{r}{6}\left(3 + \sqrt{3\left[(r_{\rm g}/r)^2 + 3\right]}\right)\left(\frac{(5+2\sqrt{6})\left[3 - 2(r_{\rm g}/r)^2\right]}{2(r_{\rm g}/r)^2 + 15 + 6\sqrt{2\left[(r_{\rm g}/r)^2 + 3\right]}}\right)^{1/\sqrt{6}}$$

$$t = 50 r_h$$



Contents

(In)stability of MP BHs

- **Separation of variables of fields around a high-D MP spacetime has been done for scalar fields and Dirac fields, but only partially done for metric perturbations.**

  Ida, Uchida and Morisawa, PRD67, 084019 (2003) [arXiv:hep-th/0212108].
  e.g., Alberta separatists

  Ohta and Yasui, arXiv:0812.1623 [hep-th].

  Murata and Soda, PTP 120, 561 (2008) [arXiv:0803.1371 [hep-th]].

- **There is a discussion that strongly indicates the instability of rapidly rotating BHs.**

  Emparan and Myers, JHEP 0309, 025 (2003).

- **Recent numerical analysis of perturbation proved that the rapidly rotating BHs are unstable at least for $D \geq 7$.**

  Dias, Figueras, Monterio, Santos and Emparan, arXiv:0907.2248 [hep-th].

Prediction by Emparan and Myers

Emparan and Myers, JHEP 0309, 025 (2003).

- **Rapidly rotating BHs become like membranes**

  ⇒ **Gregory-Laflamme instability happens.** $(D \geq 6)$

- **Thermodynamical argument**

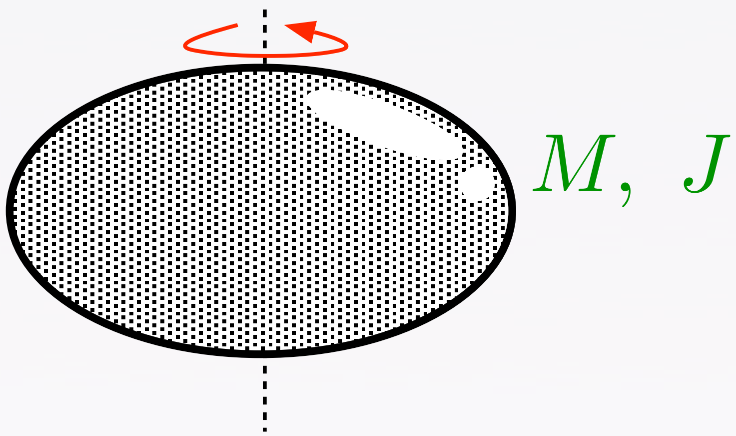
Prediction by Emparan and Myers

Emparan and Myers, JHEP 0309, 025 (2003).

- **Rapidly rotating BHs become like membranes**

  Gregory-Laflamme instability happens. $(D \geq 6)$

- **Thermodynamical argument**



$$A_0 = \Omega_{D-2} \hat{r}_k(M,J)^{D-4} \left[\hat{r}_k(M,J)^2 + a^2 \right]$$

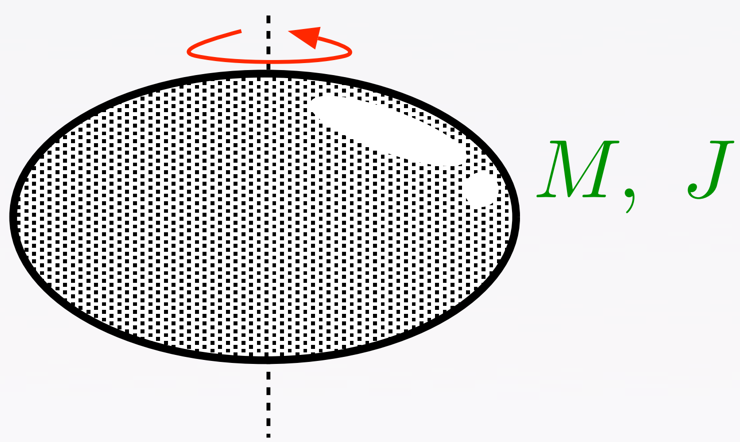
Prediction by Emparan and Myers

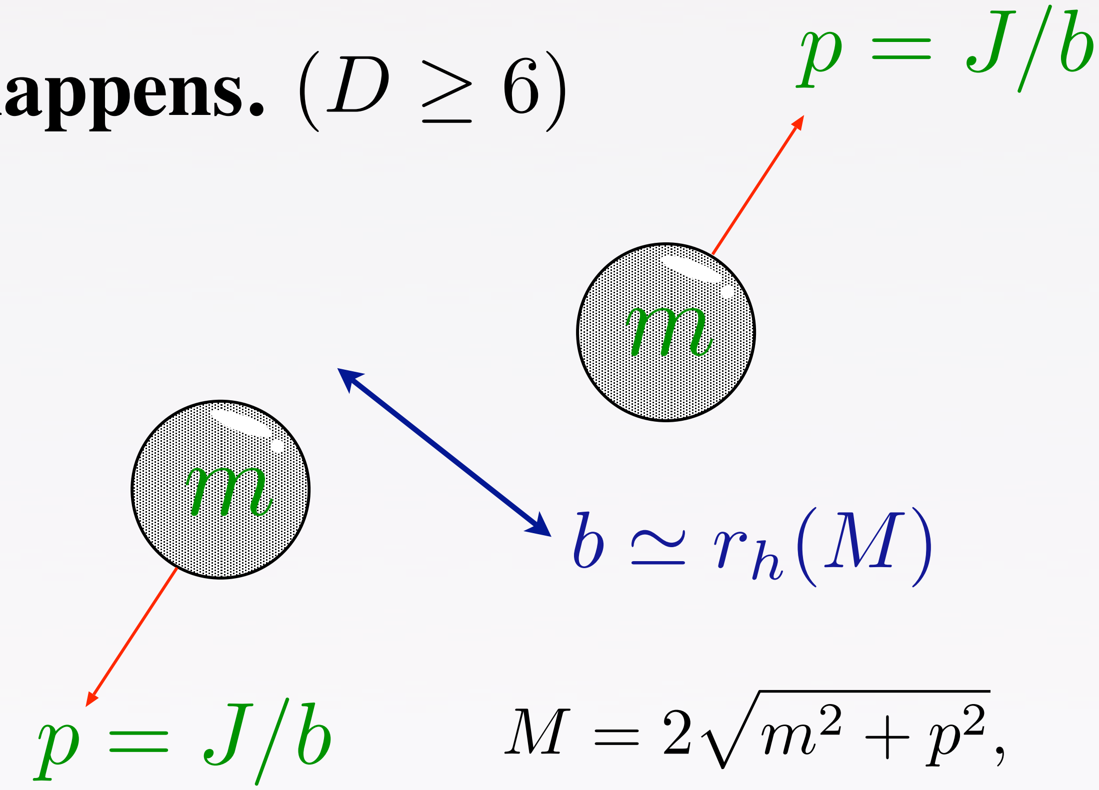
Emparan and Myers, JHEP 0309, 025 (2003).

- **Rapidly rotating BHs become like membranes**

  ⇨ **Gregory-Laflamme instability happens.** $(D \geq 6)$

- **Thermodynamical argument**





$$M = 2\sqrt{m^2 + p^2},$$

$$A_0 = \Omega_{D-2}\hat{r}_k(M,J)^{D-4}\left[\hat{r}_k(M,J)^2 + a^2\right]$$

$$A_1 = 2\Omega_{D-2} r_s(m)^{D-2}$$

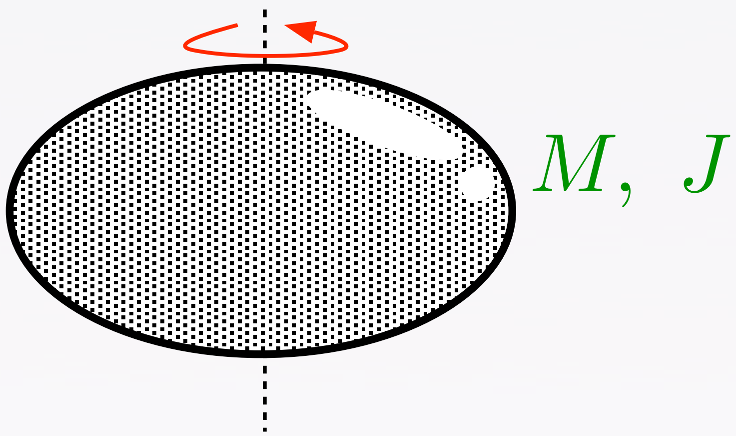
Prediction by Emparan and Myers

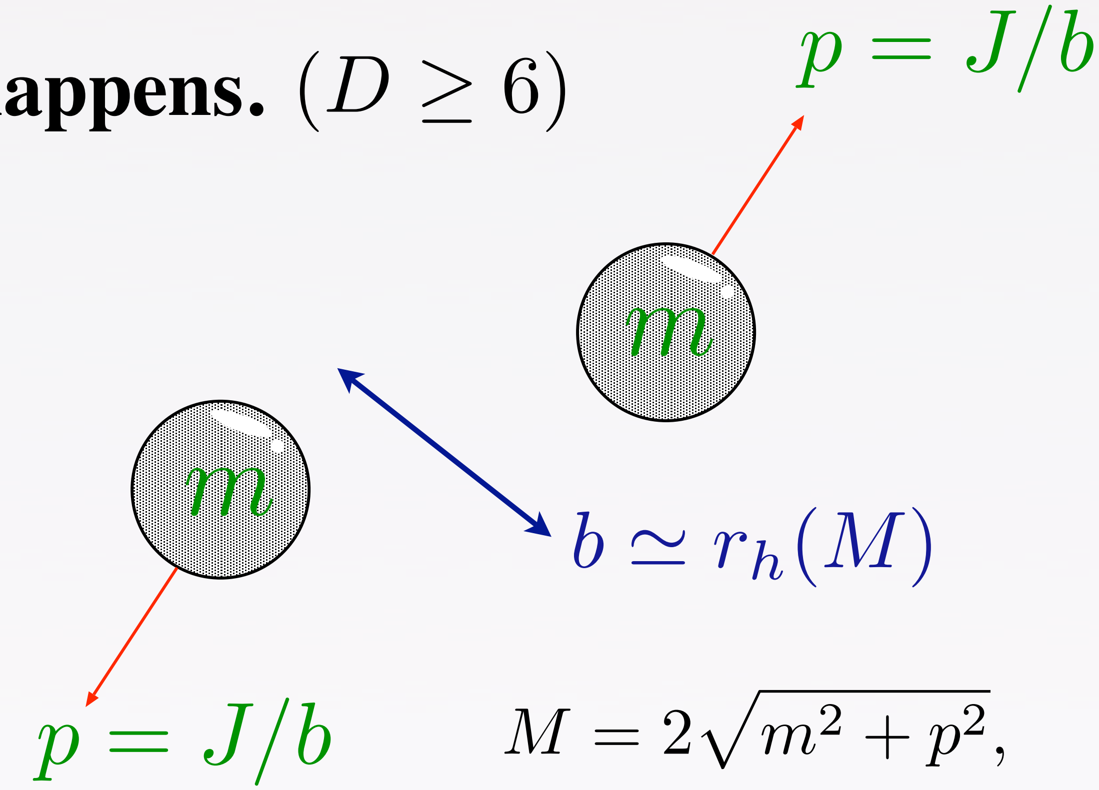
Emparan and Myers, JHEP 0309, 025 (2003).

- **Rapidly rotating BHs become like membranes**

  ➡ **Gregory-Laflamme instability happens.** $(D \geq 6)$

- **Thermodynamical argument**





$$M = 2\sqrt{m^2 + p^2},$$

$$A_0 = \Omega_{D-2}\hat{r}_k(M,J)^{D-4}\left[\hat{r}_k(M,J)^2 + a^2\right] < A_1 = 2\Omega_{D-2} r_s(m)^{D-2}$$

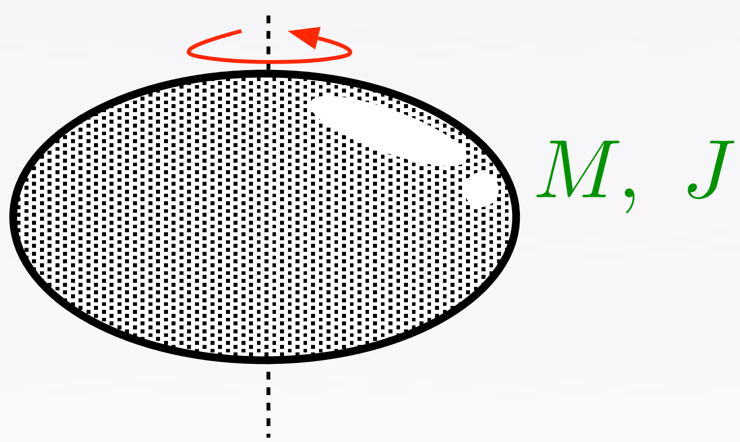
Prediction by Emparan and Myers

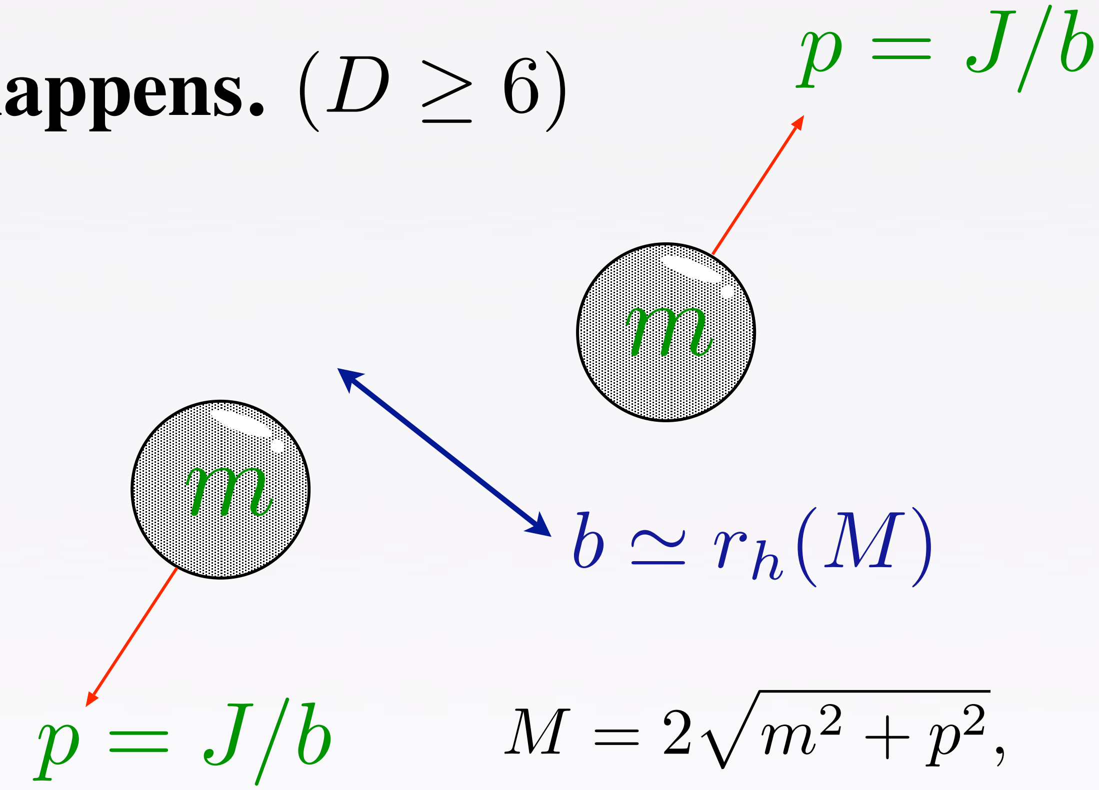
Emparan and Myers, JHEP 0309, 025 (2003).

- **Rapidly rotating BHs become like membranes**

  Gregory-Laflamme instability happens. $(D \geq 6)$

- **Thermodynamical argument**





$$M = 2\sqrt{m^2 + p^2},$$

$$A_0 = \Omega_{D-2}\hat{r}_k(M,J)^{D-4}\left[\hat{r}_k(M,J)^2 + a^2\right] < A_1 = 2\Omega_{D-2}r_s(m)^{D-2}$$

$$\Downarrow$$

5D MP BHs may be unstable for $a/\sqrt{\mu} \gtrsim 0.85$

Our problem

We simulate the sistem of a 5D Myers-Perry black hole spacetime with one rotational parameter adding nonaxisymmetric perturbation by our code.

Then, we derive the condition for the (in)stability of that black hole.

- **metric**

$$ds^2 = -dt^2 + \frac{\mu}{\Sigma}(dt - a\sin^2\theta d\varphi)^2 + \frac{\Sigma}{\Delta}d\hat{r}^2 + \Sigma d\theta^2$$
$$+(\hat{r}^2 + a^2)\sin^2\theta d\varphi^2 + \hat{r}^2\cos^2\theta d\chi^2$$

$$\Sigma = \hat{r}^2 + a^2\cos^2\theta$$
$$\Delta = \hat{r}^2 + a^2 - \mu$$

- **horizon** $\hat{r}_k = \sqrt{\mu - a^2}$

$$\begin{cases} a \le \sqrt{\mu}: & \text{black hole} \\ a > \sqrt{\mu}: & \text{no horizon} \end{cases}$$

Our problem

- **initial data (of the background)** $\hat{r} = \Phi r \qquad \Phi = 1 + \frac{\hat{r}_k^2}{4r^2}$

$$d\gamma^2 = \Phi^2 \left[\frac{\Sigma}{\hat{r}^2}(dr^2 + r^2 d\theta^2) + \frac{\rho}{\hat{r}^2\Sigma} \sin^2\theta d\varphi^2 + r^2 \cos^2\theta d\chi^2 \right]$$

$$K_{r\varphi} = \frac{\mu a \hat{r}}{r\rho^{1/2}\Sigma^{3/2}}(2\Sigma + a^2 \sin^2\theta) \sin^2\theta$$

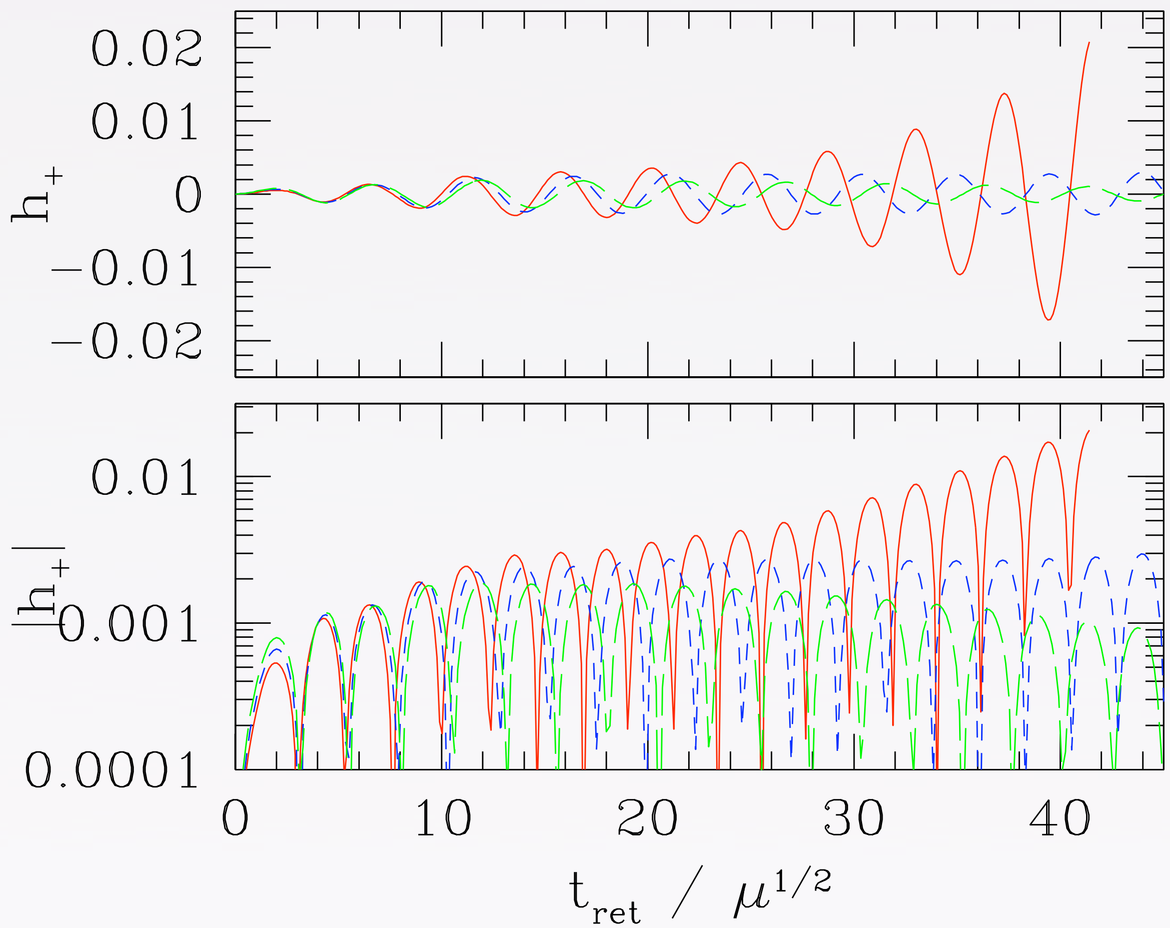
$$K_{\theta\varphi} = -\frac{\mu a^3}{\rho^{1/2}\Sigma^{3/2}} \sin^3\theta \cos\theta \left(r - \frac{\hat{r}_k^2}{4r} \right)$$

- **adding perturbation**

$$\chi = \chi_0 \left[1 + A\mu^{-1}(x^2 - y^2) \exp(-r^2/2\hat{r}_k^2) \right]$$

Simulation

- **gravitational wave**

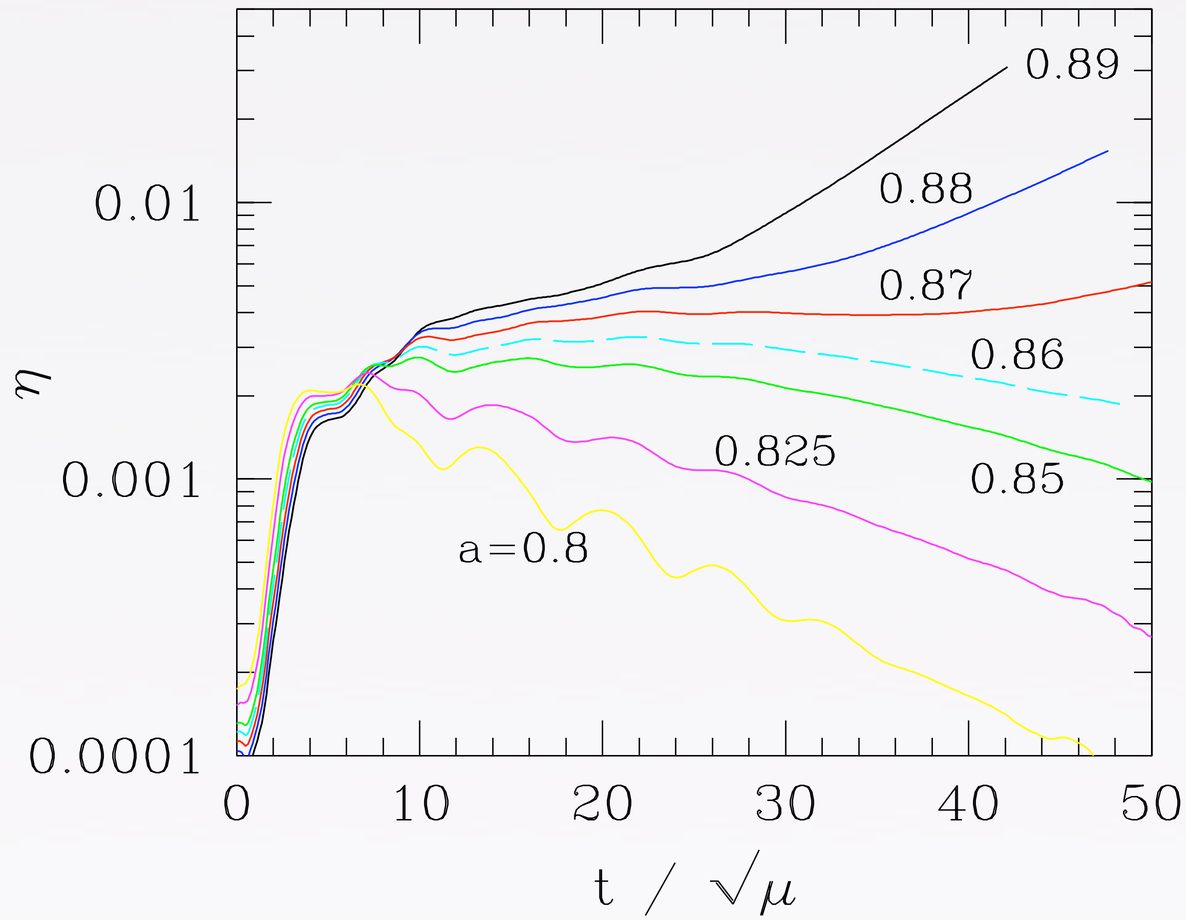


$a/\sqrt{\mu} = 0.85$

$a/\sqrt{\mu} = 0.87$

$a/\sqrt{\mu} = 0.89$

- **distortion of AH**

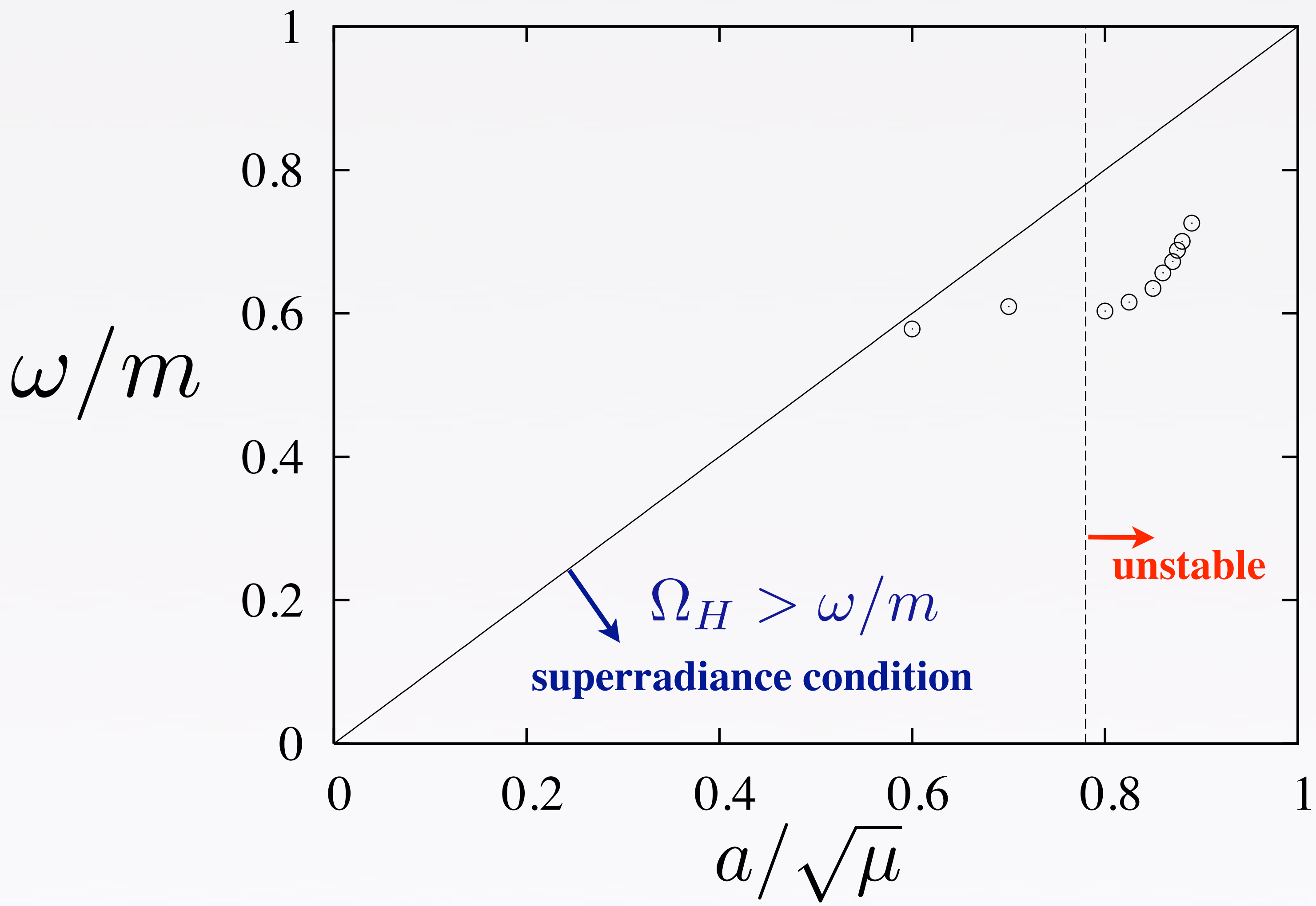


$$\eta = \frac{\sqrt{(l_0 - l_{\pi/2})^2 + (l_{\pi/4} - l_{3\pi/4})}}{l_0}$$

5D MP BHs are stable for $a/\sqrt{\mu} \lesssim 0.87$
5D MP BHs are unstable for $a/\sqrt{\mu} \gtrsim 0.87$

Simulation

- **(real part of) frequency of gravitational waves**



Gravitational waves are excited by QNM.

Wave frequency satisfies the condition for superradiance.

Angular momentum is extracted.

Open question

- Final fate of the instability?
  - settles down to a MP black hole with a smaller rotational parameter?
  - results in horizon pinch and naked singularity formation?

Contents

Summary

- We studied a formulation of higher-dimensional numerical relativity, and developed codes.
- These codes have passed several tests and thus have been validated.
- Our simulation indicates that a rapidly rotating 5D MP BH is unstable against nonaxisymmetric perturbation.

Ongoing work and next step

- **Studies on (in)stability of Myers-Perry black holes for dimensions D=6, 7, 8.**

  by Shibata and Yoshino.

- **Black hole scattering problem for D=5.**

  by Okawa, Yoshino and Shibata.

Next step

- **Final fate of instability of rapidly rotating BHs?**

- **Braneworld black holes?**

  by Tanahashi, Tanaka, ...

Appendix

Code tests

- Linear gravitational waves in a flat spacetime
    - Comparison with analytic solutions
    - Energy extraction by the Landau-Lifshitz pseudo tensor
- Schwarzschild spacetime
    - Schwarzschild spacetime in geodesic slices
    - Long term evolutions in the 1+log slicing
    - Evolution starting from a limit surface

Long-term evolution in the 1+log slicing

c.f. Hannam, Husa, Brugmann, Gonzalez, Sperhake and O'Murchadha, gr-qc/0612097 (4D case)

- **1+log slicing** $$\partial_t \alpha = -2\alpha K$$
- **Γ-driver condition** $$\partial_t \beta^i = \frac{2}{3} B^i; \qquad \partial_t B^i = \partial_t \tilde{\Gamma}^i - \eta B^i$$
- $t = 0\text{–}100 r_h$