

Brownian Motion in AdS/CFT

YITP Kyoto, 24 Dec 2009

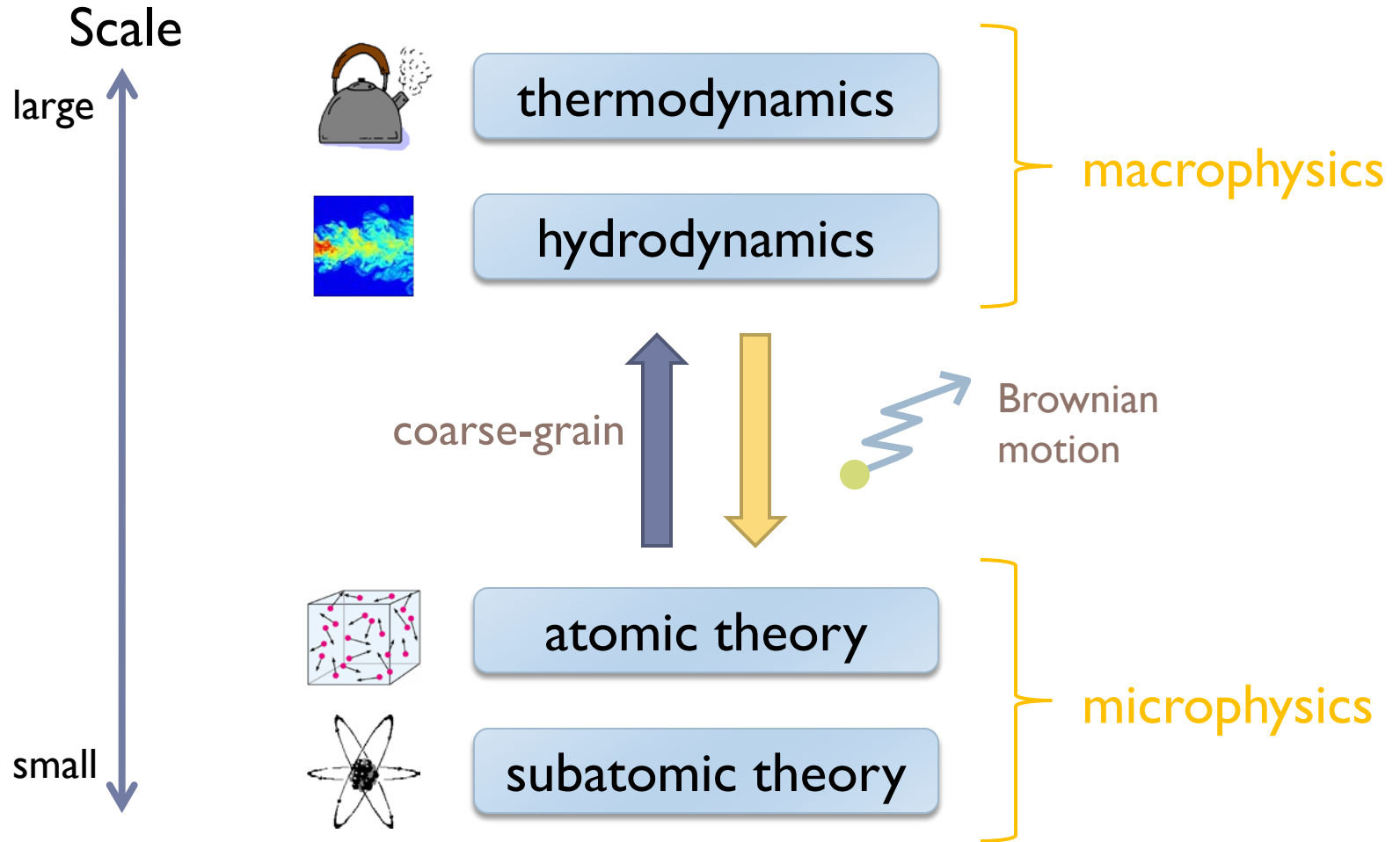
Masaki Shigemori (Amsterdam)

This talk is based on:

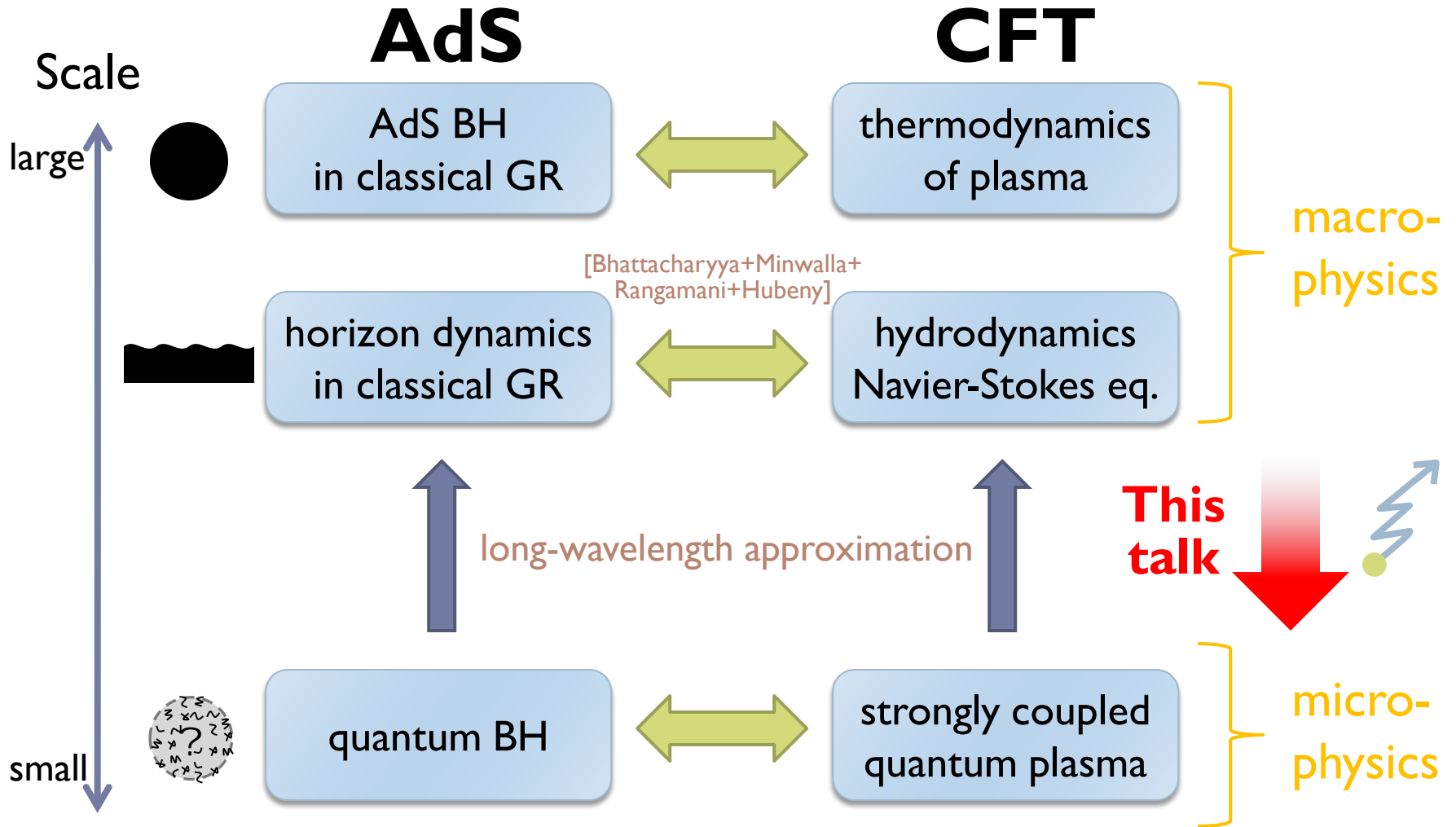
- ▶ J. de Boer, V. Hubeny, M. Rangamani, M.S.,
“Brownian motion in AdS/CFT,”
arXiv:0812.5112.
- ▶ A. Atmaja, J. de Boer, M.S., in preparation.

Introduction / Motivation

Hierarchy of scales in physics



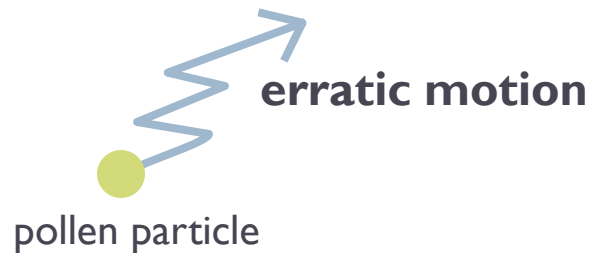
Hierarchy in AdS/CFT



Brownian motion

— Historically, a crucial step toward microphysics of nature

▶ 1827 Brown



Robert Brown (1773-1858)

- ▶ Due to collisions with fluid particles
- ▶ Allowed to determine Avogadro #: $N_A = 6 \times 10^{23} < \infty$
- ▶ Ubiquitous
- ▶ Langevin eq. (friction + random force)

Brownian motion in AdS/CFT

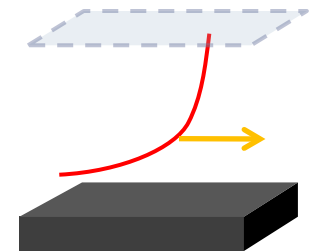
→ Do the same in AdS/CFT!

- ▶ Brownian motion of an external quark in CFT plasma
- ▶ Langevin dynamics from bulk viewpoint?
- ▶ Fluctuation-dissipation theorem
- ▶ Read off nature of constituents of strongly coupled plasma
- ▶ Relation to RHIC physics?

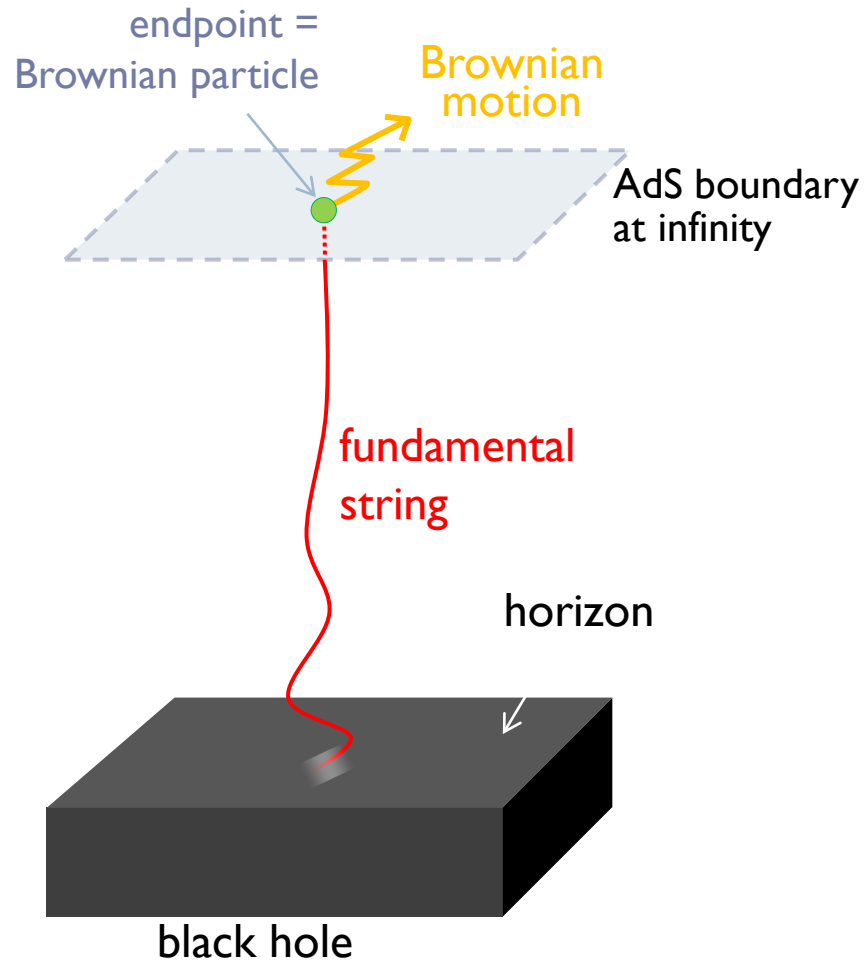
Related work:

Drag force: Herzog+Karch+Kovtun+Kozcaz+Yaffe, Gubser,
Casalderrey-Solana+Teaney

Transverse momentum broadening: Gubser, Casalderrey-Solana+Teaney

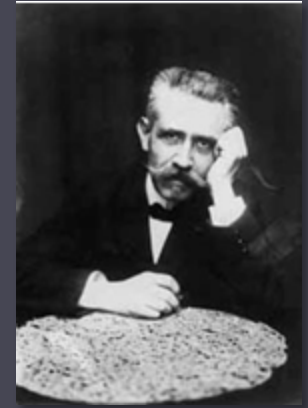


Preview: BM in AdS/CFT



Outline

- ▶ Intro/motivation
- ▶ Boundary BM
- ▶ Bulk BM
- ▶ Time scales
- ▶ BM on stretched horizon

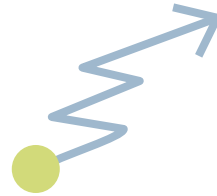


Paul Langevin (1872-1946)

Boundary BM

- Langevin dynamics

Simplest Langevin eq.



$$x, p = m\dot{x}$$

$$\dot{p}(t) = \underbrace{-\gamma_0 p(t)}_{\text{(instantaneous) friction}} + \underbrace{R(t)}_{\text{random force}}$$

$$\langle R(t) \rangle = 0, \quad \langle R(t)R(t') \rangle = \kappa_0 \delta(t - t')$$

white noise

Simplest Langevin eq.

► Displacement:

$$\langle s^2(t) \rangle \equiv \langle [x(t) - x(0)]^2 \rangle$$

$$\approx \begin{cases} \frac{T}{m} t^2 & (t \ll t_{relax}) & \text{ballistic regime} \\ & & \text{(init. velocity } \dot{x} \sim \sqrt{T/m} \text{)} \\ 2Dt & (t \gg t_{relax}) & \text{diffusive regime} \\ & & \text{(random walk)} \end{cases}$$

- Diffusion constant: $D \equiv \frac{T}{\gamma_0 m}$ (S-E relation)

- Relaxation time: $t_{relax} = \frac{1}{\gamma_0}$ FD theorem $\rightarrow \gamma_0 = \frac{\kappa_0}{2mT}$

Generalized Langevin equation

$$\dot{p}(t) = - \int_{-\infty}^t dt' \underbrace{\gamma(t-t')}_{\text{delayed friction}} p(t') + \underbrace{R(t)}_{\text{random force}} + \underbrace{F(t)}_{\text{external force}}$$

$$\langle R(t) \rangle = 0, \quad \langle R(t)R(t') \rangle = \kappa(t-t')$$

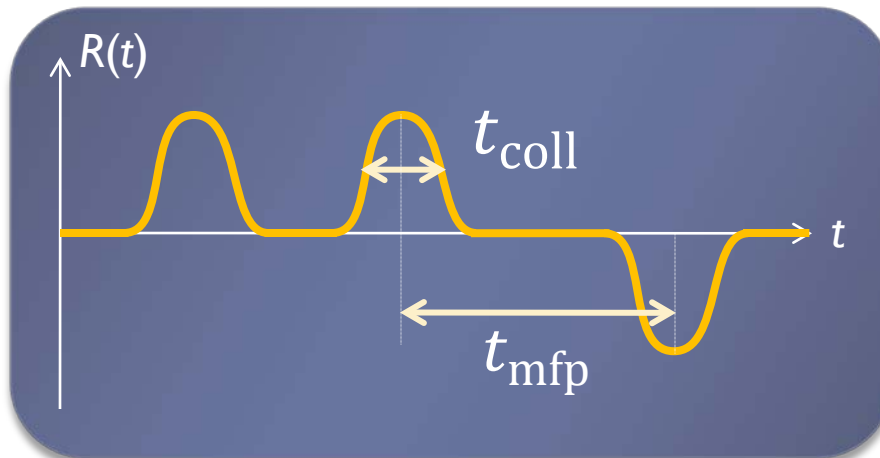
- ▶ **Qualitatively similar to simple LE**
 - ▶ ballistic regime
 - ▶ diffusive regime

Time scales

- ▶ Relaxation time $t_{\text{relax}} \equiv \frac{1}{\gamma_0}$, $\gamma_0 = \int_0^\infty dt \gamma(t)$
- ▶ Collision duration time t_{coll}

$$\langle R(t)R(0) \rangle \sim e^{-t/t_{\text{coll}}} \rightarrow \text{time elapsed in a single collision}$$

- ▶ Mean-free-path time t_{mfp} \rightarrow time between collisions



Typically

$$t_{\text{relax}} \gg t_{\text{mfp}} \gg t_{\text{coll}}$$

but not necessarily so
for strongly coupled plasma

How to determine γ, κ

$$p(\omega) = \frac{R(\omega) + F(\omega)}{\gamma[\omega] - i\omega} \equiv \underbrace{\mu(\omega)}_{\text{admittance}} [R(\omega) + F(\omega)]$$

1. Forced motion $F(t) = F_0 e^{-i\omega t}$

$$\langle p(t) \rangle = \mu(\omega) F_0 e^{-i\omega t} \rightarrow \text{read off } \mu$$

2. No external force, $F = 0$

$$\langle pp \rangle = |\mu|^2 \langle RR \rangle$$

$\underbrace{\hspace{2em}}$
measure

$\underbrace{\hspace{2em}}$
known

$\underbrace{\hspace{2em}}$
 \rightarrow read off κ

Bulk BM

Bulk setup

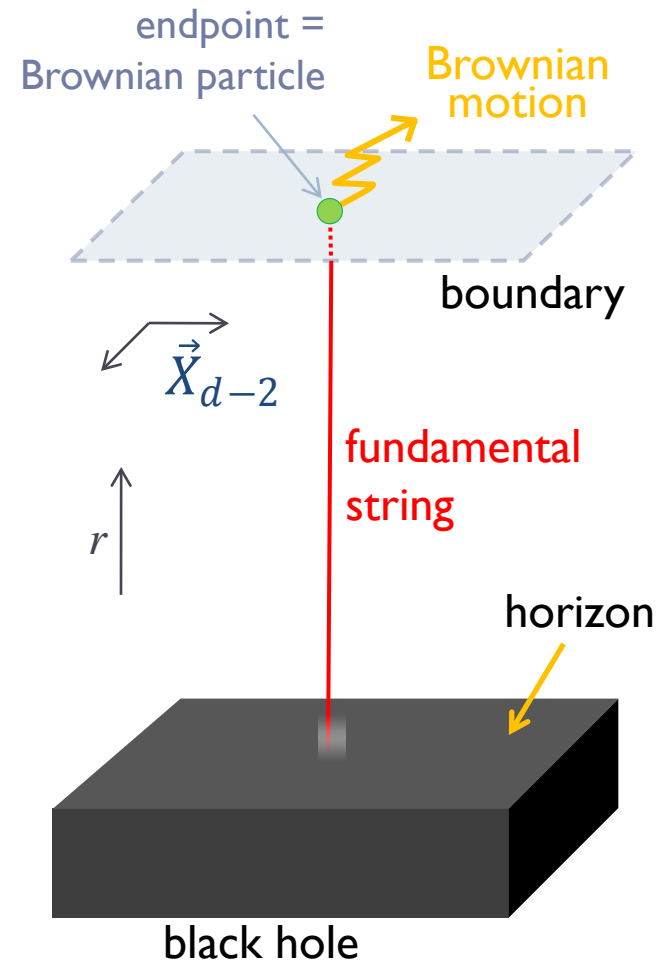
- ▶ AdS_d Schwarzschild BH (planar)

$$ds_d^2 = \frac{r^2}{l^2} (-h(r)dt^2 + d\vec{X}_{d-2}^2) + \frac{l^2 dr^2}{r^2 h(r)}$$

$$h(r) = 1 - \left(\frac{r_H}{r}\right)^{d-1}$$

$$T = \frac{1}{\beta} = \frac{(d-1)r_H}{4\pi l^2}$$

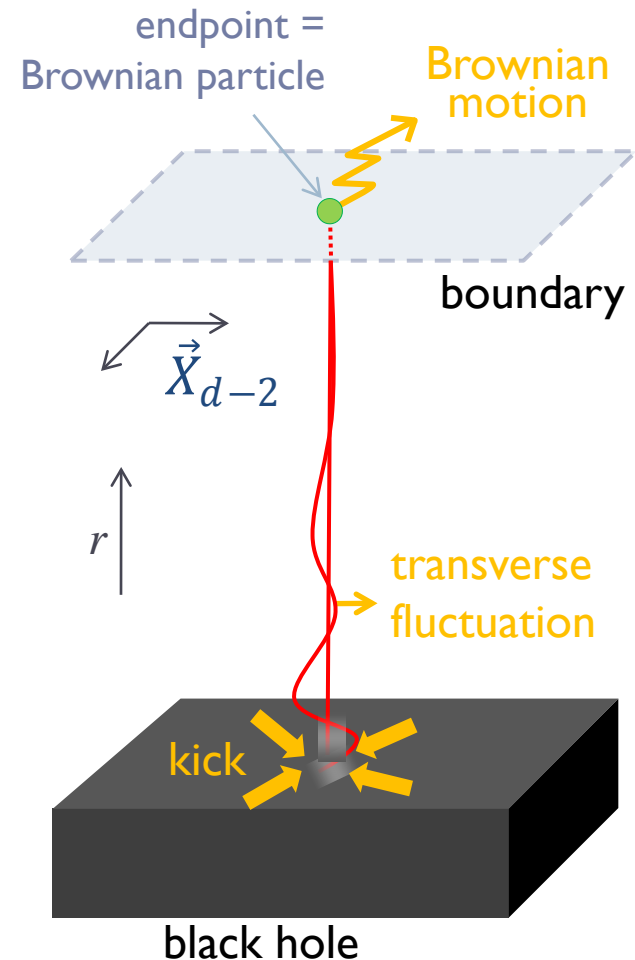
l : AdS radius



Physics of BM in AdS/CFT

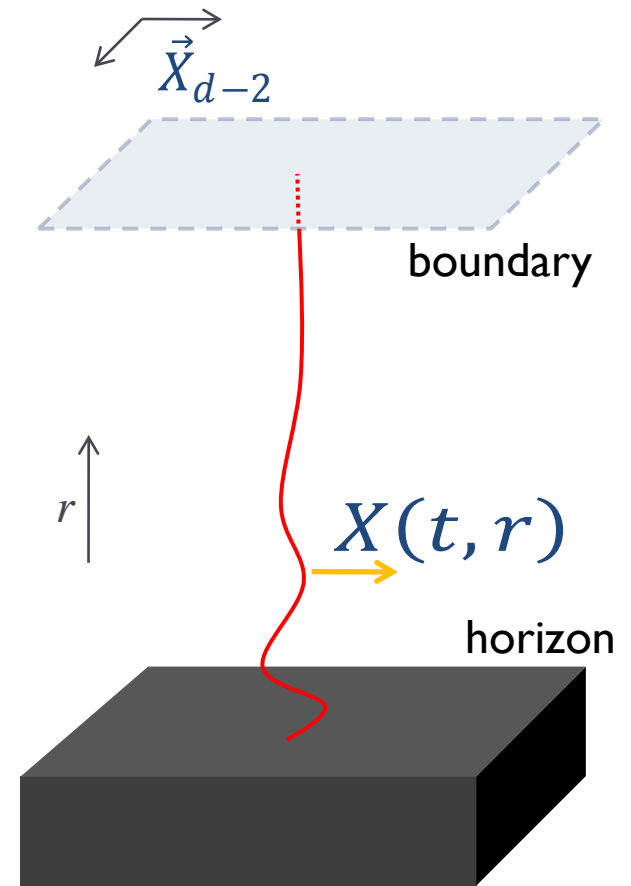
- ▶ Horizon kicks endpoint on horizon (= Hawking radiation)
- ▶ Fluctuation propagates to AdS boundary
- ▶ Endpoint on boundary (= Brownian particle) exhibits BM

Whole process is dual to quark hit by plasma particles



Assumptions

- ▶ Probe approximation
- ▶ Small g_s
 - ▶ No interaction with bulk
 - ▶ Only interaction is at horizon
- ▶ Small fluctuation
 - ▶ Expand Nambu-Goto action to quadratic order
 - ▶ Transverse positions are similar to Klein-Gordon scalar

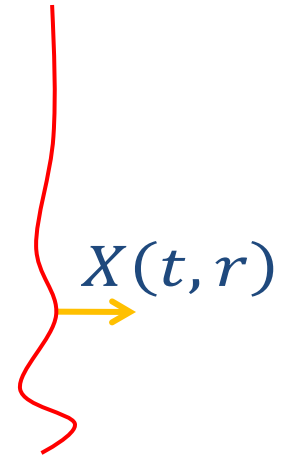


Transverse fluctuations

▶ Quadratic action

$$S_{\text{NG}} = \text{const} + S_2 + S_4 + \dots$$

$$S_2 = -\frac{1}{4\pi\alpha'} \int dt dr \left[\frac{\dot{X}^2}{h(r)} - \frac{r^4 h(r)}{l^4} X'^2 \right]$$



▶ Mode expansion

$$X(t, r) = \int_0^\infty d\omega (f_\omega(r) e^{-i\omega t} a_\omega + \text{h.c.})$$

$$\left[\omega^2 + \frac{h(r)}{l^4} \partial_r (r^4 h(r) \partial_r) \right] f_\omega(r) = 0$$

d=3: can be solved exactly

d>3: can be solved in low frequency limit

Bulk-boundary dictionary

Near horizon:

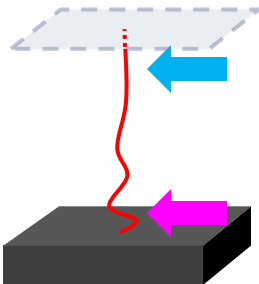
$$X(t, r) \sim \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \left[\overbrace{\left(e^{-i\omega(t-r_*)} + e^{i\theta_\omega} e^{-i\omega(t+r_*)} \right)}^{\text{outgoing mode}} a_\omega + \text{h.c.} \right]$$

phase shift
ingoing mode

r_* : tortoise coordinate

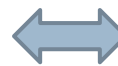
Near boundary:

$$X(t, r_c) \equiv x(t) = \int_0^\infty d\omega \left[f_\omega(r_c) e^{-i\omega t} a_\omega + \text{h.c.} \right] \quad r_c : \text{cutoff}$$



$$\langle x(t_1)x(t_2) \cdots \rangle \leftrightarrow \langle a_{\omega_1} a_{\omega_2}^\dagger \cdots \rangle$$

**observe BM
in gauge theory**



**correlator of
radiation modes**

Can learn about quantum gravity in principle!

Semiclassical analysis

- ▶ Semiclassically, NH modes are thermally excited:

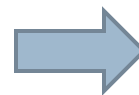
$$\langle a_\omega a_\omega^\dagger \rangle \propto \frac{1}{e^{\beta\omega} - 1}$$

➡ Can use dictionary to compute $x(t), s^2(t)$ (bulk \rightarrow boundary)

- ▶ AdS₃

$$s^2(t) \equiv \langle : [x(t) - x(0)]^2 : \rangle \approx \begin{cases} \frac{T}{m} t^2 & (t \ll t_{\text{relax}}) \quad \text{: ballistic} \\ \frac{\alpha'}{\pi l^2 T} t & (t \gg t_{\text{relax}}) \quad \text{: diffusive} \end{cases}$$

$$t_{\text{relax}} \sim \frac{\alpha' m}{l^2 T^2}$$



**Does exhibit
Brownian motion**

Semiclassical analysis

► Momentum distribution

Probability distribution for $p = m\dot{x}$

$$f(p) \propto \exp(-\beta E_p), \quad E_p = \frac{p^2}{2m}$$

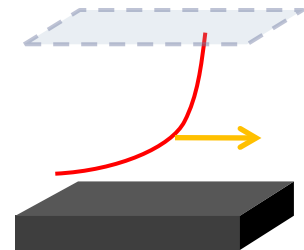
→ Maxwell-Boltzmann

► Diffusion constant

$$D = \frac{(d-1)^2 \alpha'}{8\pi l^2 T}$$

→ Agrees with drag force computation

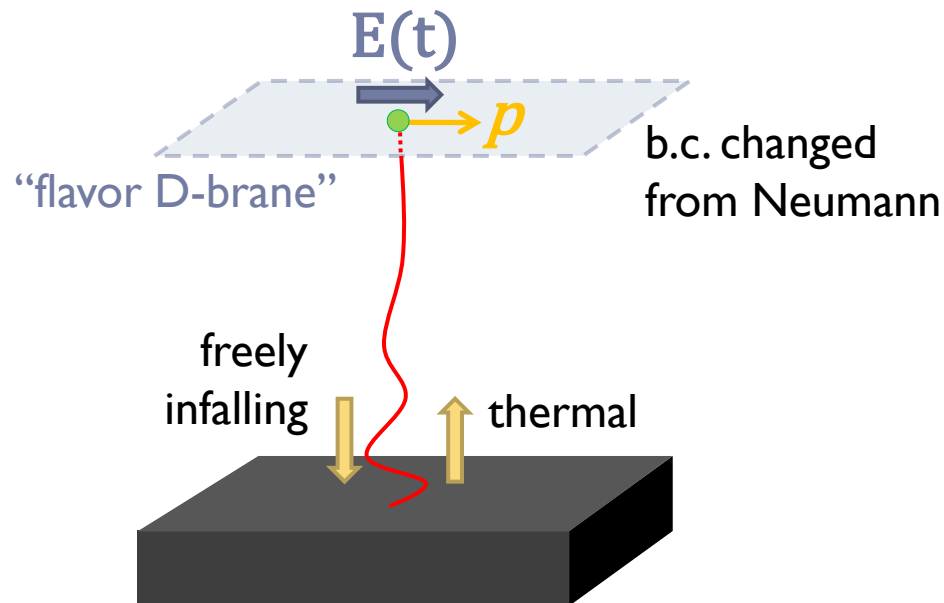
[Herzog+Karch+Kovtun+Kozcaz+Yaffe] [Liu+Rajagopal+Wiedemann]
[Gubser] [Casalderrey-Solana+Teaney]



Forced motion

- ▶ Want to determine $\mu(\omega)$, $\kappa(\omega)$
 - follow the two-step procedure

Turn on electric field $E(t) = E_0 e^{-i\omega t}$
on “flavor D-brane” and measure response $\langle p(t) \rangle$



Forced motion: results (AdS₃)

▶ Admittance

$$\mu(\omega) = \frac{1}{\gamma[\omega] - i\omega} = \frac{\alpha' \beta^2 m}{2\pi} \frac{1 - i\omega/\pi\alpha' m}{1 - \alpha' m \beta^2 \omega/2\pi}$$

▶ Random force correlator

$$\kappa(\omega) = \frac{4\pi}{\alpha' \beta^3} \frac{1 - (\beta\omega/2\pi)^2}{1 - (\omega/2\pi\alpha' m)^2} \quad \Rightarrow \quad t_{\text{coll}} \sim \frac{1}{T} \quad (\text{no } \lambda)$$

▶ FD theorem

$$2m \operatorname{Re}(\gamma[\omega]) = \beta\kappa(\omega) \quad \Rightarrow \quad \text{satisfied}$$

can be proven generally

Time scales

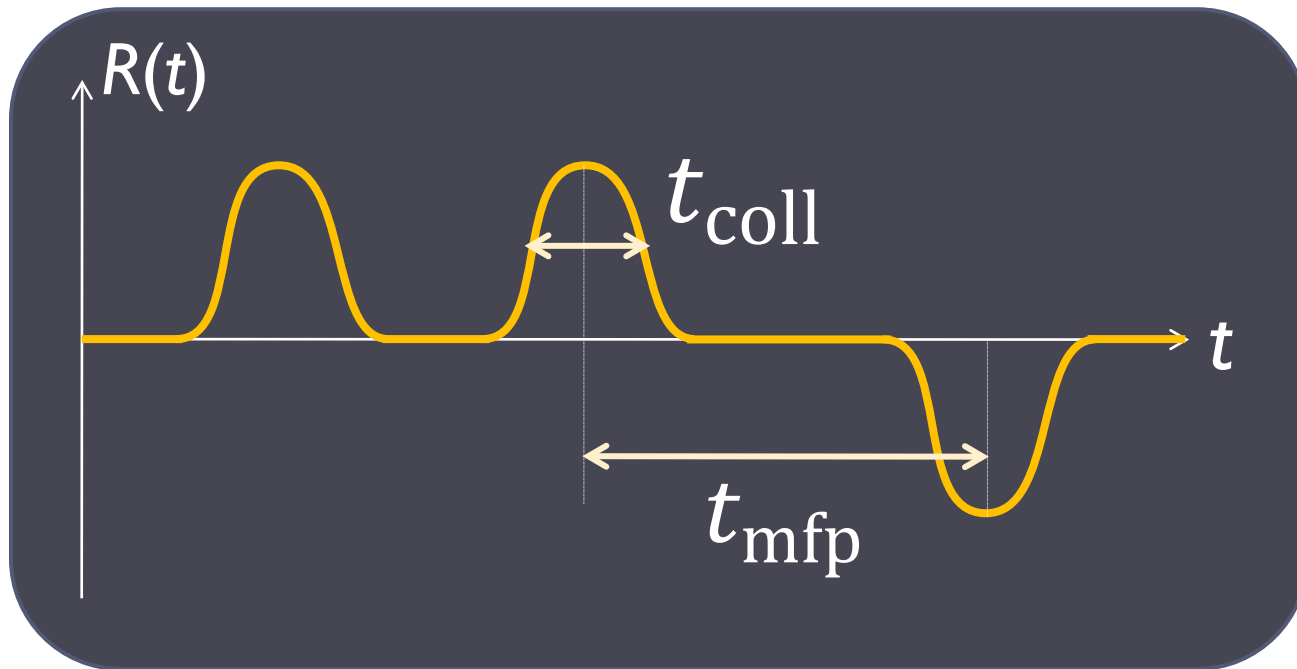
Time scales

t_{relax}

t_{mfp}

t_{coll}

information about
plasma constituents



Time scales from R-correlators

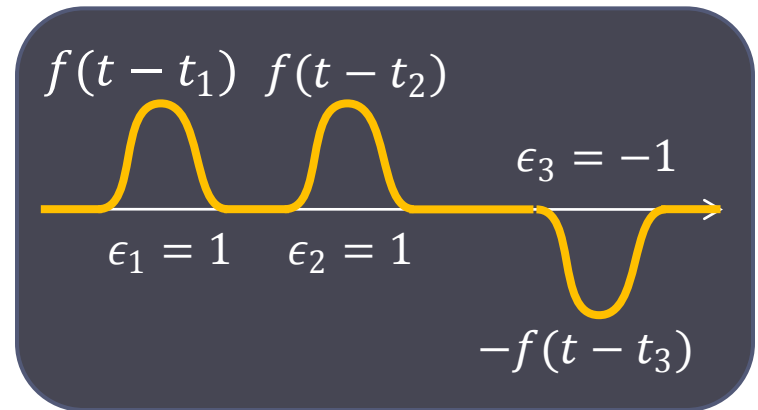
A toy model:

- ▶ $R(t)$: consists of many pulses randomly distributed

$$R(t) = \sum_{i=1}^k \epsilon_i f(t - t_i)$$

$f(t)$: shape of a single pulse

$\epsilon_i = \pm 1$: random sign



- ▶ Distribution of pulses = Poisson distribution

μ : number of pulses per unit time, $\sim 1/t_{\text{mfp}}$

Time scales from R-correlators

- 2-pt func $\langle R(t)R(0) \rangle \rightarrow t_{\text{coll}}$
- Low-freq. 4-pt func $\langle \tilde{R}(\omega_1)\tilde{R}(\omega_2)\tilde{R}(\omega_3)\tilde{R}(\omega_4) \rangle \rightarrow t_{\text{mfp}}$

$$\langle \tilde{R}(\omega_1)\tilde{R}(\omega_2) \rangle \approx 2\pi\mu\delta(\omega_1 + \omega_2)\tilde{f}(0)^2$$

$$\langle \tilde{R}(\omega_1)\tilde{R}(\omega_2)\tilde{R}(\omega_3)\tilde{R}(\omega_4) \rangle_{\text{conn}}$$

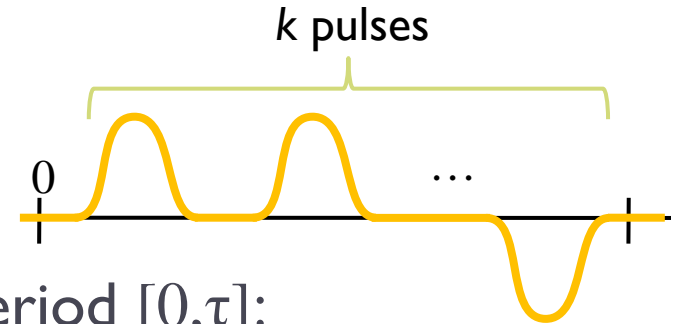
$$\approx 2\pi\mu\delta(\omega_1 + \omega_2 + \omega_3 + \omega_4)\tilde{f}(0)^4$$

tilde = Fourier transform

 Can determine μ , thus t_{mfp}

Sketch of derivation (1 / 2)

$$R(t) = \sum_{i=1}^k \epsilon_i f(t - t_i)$$



Probability that there are k pulses in period $[0, \tau]$:

$$P_k(\tau) = e^{-\mu\tau} \frac{(\mu\tau)^k}{k!} \quad (\text{Poisson dist.})$$

2-pt func:

$$\langle R(t)R(t') \rangle = \sum_{k=1}^{\infty} P_k(\tau) \sum_{i,j=1}^k \langle \epsilon_i \epsilon_j f(t - t_i) f(t' - t_j) \rangle_k$$

$$\epsilon_i = \pm 1 : \text{random signs} \rightarrow \langle \epsilon_i \epsilon_j \rangle = \delta_{ij}$$

$$\langle f(t - t_i) f(t' - t_i) \rangle_k = \frac{k}{\tau} \int_0^{\tau} du f(t - u) f(t' - u)$$

Sketch of derivation (2/2)

$$\longrightarrow \langle R(t)R(t') \rangle = \mu \int_{-\infty}^{\infty} du f(t-u)f(t'-u)$$

$$\langle \tilde{R}(\omega)\tilde{R}(\omega') \rangle = 2\pi\mu\delta(\omega + \omega')\tilde{f}(\omega)\tilde{f}(\omega')$$

Similarly, for 4-pt func,

$$\langle \tilde{R}(\omega)\tilde{R}(\omega')\tilde{R}(\omega'')\tilde{R}(\omega''') \rangle$$

$$= \langle \tilde{R}(\omega)\tilde{R}(\omega') \rangle \langle \tilde{R}(\omega'')\tilde{R}(\omega''') \rangle + (2 \text{ more terms}) \\ + 2\pi\mu\delta(\omega + \omega' + \omega'' + \omega''')\tilde{f}(\omega)\tilde{f}(\omega')\tilde{f}(\omega'')\tilde{f}(\omega''')$$

“disconnected part”



“connected part”



$$\longrightarrow \langle \tilde{R}(\omega_1)\tilde{R}(\omega_2) \rangle \approx 2\pi\mu\delta(\omega_1 + \omega_2)\tilde{f}(0)^2 \\ \langle \tilde{R}(\omega_1)\tilde{R}(\omega_2)\tilde{R}(\omega_3)\tilde{R}(\omega_4) \rangle_{\text{conn}} \approx 2\pi\mu\delta(\omega_1 + \omega_2 + \omega_3 + \omega_4)\tilde{f}(0)^4$$

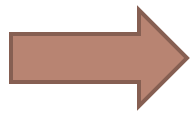
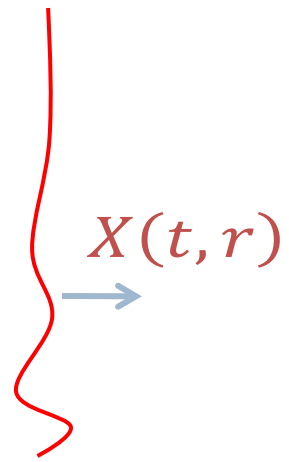
$\langle RRRR \rangle$ from bulk BM

- ▶ Can compute t_{mfp} from connected to 4-pt func.
- ▶ Expansion of NG action to higher order:

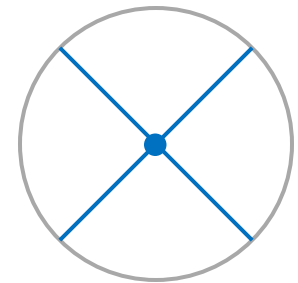
$$S_{\text{NG}} = \text{const} + S_2 + S_4 + \dots$$

$$S_4 = \frac{1}{16\pi\alpha'} \int dt dr \left[\frac{\dot{X}^2}{h(r)} - \frac{r^4 h(r)}{l^4} X'^2 \right]^2$$

4-point vertex

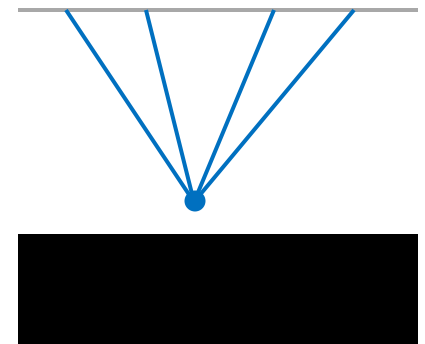


Can compute $\langle RRRR \rangle_{\text{conn}}$
and thus t_{mfp}



$\langle RRRR \rangle$ from bulk BM

- ▶ Holographic renormalization [Skenderis]
 - ▶ Similar to KG scalar, but not quite
- ▶ Lorentzian AdS/CFT [Skenderis + van Rees]
- ▶ IR divergence
 - ▶ Near the horizon, bulk integral *diverges*



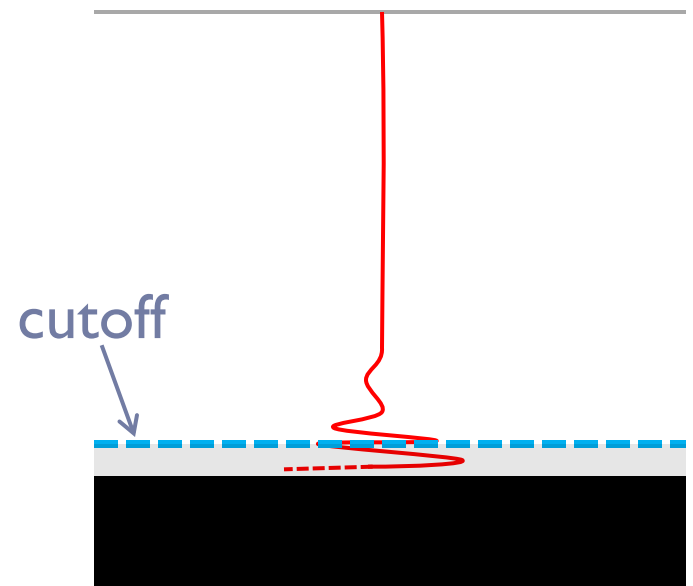
IR divergence

Reason:

- ▶ Near the horizon, local temperature is high
- ▶ String fluctuates wildly
- ▶ Expansion of NG action breaks down

Remedy:

- ▶ Introduce an IR cut off where expansion breaks down
- ▶ Expect that this gives right estimate of the full result



Times scales from AdS/CFT (weak)

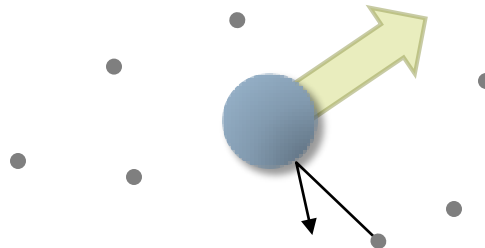
Resulting timescales:

$$t_{\text{relax}} \sim \frac{m}{\sqrt{\lambda} T^2} \quad t_{\text{coll}} \sim \frac{1}{T} \quad t_{\text{mfp}} \sim \frac{1}{T \log \lambda} \quad \lambda \equiv \frac{l^4}{\alpha'^2}$$

► weak coupling $\lambda \ll 1$

➡ $t_{\text{relax}} \gg t_{\text{mfp}} \gg t_{\text{coll}}$

➡ conventional kinetic theory is good



Times scales from AdS/CFT (strong)

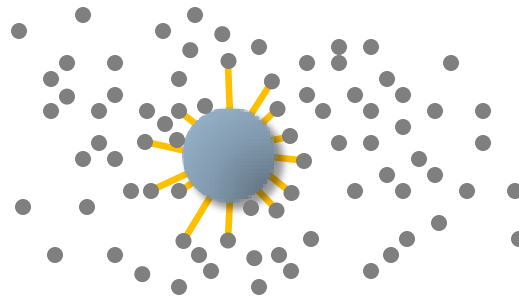
Resulting timescales:

$$t_{\text{relax}} \sim \frac{m}{\sqrt{\lambda} T^2} \quad t_{\text{coll}} \sim \frac{1}{T} \quad t_{\text{mfp}} \sim \frac{1}{T \log \lambda} \quad \lambda \equiv \frac{l^4}{\alpha'^2}$$

- strong coupling $\lambda \gg 1$

➡ $t_{\text{mfp}} \ll t_{\text{coll}}$

➡ Multiple collisions occur simultaneously.



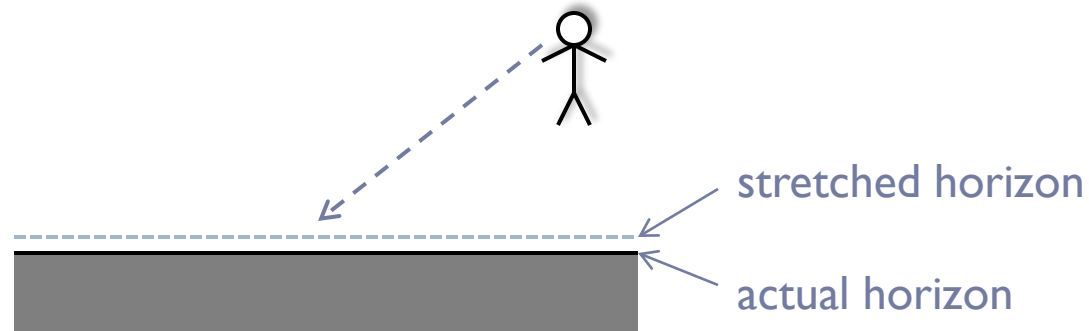
Cf. “fast scrambler”

[Hayden+Preskill]
[Sekino+Susskind]

BM on stretched horizon

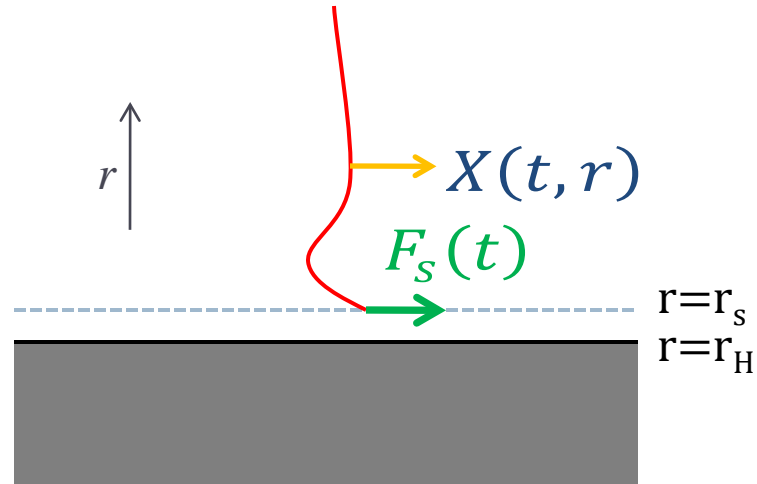
Membrane paradigm?

- ▶ Attribute physical properties to “stretched horizon”
E.g. temperature, resistance, ...



Q. Bulk BM was due to b.c. at horizon
(ingoing: thermal, outgoing: any).
Can we reproduce this by interaction
between string and “membrane”?

Langevin eq. at stretched horizon



▶ EOM for endpoint: $-\# \partial_r X(t, r_s) = F_S(t)$

▶ Postulate:

$$F_S(t) = - \int_{-\infty}^t dt' \gamma_S(t-t') \partial_t X(t', r_s) + R_S(t)$$

$$\langle R_S(t) \rangle = 0, \quad \langle R_S(t) R_S(t') \rangle = \kappa_S(t-t')$$

Langevin eq. at stretched horizon

$$X(t, r \sim r_H) = \int \frac{d\omega}{\sqrt{2\omega}} \left[\underbrace{a_\omega^{(+)} e^{-i\omega(t-r_*)}}_{\text{Ingoing (thermal)}} + \underbrace{a_\omega^{(-)} e^{-i\omega(t+r_*)}}_{\text{outgoing (any)}} + \text{h. c.} \right]$$



Plug into EOM

- ▶ Can satisfy EOM if

$$\gamma_s(t) \propto \delta(t), \quad R_s(\omega) \propto \sqrt{\omega} a_\omega^{(+)}$$

Friction precisely cancels
outgoing modes

Random force excites
ingoing modes thermally

- ▶ Correlation function:

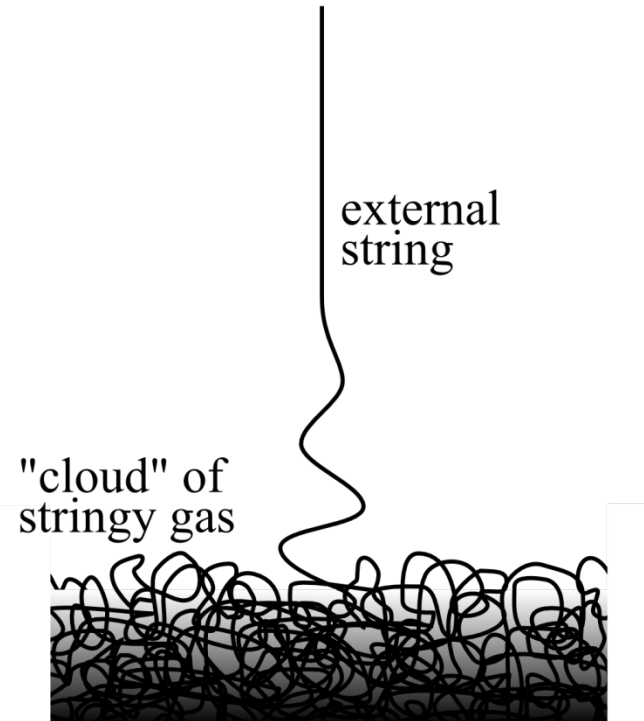
$$\langle R_s(\omega)^\dagger R_s(\omega) \rangle \propto \langle a_\omega^{(+)\dagger} a_\omega^{(+)} \rangle \propto \frac{\omega}{e^{\beta\omega} - 1}$$

Granular structure on stretched horizon

- ▶ **BH covered by “stringy gas”**

[Susskind et al.]

- ▶ Frictional / random forces can be due to this gas
- ▶ Can we use Brownian string to probe this?



Granular structure on stretched horizon

▶ AdS_d BH

$$ds_d^2 = \frac{r^2}{l^2} (-h(r)dt^2 + d\vec{X}_{d-2}^2) + \frac{l^2 dr^2}{r^2 h(r)}$$

$$h(r) = 1 - \left(\frac{r_H}{r}\right)^{d-1} \quad T = \frac{1}{\beta} = \frac{(d-1)r_H}{4\pi l^2}$$

▶ One quasiparticle / string bit per Planck area



$$\Delta X \sim \frac{l}{r_H} \ell_P$$



qp's are moving at speed of light

$$\Delta t \sim \frac{\Delta X}{\sqrt{\epsilon}} \sim \frac{l \ell_P}{\sqrt{\epsilon} r_H} \quad (\text{stretched horizon at } r_s = (1 + 2\epsilon)r_H)$$

▶ Proper distance from actual horizon: $L \sim \sqrt{\epsilon} l$

Granular structure on stretched horizon



- ▶ String endpoint collides with a quasiparticle once in time

$$\Delta t \sim \frac{\ell_P}{TL}$$

- ▶ Cf. Mean-free-path time read off R_s -correlator:

$$t_{\text{mfp}} \sim \frac{1}{T}$$



- ▶ Scattering probability for string endpoint:

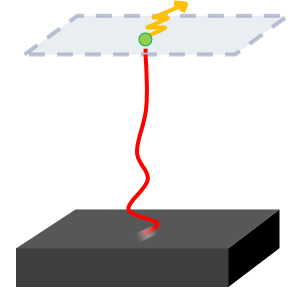
$$\sigma = \frac{\Delta t}{t_{\text{mfp}}} \sim \frac{\ell_P}{L} \sim \begin{cases} 1 & (L \sim \ell_P) \\ g_s^\# & (L \sim \ell_s) \end{cases}$$

Conclusions

Conclusions

- ▶ Boundary BM \leftrightarrow bulk “Brownian string”

Can study QG in principle



- ▶ Semiclassically, can reproduce Langevin dyn. from bulk

random force \leftrightarrow Hawking rad. (“kick” by horizon)

friction \leftrightarrow absorption

- ▶ Time scales in strong coupling QGP: t_{relax} , t_{coll} , t_{mfp}

- ▶ FD theorem

Conclusions

- ▶ BM on stretched horizon
 - ▶ Analogue of Avogadro # $N_A = 6 \times 10^{23} < \infty$?
 - ▶ Boundary: $E/T \sim \mathcal{O}(N^2) < \infty$
 - ▶ Bulk: $M/T \sim \mathcal{O}(G_N^{-1}) < \infty$
- energy of a Hawking quantum is tiny as compared to BH mass, but finite

Thanks!