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Hidden Symmetries of Charged Kerr Black Hole

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* in preparation

Motivation

• String theory implies the existence of extra dimensions and motivates us to study a gravity in a higher-dimensional framework.

• There is gravity/gauge duality which is one of the most exciting ideas in particle physics.

$(d+1)$-dim. gravitaional theory $\Leftrightarrow$ d-dim. gauge theory

• Understanding in higher-dimensional framework might give us further understanding in 4-dimension.

Black hole solutions provide important and useful gravitational backgrounds for these purposes, since black holes possess properties such as entropy and a singularity that fundamental physics aims to address.

Black hole metrics in a vacuum

- 4-dimensional black hole metric

	mass	a.m.	NUT	Λ
Schwarzschild (1915)	○			
Kerr (1963)	○	○		
Carter (1968)	○	○		○
Plebanski (1975)	○	○	○	○

- Higher-dimensional ($D \geq 4$) black hole metric

	mass	a.m.s	NUTs	Λ
Tangherlini (1963)	○			
Myers, Perry (1986)	○	○		
Gibbons, Lü, Page, Pope (2004)	○	○		○
Chen, Lü, Pope (2006)	○	○	○	○

Kerr-NUT-AdS metric in D-dimension

The most general known solution (Chen-Lü-Pope metric) is called Kerr-NUT-AdS metric, which is given by

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right]^2 + \varepsilon S \left[\sum_{k=0}^{n} A^{(k)} d\psi_k \right]^2$$

in $D = 2n + \varepsilon$ dimension, where $\varepsilon = 0$ for even dimensions and $\varepsilon = 1$ for odd dimesions.

Here the functions are

$$Q_\mu = \frac{X_\mu}{U_\mu} \ , \quad U_\mu = \prod_{\nu \neq \mu} (x_\mu^2 - x_\nu^2) \ , \quad X_\mu = \sum_{k=\varepsilon}^{n} c_k x^{2k} + b_\mu x_\mu^{1-\varepsilon} + \varepsilon \frac{(-1)^k c}{x_\mu^2} \ ,$$

$$A_\mu^{(k)} = \sum_{\substack{1 \le \nu_1 < \cdots < \nu_k \le n \\ \nu_i \neq \mu}} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \ , \quad A^{(k)} = \sum_{1 \le \nu_1 < \cdots < \nu_k \le n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \ , \quad A_\mu^{(0)} = A^{(0)} = 1 \ ,$$

$$S = \frac{c}{A^{(n)}} \ , \quad c = \text{const.} \ .$$

This metric satisfies $R_{ab} = -(D-1) c_n \, g_{ab}$ in all dimesions.

Kerr metric (4-dimension)

$$ds^2 = \frac{\Sigma}{\triangle} dr^2 + \Sigma d\theta^2 - \left(\frac{\triangle - a^2 \sin^2\theta}{\Sigma} \right) dt^2 - \frac{4Mar \sin^2\theta}{\Sigma} dt d\phi + \left[\frac{(r^2 + a^2)^2 - \triangle a^2 \sin^2\theta}{\Sigma} \right] \sin^2\theta d\phi^2$$

where

$$\Sigma = r^2 + a^2 \sin^2\theta \ , \quad \triangle = r^2 + a^2 - 2Mr$$

Kerr metric (4-dimension)

$$\begin{aligned} ds^2 =& \frac{x^2 - y^2}{X} dx^2 + \frac{y^2 - x^2}{Y} dy^2 \\ &+ \frac{X}{x^2 - y^2}(d\psi_0 + y^2 d\psi_1)^2 + \frac{Y}{y^2 - x^2}(d\psi_0 + x^2 d\psi_1)^2 \end{aligned}$$

where

$$X = x^2 - a^2 - 2Mx \ , \quad Y = y^2 - a^2$$

. . . .

Kerr-NUT metric (4-dimension)

$$\begin{aligned} ds^2 =& \frac{x^2 - y^2}{X} dx^2 + \frac{y^2 - x^2}{Y} dy^2 \\ &+ \frac{X}{x^2 - y^2}(d\psi_0 + y^2 d\psi_1)^2 + \frac{Y}{y^2 - x^2}(d\psi_0 + x^2 d\psi_1)^2 \end{aligned}$$

where

$$X = x^2 - a^2 - 2Mx \ , \quad Y = y^2 - a^2 - 2Ly$$

Ansatz metric (4-dimension)

$$
\begin{aligned}
ds^2 =& \frac{x^2 - y^2}{X(x)} dx^2 + \frac{y^2 - x^2}{Y(y)} dy^2 \\
&+ \frac{X(x)}{x^2 - y^2}(d\psi_0 + y^2 d\psi_1)^2 + \frac{Y(y)}{y^2 - x^2}(d\psi_0 + x^2 d\psi_1)^2
\end{aligned}
$$

We can determine the functions X and Y by imposing Einstein condition $R_{ab} = -3c\, g_{ab}$.

....

Kerr-NUT-AdS metric (4-dimension)

$$
\begin{aligned}
ds^2 =& \frac{x^2 - y^2}{X} dx^2 + \frac{y^2 - x^2}{Y} dy^2 \\
&+ \frac{X}{x^2 - y^2}(d\psi_0 + y^2 d\psi_1)^2 + \frac{Y}{y^2 - x^2}(d\psi_0 + x^2 d\psi_1)^2
\end{aligned}
$$

where

$$
X = cx^4 + x^2 - a^2 - 2Mx \ , \quad Y = cy^4 + y^2 - a^2 - 2Ly
$$

Kerr-NUT-AdS metric (5-dimension)

$$
\begin{aligned}
ds^2 =& \frac{x^2-y^2}{X}dx^2 + \frac{y^2-x^2}{Y}dy^2 \\
&+ \frac{X}{x^2-y^2}(d\psi_0 + y^2 d\psi_1)^2 + \frac{Y}{y^2-x^2}(d\psi_0 + x^2 d\psi_1)^2 \\
&+ \frac{c}{x^2y^2}(d\psi_0 + (x^2+y^2)d\psi_1 + x^2y^2 d\psi_2)^2
\end{aligned}
$$

$$
X = c_4 x^4 + c_2 x^2 + c_0 + b_1 + \frac{c}{x^2}\ , \quad Y = c_4 y^4 + c_2 y^2 + c_0 + b_2 + \frac{c}{y^2}
$$

Kerr-NUT-AdS metric (6-dimension)

$$
\begin{aligned}
ds^2 =& \frac{(x^2-y^2)(x^2-z^2)}{X}dx^2 + \frac{(y^2-x^2)(y^2-z^2)}{Y}dy^2 + \frac{(z^2-x^2)(z^2-y^2)}{Z}dz^2 \\
&+ \frac{X}{(x^2-y^2)(x^2-z^2)}(d\psi_0 + (y^2+z^2)d\psi_1 + y^2z^2d\psi_2)^2 \\
&+ \frac{Y}{(y^2-x^2)(y^2-z^2)}(d\psi_0 + (z^2+x^2)d\psi_1 + z^2x^2d\psi_2)^2 \\
&+ \frac{Z}{(z^2-x^2)(z^2-y^2)}(d\psi_0 + (x^2+y^2)d\psi_1 + x^2y^2d\psi_2)^2
\end{aligned}
$$

where

$$
\begin{aligned}
X &= c_6 x^6 + c_4 x^4 + c_2 x^2 + c_0 + b_1 x \ , \\
Y &= c_6 y^6 + c_4 y^4 + c_2 y^2 + c_0 + b_2 y \ , \\
Z &= c_6 z^6 +_c 4 z^4 + c_2 z^2 + c_0 + b_3 z
\end{aligned}
$$

Kerr-NUT-AdS metric (7-dimension)

$$\begin{aligned} ds^2 =& \frac{(x^2-y^2)(x^2-z^2)}{X}dx^2 + \frac{(y^2-x^2)(y^2-z^2)}{Y}dy^2 + \frac{(z^2-x^2)(z^2-y^2)}{Z}dz^2 \\ &+ \frac{X}{(x^2-y^2)(x^2-z^2)}(d\psi_0 + (y^2+z^2)d\psi_1 + y^2z^2d\psi_2)^2 \\ &+ \frac{Y}{(y^2-x^2)(y^2-z^2)}(d\psi_0 + (z^2+x^2)d\psi_1 + z^2x^2d\psi_2)^2 \\ &+ \frac{Z}{(z^2-x^2)(z^2-y^2)}(d\psi_0 + (x^2+y^2)d\psi_1 + x^2y^2d\psi_2)^2 \\ &+ \frac{c}{x^2y^2z^2}(d\psi_0 + (x^2+y^2+z^2)d\psi_1 + (x^2y^2+y^2z^2+x^2z^2)d\psi_2 + x^2y^2z^2d\psi_3)^2 \end{aligned}$$

where

$$X = c_6x^6 + c_4x^4 + c_2x^2 + c_0 + b_1 - \frac{c}{x^2} \ ,$$
$$Y = c_6y^6 + c_4y^4 + c_2y^2 + c_0 + b_2 - \frac{c}{y^2} \ ,$$
$$Z = c_6z^6 +_c 4z^4 + c_2z^2 + c_0 + b_3 - \frac{c}{z^2}$$

We can assume the ansatz metric

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right]^2 + \varepsilon S \left[\sum_{k=0}^{n} A^{(k)} d\psi_k \right]^2$$

in $D = 2n + \varepsilon$ dimension, where $\varepsilon = 0$ for even dimensions and $\varepsilon = 1$ for odd dimesions.

Here the functions are

$$Q_\mu = \frac{X_\mu}{U_\mu} \ , \quad U_\mu = \prod_{\nu \neq \mu} (x_\mu^2 - x_\nu^2) \ , \quad X_\mu = X_\mu(x_\mu) \ ,$$

$$A_\mu^{(k)} = \sum_{\substack{1 \leq \nu_1 < \cdots < \nu_k \leq n \\ \nu_i \neq \mu}} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \ , \quad A^{(k)} = \sum_{1 \leq \nu_1 < \cdots < \nu_k \leq n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \ , \quad A_\mu^{(0)} = A^{(0)} = 1 \ ,$$

$$S = \frac{c}{A^{(n)}} \ , \quad c = \text{const.} \ .$$

Imposing Einstein condition $R_{ab} = \lambda g_{ab}$, we can determine the form of the functioin X_μ

$$X_\mu = \sum_{k=\varepsilon}^{n} c_k x^{2k} + b_\mu x_\mu^{1-\varepsilon} + \varepsilon \frac{(-1)^k c}{x_\mu^2} \ .$$

Separabilities of Kerr-NUT-AdS spacetime in higher-dimensions

It is known that the separation of variables for various field equations on Kerr-NUT-AdS background.

- Geodesic equation

Frolov-Krtous-Kubiznak-Page(2006)

- Klein-Gordon equation

Kubiznak-Krtous-Kubiznak(2006)

- Dirac equation

Oota-Yasui(2008), Wu(2009)

- gravitational perturbation equation (tensor modes)

Kundri-Lucietti-Reall(2006), Oota-Yasui(2008)

- Maxwell equation ?

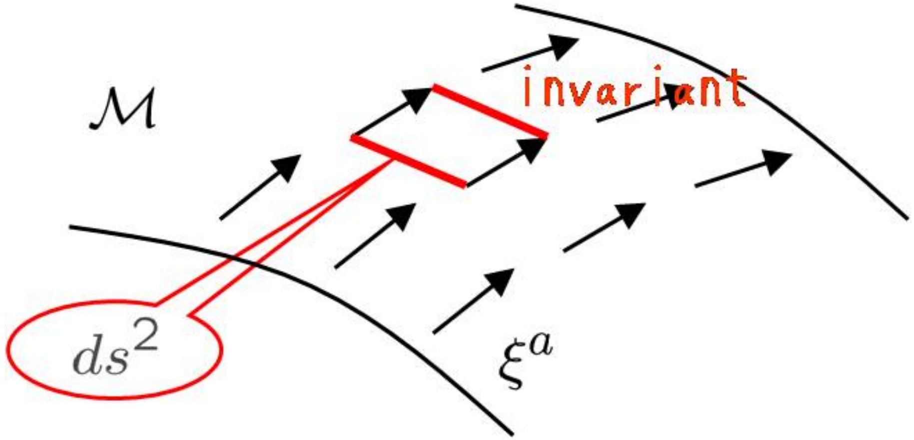
Killing vector

Def. A generator of isometry of spacetime ξ, i.e.,

$$\nabla_{(a}\xi_{b)} = 0 \qquad (\mathcal{L}_\xi\, g = 0)\ ,$$

is called **Killing vector**.

If the orbit of Killing vector is closed, it generates axial symmetry. If not, it generates translation symmetry.



Conformal Killing vector

Def. A generator of conformal symmetry of spacetime ξ, i.e.,

$$\nabla_{(a}\xi_{b)} = \phi\, g_{ab} \qquad (\mathcal{L}_\xi\, g = 2\phi\, g)\ ,$$

is called **conformal Killing vector**.

Geodesic integrability

For geodesic Hamiltonian $H = \frac{1}{2} g_{ab} p^a p^b$, E.O.M. gives geodesic equation

$$p^b \nabla_b p^a = 0 \qquad (\ddot{x}^a + \Gamma^a{}_{bc} \dot{x}^b \dot{x}^c = 0) \ .$$

We assume that a C.O.M. is written as $C = K_{a_1 \dots a_n} p^{a_1} \cdots p^{a_n}$. Then the condition

$$\{C, H\}_P = 0$$

leads to the equation

$$\nabla_{(b} K_{a_1 \dots a_n)} = 0 \ .$$

This equation is called **Killing equation** and K is called **Killing tensor of rank-n**. When $n = 1$, K is a Killing vector.

Since Killing tensor gives C.O.M. along geodesic, geodesic equation is integrable if there are the dimension number of Killing vectors and Killing tensors totally.

Contents

	Killing vector	conformal Killing vector
symmetric	Killing tensor	conformal Killing tensor
anti-symmetric	Killing-Yano tensor	conformal Killing-Yano tensor

Geodesic integrability of Kerr spacetime in 4-dimension

Carter (1968) ... There exists an nontrivial Killing tensor K, so there are four constants of motion.

$$\xi = \partial_t \ , \quad \eta = \partial_\phi \ , \quad g \ , \quad K$$

Penrose and Floyd (1973) ... Killing tensor K is written as the square of rank-2 Killing-Yano tensor f.

$$^{\exists}f \quad \text{s.t.} \quad K_{ab} = f_a{}^c f_{bc} \ , \ \underbrace{f_{ba} = -f_{ab} \ , \ \nabla_{(a} f_{b)c} = 0}_{\text{KY equation}}$$

Hughston and Sommers (1987) ... Two Killing vectors, ξ and η, are also constructed from the Killing-Yano tensor f.

$$\xi^a = \nabla_b (*f)^{ba} \ , \quad \eta^a = K^a{}_b \xi^b$$

⇒ KY tensor is more fundamental.

Killing tensor

Def. When a rank-n symmetric tensor K satisfies the equation

$$\nabla_{(b}K_{a_1 \ldots a_n)} = 0 \ ,$$

K is called **Killing tensor**.

Killing-Yano tensor

Def. When a rank-n anti-symmetric tensor f satisfies the equation

$$\nabla_{(b}f_{a_1)a_2 \ldots a_n} = 0 \ ,$$

f is called **Killing-Yano (KY) tensor**.

Geodesic integrability of Kerr-NUT-AdS spacetime in D-dimension

Page, Frolov, Kubizňák, Krtous and Vasdevan (2006)

There exist $n-1$ nontrivial Killing tensors $K^{(j)}$ in D-dimension, so there are the dimension number of constants of motion, which are mutually commuting.

$$\xi = \partial_t \ , \quad \eta^{(j)} = \partial_{\phi_i} \ , \quad g \ , \quad K^{(j)} \ \text{ and } \ \eta^{(n)} \quad (j = 1, \ldots n-1)$$

# Dimension	# Killing vector	# Killing tensor
$D = 2n$	n	n
$D = 2n+1$	$n+1$	n

As the 4-dimension, Killing vectors and tensors, ξ, $\eta^{(j)}$ and $K^{(j)}$, are constructed from rank-$(D-2j)$ Killing-Yano tensors $f^{(j)}$.

$$K^{(j)}_{ab} = f^{(j)}{}_{a\cdots} f^{(j)}{}_{b}{}^{\cdots} \ , \quad \xi^a = \nabla_b (*f^{(1)})^{ba} \ , \quad \eta^{(j)a} = K^{(j)a}{}_b \xi^b$$

Geodesic integrability of Kerr-NUT-AdS spacetime in D-dimension

Futhermore, $n-1$ Killing-Yano tensors $f^{(j)}$ are constructed from a single rank-2 CKY tensor h.

$$f^{(j)} = * h^{(j)} \ , \quad h^{(j)} = h \wedge h \wedge \cdots \wedge h$$

(j times)

$\Rightarrow$ **CKY tensor is the most fundamental.**

Conformal Killing-Yano tensor

Def. For a rank-n anti-symmetric tensor h, when there exists a rank-$(n-1)$ anti-symmetric tensor ξ such that

$$\nabla_{(a}h_{b)c_1\dots c_{n-1}} = g_{ab}\xi_{c_1\dots c_{n-1}} + \sum_{i=1}^{n-1}(-1)^i g_{c_i(a}\xi_{b)c_1\dots\hat{c}_i\dots c_{n-1}} \,,$$

h is called **conformal Killing-Yano (CKY) tensor** and ξ is called associated tensor of h,

$$\xi_{c_1\dots c_{n-1}} = \frac{1}{D-n+1}\nabla^a h_{ac_1\dots c_{n-1}} \,.$$

In particular, if $\xi = 0$ then h is called **Killing-Yano (KY) tensor**.

Tachibana and Kashiwada (1968)

Closed conformal Killing-Yano tensor

Def. Let h be a p-form. If h satifies the equations

$$\nabla_X h = -\frac{1}{D-p+1} X^\flat \wedge \delta h \qquad \text{and} \qquad dh = 0$$

for $^\forall X \in TM$, then we call h **rank-p closed conformal Killing-Yano (CCKY) tensor**.

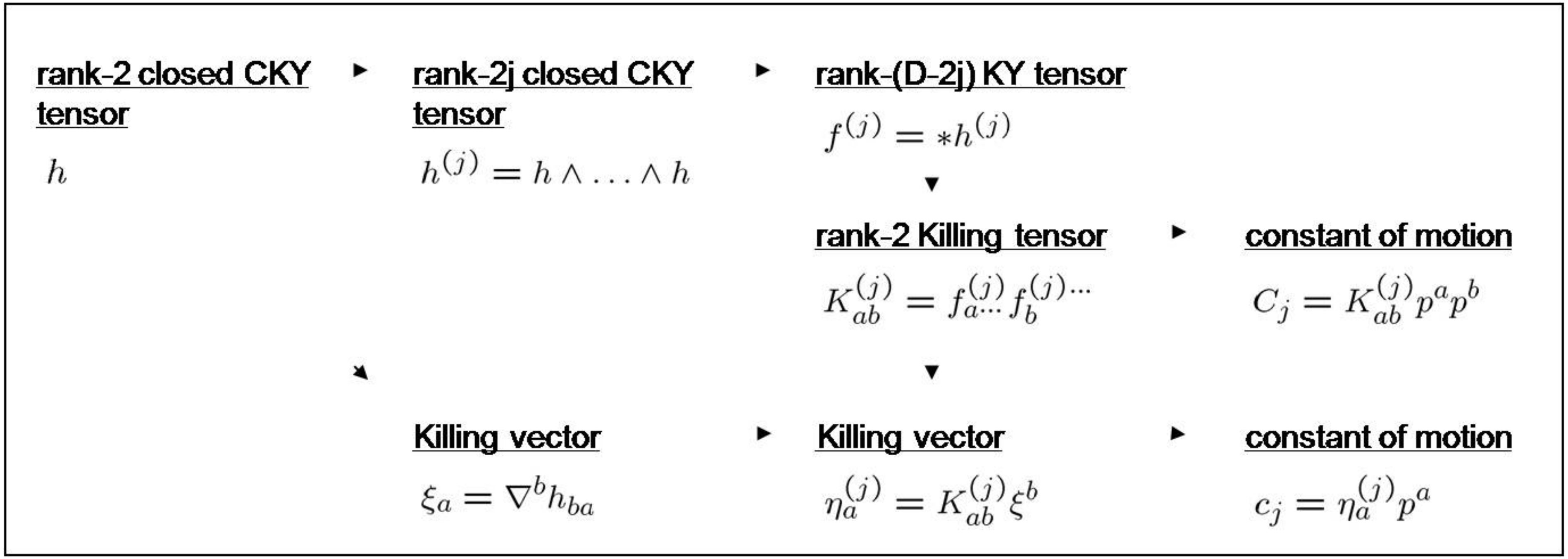
∇ : Levi-Civita connection, $\wedge$: wedge product, d : exterior derivative,
δ : coderivative operator $(= *d*)$

	Killing vector	conformal Killing vector
symmetric	Killing tensor	conformal Killing tensor
anti-symmetric	Killing-Yano tensor	conformal Killing-Yano tensor

Prop. Suppose that a spacetime admits a rank-2 non-degenerate CCKY tensor. Then the geodesic equation is integrable, namely there are the dimension number of Killing vectors and rank-2 Killing tensors totally.

Houri, Oota and Yasui (2007), Krtous, Frolov and Kubizňák (2008)

# Dimension	# Killing vector	# Killing tensor
$2n$	n	n
$2n+1$	$n+1$	n



We can prove that

$$\{C_i, C_j\}_P = 0 \ , \quad \{C_i, c_j\}_P = 0 \ , \quad \{c_i, c_j\}_P = 0 \ .$$

Theor. We assume that a spacetime admits a rank-2 non-degenerate CCKY tensor. Then such a spacetime is given only by the metric of Kerr-NUT-AdS type. (Einstein equation is not imposed.)

Houri, Oota and Yasui (2007), Krtous, Frolov and Kubizňák (2008)

Kerr-NUT-AdS-type metric in $D = 2n + \varepsilon$ dimension

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right]^2 + \varepsilon S \left[\sum_{k=0}^{n} A^{(k)} d\psi_k \right]^2$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu} \;, \quad U_\mu = \prod_{\nu \neq \mu} (x_\mu^2 - x_\nu^2) \;, \quad X_\mu = X_\mu(x_\mu) \;,$$

$$A_\mu^{(k)} = \sum_{\substack{1 \leq \nu_1 < \cdots < \nu_k \leq n \\ \nu_i \neq \mu}} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \;, \quad A^{(k)} = \sum_{1 \leq \nu_1 < \cdots < \nu_k \leq n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \;, \quad A_\mu^{(0)} = A^{(0)} = 1 \;,$$

$$S = \frac{c}{A^{(n)}} \;, \quad c = \text{const.} \;.$$

Solutions admitting a rank-2 closed CKY tensor

4-dimensional black hole metric

	mass	a.m.	NUT	Λ
Schwarzschild (1915)	○			
Kerr (1963)	○	○		
Carter (1968)	○	○		○
Plebanski (1975)	○	○	○	○

Higher-dimensional ($D \geq 4$) black hole metric

	mass	a.m.s	NUTs	Λ
Tangherlini (1963)	○			
Myers, Perry (1986)	○	○		
Gibbons, Lü, Page, Pope (2004)	○	○		○
Chen, Lü, Pope (2006)	○	○	○	○

4-dimensional Kerr-Newman metric

Theorem

We assume that D-dimensional spacetime (M, g) admits a single rank-2 closed CKY tensor. Then (M, g) is the only generalized Kerr-NUT-AdS spacetime. (Here Einstein condition is not imposed.)

Houri, Oota and Yasui (2008)

rank-2 non-degenerate closed conformal Killing-Yano tensor	unique $\Longrightarrow$	Kerr-NUT-AdS metric
rank-2 closed conformal Killing-Yano tensor	unique $\Longrightarrow$	generalized Kerr-NUT-AdS metric

$$
\begin{aligned}
h &= \sum_{\mu=1}^{n} x_\mu \, e^\mu \wedge e^{n+\mu} + \xi_1 \sum_{\alpha_1=1}^{m_1} e^{\alpha_1} \wedge e^{m_1+\alpha_1} + \cdots + \xi_N \sum_{\alpha_N=1}^{m_N} e^{\alpha_N} \wedge e^{m_N+\alpha_N} \\
&= \sum_{\mu=1}^{n} x_\mu \, e^\mu \wedge e^{n+\mu} + \sum_{j=1}^{N} \left(\xi_j \sum_{\alpha_j=1}^{m_j} e^{\alpha_j} \wedge e^{m_j+\alpha_j} \right)
\end{aligned}
$$

It is convenient to write eigenvalues of a rank-2 closed CKY tensor by introducing $Q^a{}_b = -h^a{}_c h^c{}_b$.

$$V^{-1}(Q^a{}_b)V = \{\underbrace{-x_1^2, -x_1^2, \dots, -x_n^2, -x_n^2}_{2n}, \underbrace{-\xi_1^2, \dots, -\xi_1^2}_{2m_1}, \dots, \underbrace{-\xi_N^2, \dots, -\xi_N^2}_{2m_N}, \underbrace{0, \dots, 0}_{K}\}$$

Then D-dimensional generalized Kerr-NUT-AdS metric is

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{P_\mu} + \sum_{\mu=1}^{n} P_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} \theta_k\right]^2 + \sum_{j=1}^{N} \prod_{\mu=1}^{n} (x_\mu^2 - \xi_j^2) g^{(j)} + \Big(\prod_\mu x_\mu^2\Big) g^{(0)}$$

where $g^{(0)}$ is arbitrary K-dim. metric and $g^{(j)}$ is $2m_j$-dim. Kähler metric with the Kähler form $\omega^{(j)}$.

$$P_\mu = \frac{X_\mu(x_\mu)}{x_\mu^K \prod_{j=1}^{N}(x_\mu^2 - \xi_j^2)^{m_j} \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n}(x_\mu^2 - x_\nu^2)}\,, \quad A_\mu^{(k)} = \sum_{\substack{1\leq\nu_1<\cdots<\nu_k\leq n \\ \nu_i\neq\mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2\,,$$

$$d\theta_k + 2\sum_{j=1}^{N} (-1)^{n-k} \xi_j^{2n-2k-1} \omega^{(j)} = 0\ .$$

- D**-dimensional generalized Kerr-NUT-AdS metric**

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{P_\mu} + \sum_{\mu=1}^{n} P_\mu \left[\sum_{k=0}^{n-1} A_\mu^{(k)} \theta_k \right]^2 + \sum_{j=1}^{N} \prod_{\mu=1}^{n} (x_\mu^2 - \xi_j^2) g^{(j)} + \Big(\prod_{\mu} x_\mu^2 \Big) g^{(0)}$$

where

$$P_\mu = \frac{X_\mu(x_\mu)}{x_\mu^K \prod_{j=1}^{N} (x_\mu^2 - \xi_j^2)^{m_j} \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{n} (x_\mu^2 - x_\nu^2)} \ , \quad A_\mu^{(k)} = \sum_{\substack{1 \le \nu_1 < \cdots < \nu_k \le n \\ \nu_i \neq \mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2 \ ,$$

$$d\theta_k + 2 \sum_{j=1}^{N} (-1)^{n-k} \xi_j^{2n-2k-1} \omega^{(j)} = 0 \ .$$

When $g^{(0)}$ is K-dim. Einstein metric, $g^{(j)}$ is $2m_j$-dim. Einstein-Kähler metric with the Kähler form $\omega^{(j)}$ and

$$X_\mu = x_\mu \int dx_\mu \chi(x_\mu) x_\mu^{K-2} \prod_{i=1}^{N} (x_\mu^2 - \xi_i^2)^{m_i} + d_\mu x_\mu$$

where

$$\chi(x_\mu) = \sum_{i=0}^{n} \alpha_i x^{2i} \ , \quad \alpha_0 = (-1)^{n-1} \lambda^{(0)}$$

This metric satisfies Einstein equation $R_{ab} = -(D-1)\alpha_n \, g_{ab}$.

generalized Kerr-NUT-AdS metric

Spacetime described by generalaized Kerr-NUT-(A)dS metric has a fiber bundle structure such that

base space :	direct products of n Kähler-Einstein spaces
fiber :	Kerr-NUT-AdS spacetime

Such a structure of spacetime appears in higher dimensional black holes with equal angular momenta.

For example, $(2m+3)$-dimensional Kerr-AdS black hole metric with equal angular momenta has the follwing structure:

base space :	$CP(m)$
fiber :	3-dimensional Kerr-NUT-AdS spacetime

Charged rotating black holes in supergravity theory

Let us consider the following (Einstein-frame) Lagrangian :

$$\mathcal{L}_D = R * 1 + \frac{1}{2} * d\varphi \wedge d\varphi - X^{-2} * F_{(2)} \wedge F_{(2)} - \frac{1}{2} X^{-4} * H_{(3)} \wedge H_{(3)} \ ,$$

where

$$X = e^{-\varphi/\sqrt{2(D-2)}} \ , \quad F_{(2)} = dA_{(1)} \ , \quad H_{(3)} = dB_{(2)} - A_{(1)} \wedge dA_{(1)} \ .$$

This is a system consisted of gravitational field g, scalar field φ, 1-form potential $A_{(1)}$ and 2-form potential $B_{(2)}$.

* This Lagrangian appears as a truncation of the bosonic part of various supergravity theories, for example of heterotic supergravity compactified on a torus, and also as the ungauged limit of truncations of certain gauged supergravity theories.

Charged Kerr-NUT solution in $D = 2n + \varepsilon$ dimension

Chow (2008)

$$g_D = H^{2/(D-2)}\left\{\sum_{\mu=1}^{n}\frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu\Big(\mathcal{A}_\mu - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\Big)^2 + \varepsilon S\Big(\mathcal{A} - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\Big)^2\right\}$$

$$X = H^{-1/(D-2)}\ ,\quad A_{(1)} = \sum_{\mu=1}^{n}\frac{2N_\mu sc}{HU_\mu}\mathcal{A}_\mu\ ,\quad B_{(2)} = d\psi_0 \wedge \Big(\sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\Big)\ .$$

Here the 1-forms and the functions are

$$\mathcal{A}_\mu = \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k\ ,\quad \mathcal{A} = \sum_{k=0}^{n} A^{(k)} d\psi_k\ ,\quad H = 1 + \sum_{\mu=1}^{n}\frac{2N_\mu s^2}{U_\mu}\ ,\quad N_\mu = m_\mu x_\mu^{1-\varepsilon}\ ,$$

$$Q_\mu = \frac{X_\mu}{U_\mu}\ ,\quad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(x_\mu^2 - x_\nu^2)\ ,\quad X_\mu = X_\mu(x_\mu)\ ,$$

$$A_\mu^{(k)} = \sum_{\substack{1\le\nu_1<\cdots<\nu_k\le n\\ \nu_i\neq\mu}} x_{\nu_1}^2\cdots x_{\nu_k}^2\ ,\quad A^{(k)} = \sum_{1\le\nu_1<\cdots<\nu_k\le n} x_{\nu_1}^2\cdots x_{\nu_k}^2\ ,\quad A_\mu^{(0)} = A^{(0)} = 1\ ,$$

$$S = \frac{c}{A^{(n)}}\ ,\quad c = \text{const.}\ .$$

From the viewpoint of hidden symmetries, it is convenient to use a string-frame metric g_s which is conformally related to a Einstein-frame metirc g_E by

$$g_E = X^{-2} g_s \ .$$

Then it leads to the string-frame Lagrangian

$$\mathcal{L}_D = X^{-(D-2)} \Big\{ * R_s + \frac{1}{2} * d\varphi \wedge d\varphi - *F_{(2)} \wedge F_{(2)} - \frac{1}{2} * H_{(3)} \wedge H_{(3)} \Big\} \ .$$

In string frame the metric g_s is written as

$$g_s = \sum_{\mu=1}^{n} (e^{\mu} e^{\mu} + e^{\hat{\mu}} e^{\hat{\mu}}) + \varepsilon e^0 e^0 \ ,$$

where the vielbeins for Chow's solution are

$$e^{\mu} = \frac{dx_{\mu}}{\sqrt{Q_{\mu}}} \ , \quad e^{\hat{\mu}} = \sqrt{Q_{\mu}} \Big(\mathcal{A}_{\mu} - \sum_{\nu=1}^{n} \frac{2N_{\nu} s^2}{HU_{\nu}} \mathcal{A}_{\nu} \Big) \ , \quad e^0 = \sqrt{S} \Big(\mathcal{A} - \sum_{\nu=1}^{n} \frac{2N_{\nu} s^2}{HU_{\nu}} \mathcal{A}_{\nu} \Big) \ .$$

As we find soon, there are $n + \varepsilon$ Killing vectors given by $\partial/\partial\psi_k$, $k = 0, \ldots, n-1+\varepsilon$. In addition, it is known that there are $n-1$ rank-2 Killing tensors $K^{(j)}$ given by

$$K^{(j)} = \sum_{\mu=1}^{n} A_{\mu}^{(j)} (e^{\mu} e^{\mu} + e^{\hat{\mu}} e^{\hat{\mu}}) + \varepsilon A^{(j)} e^0 e^0 \ ,$$

where $j = 1, \ldots, n-1$. Consequently, there are in Einstein frame $n-1$ rank-2 conformal Killing tensors $Q^{(j)}$ given by

$$Q^{(j)} = H^{2/(D-2)} K^{(j)} \ .$$

Generalized Closed Conformal Killing-Yano Tensor

Kubizňák, Kunduri and Yasui (2008)

Def. Let h be a p-form and T be a 3-form. If a pair of (h, T) satifies the equations

$$\nabla^T_X h = -\frac{1}{D-p+1} X^\flat \wedge \delta^T h \quad \text{and} \quad d^T h = 0$$

for $^\forall X \in TM$, then we call h **rank-p generalized closed conformal Killing-Yano (GCCKY) tensor with 3-form T.**

∇ : Levi-Civita connection, $\wedge$: wedge product, d : exterior derivative, δ : coderivative operator $(= *d*)$, $\lrcorner$: inner product

$$\nabla^T_X h := \nabla_X h - \frac{1}{2}\sum_a (X \lrcorner e_a \lrcorner T) \wedge (e_a \lrcorner h) \ ,$$
$$d^T h := \sum_a e^a \wedge \nabla^T_{e_a} h \ , \quad \delta^T h := -\sum_a e_a \lrcorner \nabla^T_{e_a} h \ .$$

Prop. Let $(M,\ g)$ be a D-dimensional spacetime. If $(M,\ g)$ admits a rank-2 non-degenerate GCCKY tensor h with a 3-form T then there exist $n-1$ rank-2 Killing tensors $K^{(j)}$ $(j=1,\ldots,n-1)$.

$$h=\sum_{\mu=1}^{n} x_\mu\, e^\mu \wedge e^{\hat{\mu}}\ ,\quad K^{(j)}=\sum_{\mu=1}^{n} A^{(j)}_\mu (e^\mu e^\mu + e^{\hat{\mu}} e^{\hat{\mu}}) + \varepsilon A^{(j)} e^0 e^0$$

$\{e^a\}$: orthonormal basis

Difference 1

With $T=0$ all commutators of Killing tensors vanish automatically, but with $T\neq 0$ it doesn't occur.

Difference 2

With $T=0$ rank-2 CCKY tensor leads to $n+\varepsilon$ Killing vectors, but it doesn't occur with $T\neq 0$.

For geodesic integrability we need some additional condition for T.

Charged Kerr-NUT solution in $D = 2n + \varepsilon$ dimension

Chow (2007)

$$g_D = H^{2/(D-2)}\left\{ \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu\Big(\mathcal{A}_\mu - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\Big)^2 + \varepsilon S\Big(\mathcal{A} - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\Big)^2 \right\}$$

$$X = H^{-1/(D-2)} \ , \quad A_{(1)} = \sum_{\mu=1}^{n} \frac{2N_\mu sc}{HU_\mu}\mathcal{A}_\mu \ , \quad B_{(2)} = d\psi_0 \wedge \Big(\sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\mathcal{A}_\nu\Big) \ .$$

Here the 1-forms and the functions are

$$\mathcal{A}_\mu = \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \ , \quad \mathcal{A} = \sum_{k=0}^{n} A^{(k)} d\psi_k \ , \quad H = 1 + \sum_{\mu=1}^{n} \frac{2N_\mu s^2}{U_\mu} \ , \quad N_\mu = m_\mu x_\mu^{1-\varepsilon} \ ,$$

$$Q_\mu = \frac{X_\mu}{U_\mu} \ , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\mu^2 - x_\nu^2) \ , \quad X_\mu = X_\mu(x_\mu) \ ,$$

$$A_\mu^{(k)} = \sum_{\substack{1\le\nu_1<\cdots<\nu_k\le n \\ \nu_i\neq\mu}} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \ , \quad A^{(k)} = \sum_{1\le\nu_1<\cdots<\nu_k\le n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \ , \quad A_\mu^{(0)} = A^{(0)} = 1 \ ,$$

$$S = \frac{c}{A^{(n)}} \ , \quad c = \text{const.} \ .$$

For Chow's solution in string frame, we find a rank-2 GCCKY tensor

$$h = \sum_{\mu=1}^{n} x_\mu \, e^\mu \wedge e^{\hat{\mu}}$$

with a 3-form

$$h = \sum_{\rho=1}^{n} \sum_{\substack{\mu=1 \\ \mu\neq\rho}}^{n} \sqrt{Q_\mu}(\partial_\rho \ln H) \, e^\rho \wedge e^{\hat{\mu}} \wedge e^{\hat{\rho}}$$
$$- \varepsilon \sum_{\rho=1}^{n} \sqrt{S}(\partial_\rho \ln H) \, e^\rho \wedge e^{\hat{\rho}} \wedge e^0 + \varepsilon \sum_{\rho=1}^{n} \frac{f}{x_\rho} \, e^\rho \wedge e^{\hat{\rho}} \wedge e^0 \ ,$$

where f is an arbitrary function.

When $f = 0$, we can write the 3-form T as

$$T = k \ X^{D-6} H_{(3)} \ ,$$

where $H_{(3)} = dB_{(2)} - A_{(1)} \wedge dA_{(1)}$ and k is some constant.

Thus in string frame there are $n-1$ rank-2 Killing tensors $K^{(j)}$ given by

$$K^{(j)} = \sum_{\mu=1}^{n} A_{\mu}^{(j)} (e^{\mu} e^{\mu} + e^{\hat{\mu}} e^{\hat{\mu}}) + \varepsilon A^{(j)} e^{0} e^{0} \ ,$$

where $j = 1, \ldots, n-1$.

One can check that the torsion T satisfies a condition on which Killing tensors are mutually commuting.

Consequently, there are in Einstein-frame $n-1$ rank-2 conformal Killing tensors $Q^{(j)}$ given by

$$Q^{(j)} = H^{2/(D-2)} K^{(j)} \ .$$

Summary

• We have introduced the notion of (G)CCKY tensor and showed the relation to geodesic integrabilty.

• By imposing a rank-2 non-degenerate CCKY tensor we have constructed a metric ansatz which has geodesic integrability and examined solutions to (vacuum) Einstein equation.

• We have considered the charged Kerr-NUT spacetime given by Chow's solution, which includes ...

Kerr-Sen black hole in 4 dimension,

charged rotating black hole with $\delta_1 = \delta_2$ **and** $\delta_3 = 0$ **in 5-dim. U(1)**^{3} **ungaged supergravity,** etc.

• We have understood that the Killing tensors for the charged Kerr-NUT spacetime (in string frame) come from a rank-2 GCCKY tensor.

Discussion - small questions -

- Properties of charged Kerr-NUT spacetime

  relation between GCCKY tensor and Killing vectors?

  separability of Klein-Gordon equation, Dirac equation, etc?

- How about other known solutions?

- General properties of GCCKY tensor

  What's the condition that Killing vectors can be constructed from a GCCKY tensor?

  Are symmetry operators which commute with raplacian, Dirac operator, etc constructed from it?

Discussion - large questions -

- Can we construct new solution?

  e.g. vacuum black hole solutions, Houri, Oota and Yasui (2007)
  black hole solution in 5-dim. minimal gauged supergravity Ahmedov and Aliev (2009)

- What's the physical meaning?

- Why many known black hole solutions have such a symmetry?

  e.g. black ring solution doesn't admit CCKY tensor.

It seems to me that these questions are deeply related each other...