$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$

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SML Graph Theory

Claim

For a finite graph G, $rad(G) \leq diam(G) \leq 2rad(G)$

Left Inequality

The proof that $\mathrm{rad}(G) \leq \mathrm{diam}(G)$ is trivial. By definition, rad and diam are defined by:

$$\mathrm{rad}(G) = \min_{x \in V} \mathrm{ecc}(x) \tag{1}$$
$$\mathrm{diam}(G) = \max_{x \in V} \mathrm{ecc}(x) \tag{2}$$

Hence (1) $\leq$ (2) by definition of min and max. $\square$

Right Inequality

In order to prove the right inequality, it is helpful to use the triangle inequality as defined for distances in graphs. As this is not covered in the lecture notes (yet), let us give a quick proof.

Triangle Inequality

Claim. Given 3 vertices $a, b, c \in V$, we have that $\mathrm{d}(a, c) \leq \mathrm{d}(a, b) + \mathrm{d}(b, c)$

Proof. If there is no path $a...b$ or path $b...c$, then $\mathrm{d}(a, b)$ or $\mathrm{d}(b, c) = \infty$, so the inequality is trivially satisfied. Otherwise, we have that there is a path $a...b$ of length $\mathrm{d}(a, b)$ and a path $b...c$ of length $\mathrm{d}(b, c)$. We may compose these two paths to form a longer walk $a...b...c$ of length $\mathrm{d}(a, b) + \mathrm{d}(b, c)$. However, the distance $\mathrm{d}(a, c)$ is the length of the shortest path $a...c$, so that any walk $(a...c)$ is equal in length or longer than the shortest path. One such walk is the walk defined by the composition of the paths $a...b$ and $b...c$, which completes the proof. $\square$

Now we prove that $\mathrm{diam}(G) \leq 2\mathrm{rad}(G)$.

Proof. Let $a...b$ be the longest path in G. Then there always exists a vertex c that is a central vertex. Then by triangle inequality, we have

$$\mathrm{d}(a,b) \leq \mathrm{d}(a,c) + \mathrm{d}(c,b) \tag{3}$$

Since $a...b$ is the longest path, $\mathrm{d}(a,b) = \mathrm{diam}(G)$. Now

$$\mathrm{d}(a,c) \leq \max_{x \in V} \mathrm{d}(x,c) = \mathrm{ecc}(c) = \mathrm{rad}(G) \tag{4}$$

where the rightmost equality holds since c is central.

The same argument holds for $\mathrm{d}(c,b)$.

Then we may finish the proof as follows

$$\mathrm{diam}(G) = \mathrm{d}(a,b) \leq \mathrm{d}(a,c) + \mathrm{d}(c,b) \leq \mathrm{rad}(G) + \mathrm{rad}(G) = 2\mathrm{rad}(G) \tag{5}$$

as required. $\square$