On Eulerian Graphs

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This report aims to prove the sufficient and necessary condition for Eulerian graphs (**Theorem 1.25**) and give a classical example related to the Eulerian graph, namely the problem of The Seven Bridges of Königsberg.

We first define a convenient map for later use. For a given undirected graph G, let $\mathcal{CT}(G)$ denote the set of all closed trails in that graph, and $\mathcal{P}(V)$ denote the power set of the vertices of the graph G.

We first set

$$\mathcal{E}_v := \{e \in E \mid i(e) = (v, y)\ \forall y\},$$

and define the map A as:

$$\begin{aligned} \mathcal{A} : \mathcal{CT}(G) &\longrightarrow \mathcal{P}(V) \\ C &\longmapsto \{v \in V \mid \mathcal{E}_v \subset C\}. \end{aligned}$$

In simple terms, the map $\mathcal{A}$ takes a closed trail C and outputs the vertices that have no edges that do not belong to C. We shall also define another map $\mathcal{D} : \mathcal{CT}(G) \to \mathcal{P}(E)$, which is defined by

$$\mathcal{D}(C) = \{e \in E \mid e \in C\}.$$

While the variable for both maps are C, both maps have an an implicit dependence on V and E through G.

1 Condition for Partitioning of a Graph into Cycles

The aim of this section is to present a preliminary result that will be useful to prove **Theorem 1.25**.

Theorem 1

The edge set E of a finite, connected, undirected graph $G = (V, E)$ can be partitioned into cycles C_i, where no two cycle share an edge, if and only if $\forall v \in V$, $\deg(v)$ is even and greater than or equal to 2.

Forward Statement

For all vertices in a cycle, one edge is used to enter the vertex and one edge is used to exit. As such, (edges in) a cycle contributes 2 to the degree of each vertex the cycle passes through (since no repeated vertices are allowed in a cycle, and a cycle is closed).

Since the edge set E of the graph G can be fully partitioned into cycles with no two cycles sharing an edge, and a cycle contributes two to the degree of all vertices it passes through, it means that any vertex v in the graph G must have a degree equal to two times the number of cycles that passes through it.

As such, the degree of any vertex v in such a graph must be even. Furthermore, since the graph is connected, there are no vertices with degree 0, so all vertices of this graph will have degree ≥ 2. □

Backward Statement

If the graph only consists of one point and loop(s), then the proof is trivial (since the loop(s) will be the cycle(s)).

Therefore, we consider the case when $|G| \geq 2$. We first start on any vertex x in the graph. Since this graph is connected and $|G| \geq 2$, there exists an edge e such that $i(e) = (x, y)$ where $y \in N(x)$ is a neighbor of the point x. Thus, (x, e, y) represents a trail from x to y.

Since $\deg(y) \geq 2$ and $\deg(y)$ is even, there must exist some edge $e_1 \neq e$ such that $i(e_1) = (y, z)$ for at least one $z \in V$ (it is possible that $z = x$). We then continue our trail through the edge e_1.

If $z = x$, then our trail is a cycle. Otherwise, we continue our trail from z via another edge that we have not yet passed through until we arrive at a vertex v that we have passed through before. The segment of the trail between the two times v appears must then form a cycle. We call this cycle C_1.

If $E \subset C_1$, then our proof is complete. If not, then we consider the graph G' (not necessarily a connected graph, but is a disjoint union of connected graphs) defined by

$$G' = (V', E') = (V \backslash \mathcal{A}(C_1), E \backslash \mathcal{D}(C_1)) \,.$$

We then start at any vertex of G' and repeat the above procedure, forming another cycle C_2.

If C_1 and C_2 contains all edges of G, i.e.,

$$E \subset (C_1 \cup C_2),$$

then the proof is complete. If not, then we repeat the procedure with the graph

$$G'' = (V'', E'') = (V' \backslash \mathcal{A}'(C_2), \mathcal{D}'(C_2)) \,.$$

Here, the primes on $\mathcal{A}$ and $\mathcal{D}$ is there to indicate that their domain is not $\mathcal{CT}(G)$ but is $\mathcal{CT}(G')$.

We then repeat this procedure until we get n cycles $C_1, \ldots, C_n$ such that

$$E \subset \bigcup_{i=1}^{N} C_i.$$

By construction, no edges will be shared between any two cycles. □

We now illustrate this process with the graph G below:

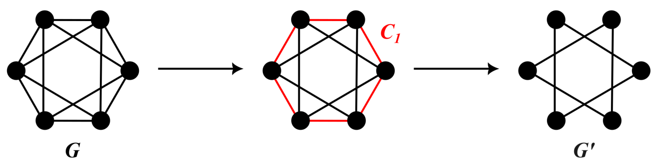

Figure 1. The Graph G (left), a cycle C_1 in graph G, drawn in red (middle), and the graph G' obtained from G by removing C_1 (right).

We then iterate the procedure two more times to obtain:

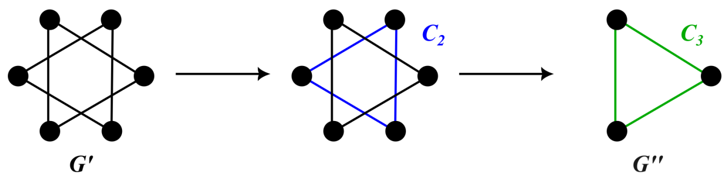

Figure 2. The Graph G' (left), cycle C_2 in graph G, drawn in blue (middle), and the graph G'' obtained from G' by removing C_2 and the cycle C_3 containing all its edges (right).

It is clear that the edge set E of the graph G is partitioned into the three cycles C_1, C_2 and C_3:

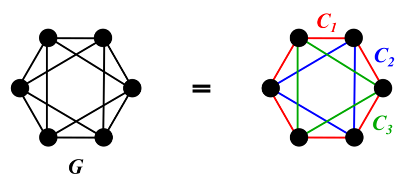

Figure 3. The edge set of the graph G partitioned into three cycles.

2 Proof of Theorem 1.25

Theorem 1.25 (Eulerian Graphs)

A connected, undirected and finite graph is Eulerian if and only if every vertex has even degree.

Forward Statement

If G is Eulerian, it possesses an Eulerian tour C. We start at a vertex x, and follow the Eulerian tour C. Each time we pass a vertex, we use one edge going towards the vertex and one edge exiting it. As we only use an edge exactly once, the degree of a vertex in G is simply two times the number of times we pass this vertex in the Eulerian tour C. Furthermore, since C is closed, then the degree of the vertex x is also even. $\square$

Backward Statement

If the graph only consists of one point and loops, then the Eulerian circuit can be constructed by going through all the loops (without going through any loop twice).

We now consider the case when $|G| \geq 2$. We start from any vertex x_0. Since G is connected and $|G| \geq 2$, there exists an edge e such that $i(e) = (x_0, y)$ and $x_0 \neq y$. Since $\deg(y) \geq 2$ and is even, there must exist another edge $e_1 \neq e$ such that $i(e_1) = (y, z)$ for some vertex z. We continue this trail until we return back to x_0 (note here that we can visit the same vertex twice, since this is a trail and not a path). This is always possible since x_0 is in a cycle by **Theorem 1**. We call this closed trail C_1. If $E \subset C_1$, then C_1 is an Eulerian tour, and our proof is complete. If not, consider the graph G' defined by

$$G' = (V', E') = (V \backslash \mathcal{A}(C_1), E \backslash \mathcal{D}(C_1)) .$$

Since G is connected, then there exists a vertex x_1 that belongs in both G' and C_1. Since we removed a closed trail C_1, the degrees of all vertices in G' are still even numbers. We then repeat the process from x_1 to obtain another closed trail C_2.

If $E \subset (C_1 \cup C_2)$, then we can construct the Eulerian trail by the process shown below. If not, we repeat this process from a vertex x_2 belonging to both

$$G'' = (V'', E'') = (V' \backslash \mathcal{A}'(C_2), E' \backslash \mathcal{D}'(C_2))$$

and $C_1 \cup C_2$ to obtain another closed trail C_3. We repeat this process until we obtain n closed trails $C_1, \ldots, C_n$ such that all edges are used, i.e.,

$$E \subset \bigcup_{i=1}^{n} C_i.$$

By construction, no two closed tours will share an edge. Thus, each edge will only be used once.

We then construct our Eulerian tour by starting at x_0, going to x_1 along C_0, and then traversing C_1. Then, we go along C_0 (or C_1, in case when $x_2 \notin C_0$) from x_1 to x_2 and then traversing C_2. Since the graph is finite, at one point, we would reach the starting vertex x_0, completing the Eulerian tour. As G possesses an Eulerian tour, G is an Eulerian Graph. $\square$

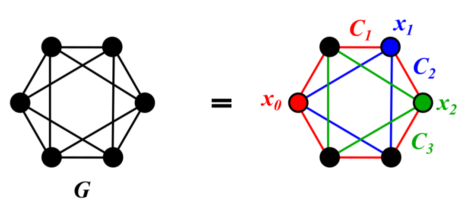

Figure 4. The Eulerian tour of graph G, generated by starting at x_0, moving to x_1 along C_1, then following C_2 back to x_1, following C_1 to x_2, following C_3, and going back to x_0 through C_1.

3 The Seven Bridges of Konigsberg

This section aims to introduce the famous problem that gave rise to the idea of graph theory.

The city of Königsberg in Prussia was separated into two by the Pregel river, with two islands floating between the two opposite sides of the city. Seven bridges in total connect the two islands and the two large landmasses.

This city was made famous by the formulation of a problem: is there a trail that goes through every bridge in the city exactly once? In 1736, Leonhard Euler solved this problem by showing that no such trail exists. The development of a method of analyzing this problem by Euler led to the foundation of graph theory being laid.

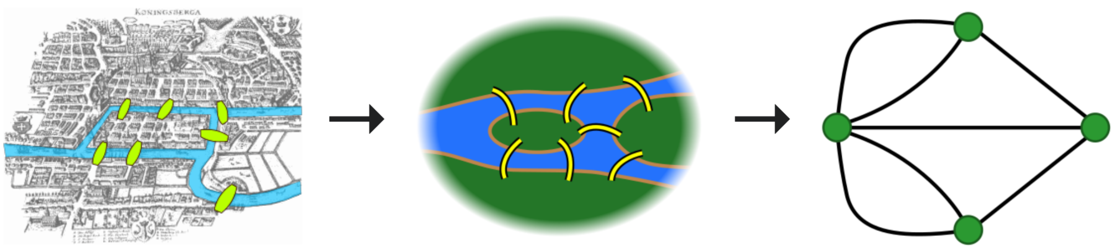

Figure 5. The seven bridges in Konigsberg (left), a simplified picture (middle), and the equivalent graph G, with landmasses as vertices and bridges as edges (right). Picture taken from Wikipedia.

We will now solve this problem using graph theory.

We observe that any path taken inside an island/landmass does not have any effect on the result, and we only need to consider the time(s) we cross a bridge. As such, it is appropriate to draw a graph where the landmasses are taken as vertices and the bridges as edges (Figure 5, right).

We observe that all vertices has odd degree, and as such, it does not satisfy the conditions for an Eulerian graph. As such, no Eulerian tour exists in this graph, and there is no closed trail that contains every single edge in this graph.

While this does not answer the original problem of having a (not necessarily closed) trail that traverses every edge in this graph, this is still an interesting result to mention when considering the historical significance of this problem.