Graphic Sequence and Havel-Hakimi Theorem

FIRDAUS Rafi Rizqy (062101889)

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In this report, we investigate the Havel-Hakimi algorithm's capacity to discern graphical sequences that represent simple graphs. We will review the essential definitions, demonstrate the algorithm's recursive nature, and provide a clear example of its practical application.

Definition 1 ([1], pg. 9)**.** *A non-increasing sequence* $(d_1, d_2, \ldots, d_n)$ *is said to be graphic if it is the degree sequence of some simple graph. That simple graph is said to realize the sequence.*

Remark. *If* $(d_1, d_2, \ldots, d_n)$ *is the degree sequence of a simple graph, then, clearly,* $d_1 \leq n - 1$*.*

Theorem 1 ([1], pg. 9)**.** *Let* $(d_1, d_2, \ldots, d_n)$ *be a graphic sequence, with* $d_1 \geq d_2 \geq \ldots \geq d_n$*. Then there is a simple graph with vertex-set* $\{v_1, \ldots, v_n\}$ *satisfying* $\deg(v_i) = d_i$ *for* $i = 1, 2, \ldots, n$*, such that* v_1 *is adjacent to vertices* $v_2, \ldots, v_{d_1+1}$*.*

Proof. Consider a collection of simple graphs with a vertex set $\{v_1, v_2, \ldots, v_n\}$ where each vertex v_i has a degree d_i. Select graph G such that $r = |N_G(v_1) \cap \{v_2, \ldots, v_{d_1+1}\}|$ is at its maximum, where $N_G(v_1)$ signifies the neighborhood of v_1. If r matches d_1, the proof is evident. When r is less than d_1, it implies the existence of a vertex v_s within the range $2 \leq s \leq d_1 + 1$ that does not share an edge with v_1, and conversely, a vertex v_t where $t > d_1 + 1$ that is adjacent to v_1 given $\deg(v_1) = d_1$. Furthermore, since $\deg(v_s) \geq \deg(v_t)$, we deduce the presence of a vertex v_k adjacent to v_s but not to v_t. Define G' as the graph formed from G by interchanging the edges $v_1 v_t$ and $v_s v_k$ with $v_1 v_s$ and $v_t v_k$ as illustrated in Figure 1. This action keeps all vertex degrees constant and positions $v_s \in N_{G'}(v_1) \cap \{v_2, \ldots, v_{d_1+1}\}$. Consequently, $|N_{G'}(v_1) \cap \{v_2, \ldots, v_{d_1+1}\}| = r + 1$, contradicting the initial selection of G and thus finalizing the argument. □

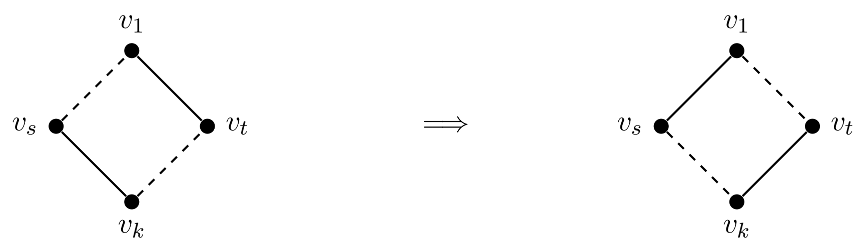

Figure 1: Switching adjacencies while preserving all degrees.

Corollary 1.1 ([1], pg. 10)**.** *(Havel (1955) and Hakimi (1961)) A sequence* $S := (d_1, d_2, \ldots, d_n)$ *of nonnegative integers such that* $d_1 \leq n-1$ *and* $d_1 \geq d_2 \geq \ldots \geq d_n$ *is graphic if and only if the sequence* $S_1 := (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ *is graphic.*

Proof. Sufficiency ($\Leftarrow$): Suppose S_1 is graphical. Then by definition, there exists a graph G_1, of order $n-1$ with degree sequence S_1. Hence we can label $V(G_1)$ with $v_2, v_3, \ldots, v_n$ such that

$$\deg(v_i) = \begin{cases} d_i - 1 & \text{for } i = 2, 3, \ldots, d_1 + 1 \\ d_i & \text{for } i = d_1 + 2, \ldots, n \end{cases}$$

We can construct a new graph G as follows:

- Start with the graph G_1.
- Add a new vertex v_1.
- Add d_1 new edges $v_1 v_i$ for $i = 2, 3, \ldots, d_1 + 1$.

The sketch of the construction is given in **Figure 2**. Thus, in the new graph G the degree of the new sequence is given by

$$\deg(v_i) = d_i \quad \text{for } i = 1 + 2, \ldots, n$$

As a result, the sequence $S := (d_1, d_2, \ldots, d_n)$ is graphic.

Necessity ($\Rightarrow$): Suppose sequence $S := (d_1, d_2, \ldots, d_n)$ is graphical. Then, by **Theorem 1**, there exists a simple graph G with vertex-set $\{v_1, \ldots, v_n\}$ satisfying $\deg(v_i) = d_i$ for $i = 1, 2, \ldots, n$, such that v_1 is adjacent to vertices $v_2, \ldots, v_{d_1+1}$. Thus, by removing vertex v_1 from graph G yields a graph $G - v_1$ which has degree sequence $(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$. Therefore the sequence $S_1 := (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is a graphical sequence. □

Remark. *Corollary 1.1 yields a recursive algorithm that decides whether a non-increasing sequence is graphic.*

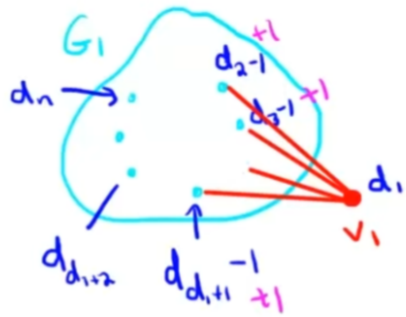

Figure 2: A sketch for the proof of Havel-Hakimi Theorem.

Algorithm 1.1.1: GraphicSequence($d_1, d_2, \ldots, d_n$)
Input: a non-increasing sequence $(d_1, d_2, \ldots, d_n)$.
Output: TRUE if the sequence is graphic, FALSE if it is not.

- If $d_1 \geq n$ or $d_1 < 0$
  - Return FALSE
- Else
  - If $d_1 = 0$
    - Return TRUE
  - Else
    - Let $(a_1, a_2, \ldots, a_{n-1})$ be a non-increasing permutation of $(d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$.
    - Return GraphicSequence($a_1, a_2, \ldots, a_{n-1}$)

Remark. *Given a graphic sequence, the steps of the iterative version can be reversed to construct a graph realizing the sequence. However many zeros you get at the end of the forward pass, start with that many isolated vertices. Then backtrack the algorithm, adding a vertex each time. The following example illustrates these ideas.*

Example. *We start with the sequence* $(3, 3, 2, 2, 2, 1, 1)$. *Figure 1.1.13 illustrates an iterative version of the algorithm GraphicSequence and then illustrates the backtracking steps leading to a graph that realizes the original sequence. The hollow vertex shown in each backtracking step is the new vertex added at that step.*

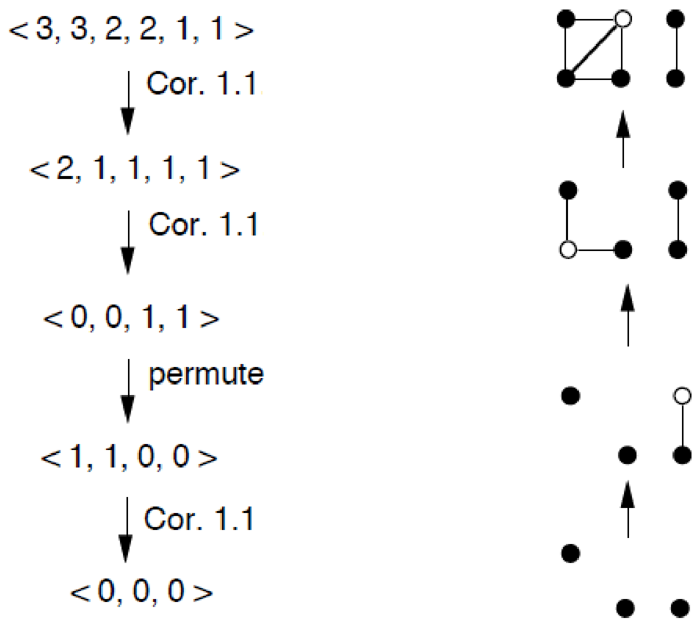

Figure 3: Testing and realizing the sequence $(3, 3, 2, 2, 1, 1)$ ([1], pg. 11).

References

[1] Anderson Gross, Yellen. *Graph Theory and Its Applications*. 2019.