Report 2

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Contents

1 Continuation of the Exercises of Report 1

1.1 Exercise 7

Let $U : \mathcal{H} \to \mathcal{H}$ be an unitary operator. Prove $\sigma(U) \subset S$, where $S = \{z \in \mathbb{C} \mid |z| = 1\}$.

To solve this exercise, we'll prepare the following lemma.

Lemma 1.1.
Suppose $X \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\|X\| < 1$. Then, $(1 - X)^{-1}$ exists and is bounded on $\mathcal{H}$. In particular, $\mathrm{Ker}(1 - X) = \{0\}$.

【Proof】
For each $k \in \mathbb{N} \cup \{0\}$, we have $\|X^k\| = \|XX \cdots X\| \leqq \|X\|\|X\| \cdots \|X\| = \|X\|^k$ since X is bounded. Thus we get

$$\sum_{k=0}^{\infty} \|X^k\| \leqq \sum_{k=0}^{\infty} \|X\|^k < \infty$$

due to $\|X\| < 1$. Therefore $\sum_{k=0}^{\infty} X^k$ is absolutely convergent. Moreover, $\mathcal{B}(\mathcal{H}, \mathcal{H})$ is complete since $\mathcal{H}$ is complete. These facts indicate that $\sum_{k=0}^{\infty} X^k$ is convergent, because absolute convergence implies convergence in complete spaces. As a side note, $\sum_{k=0}^{\infty} X^k$ is called the **Neumann series**.

Now, simple calculation yields

$$\left(\sum_{k=0}^{N} X^k\right)(1 - X) = (1 - X)\left(\sum_{k=0}^{N} X^k\right) = 1 - X^{N+1},$$

and as $N \to \infty$, we have $X^{N+1} \to 0$ since $\|X^{N+1}\| \leqq \|X\|^{N+1} \to 0$, and hence

$$\left(\sum_{k=0}^{\infty} X^k\right)(1 - X) = (1 - X)\left(\sum_{k=0}^{\infty} X^k\right) = 1.$$

Thus $(1 - X)^{-1} = \sum_{k=0}^{\infty} X^k$. To see this is bounded, let $f \in \mathcal{H}$ and observe that

$$\left\|\left(\sum_{k=0}^{N} X^k\right) f\right\| \leqq \left\|\sum_{k=0}^{N} X^k\right\| \|f\| \leqq \sum_{k=0}^{N} \|X\|^k \|f\| = \frac{1 - \|X\|^{N+1}}{1 - \|X\|} \|f\|$$

and letting $N \to \infty$, we get $\|(1 - X)^{-1} f\| \leqq \dfrac{1}{1 - \|X\|}\|f\|$.

$\mathrm{Ker}(1 - X) = \{0\}$ follows by

$$(1 - X)f = 0 \implies f = (1 - X)^{-1}(1 - X)f = (1 - X)^{-1}0 = 0.$$

□

So, let us make a start on the exercise 7.

【Solution of Exercise 7】

It suffices to show that $\mathbb{C} \setminus S \subset \rho(U)$. Let $\lambda \in \mathbb{C} \setminus S$.

First, assume $|\lambda| < 1$. In order to see $\lambda \in \rho(U)$, we have to check $\mathrm{Ker}(U - \lambda \cdot 1) = \{0\}$ and $(U - \lambda \cdot 1)^{-1}$ is bounded. We have $\mathrm{Ker}(1 - \lambda U^{-1}) = \{0\}$ by $\|\lambda U^{-1}\| = |\lambda| \|U^{-1}\| = |\lambda| < 1$ and by the lemma 1.1. Moreover, $\mathrm{Ker}(U) = \{0\}$ since U is unitary. Now, noting that

$$U - \lambda \cdot 1 = U(1 - \lambda U^{-1}),$$

we can see $\mathrm{Ker}(U - \lambda \cdot 1) = \{0\}$ and $(U - \lambda \cdot 1)^{-1} = (1 - \lambda U^{-1})^{-1} U^{-1}$, which is bounded by the lemma 1.1 and by the unitarity of U^{-1}. Thus $\lambda \in \rho(U)$.

Next, assume $|\lambda| > 1$. We have

$$U - \lambda \cdot 1 = -\lambda \left(1 - \frac{1}{\lambda} U\right).$$

Since $\left\|\frac{1}{\lambda} U\right\| = \left|\frac{1}{\lambda}\right| \|U\| < 1$, we see that $\mathrm{Ker}\left(1 - \frac{1}{\lambda} U\right) = \{0\}$ and $\left(1 - \frac{1}{\lambda} U\right)^{-1}$ exists and is bounded by the lemma 1.1. Hence $\mathrm{Ker}(U - \lambda \cdot 1) = \{0\}$ and $(U - \lambda \cdot 1)^{-1} = -\frac{1}{\lambda} \left(1 - \frac{1}{\lambda} U\right)^{-1}$ is bounded. Therefore, $\lambda \in \rho(U)$. □

2 The Cayley Transform

This section 2 will introduce a special operator, the **Cayley transform**. Let $\mathcal{H}$ be a Hilbert space on $\mathbb{C}$.

Lemma 2.1.

Suppose $\mathcal{H} \neq \{0\}$, and let $A : D(A) \to \mathcal{H}$ be a bounded linear operator. If $D(A) = \mathcal{H}$ and $\langle Af, f \rangle = 0$ for all $f \in D(A)$, then $A = 0$.

【Proof】

We have to show that $Af = 0$ for all $f \in D(A)$. Let $f \in D(A)$, and put $g := Af$. Now, fix $c \in \mathbb{C}$ arbitrarily. Since $cf + g \in \mathcal{H} = D(A)$, by the supposition, we have $\langle A(cf + g), cf + g \rangle = 0$. Moreover, the supposition gives $\langle Af, f \rangle = \langle Ag, g \rangle = 0$. Thus

$$\begin{aligned} 0 &= \langle A(cf + g), cf + g \rangle \\ &= |c|^2 \langle Af, f \rangle + c \langle Af, g \rangle + \overline{c} \langle Ag, f \rangle + \langle Ag, g \rangle \\ &= c \langle Af, g \rangle + \overline{c} \langle Ag, f \rangle. \end{aligned}$$

Since c is arbitrary, we can consider the cases $c = 1$ and $c = i$. Then, we get

$$\langle Af, g \rangle + \langle Ag, f \rangle = 0$$

and

$$\langle Af, g \rangle - \langle Ag, f \rangle = 0.$$

Adding these two equalities gives us $\langle Af, g\rangle = 0$. Recalling that $g = Af$, we get $\|Af\|^2 = 0$, i.e., $Af = 0$. □

Lemma 2.2.

Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator. Then, $A + i \cdot 1 : D(A) \to \mathcal{H}$ is injective. In particular, $(A + i \cdot 1)^{-1}$ exists with the domain $\mathrm{Ran}(A + i \cdot 1)$ and the codomain $D(A)$.

【Proof】

For each $f \in D(A)$, simple calculation and the self-adjointness of A yield

$$\begin{aligned}
\|(A + i \cdot 1)f\|^2 &= \langle (A + i \cdot 1)f, (A + i \cdot 1)f\rangle \\
&= \langle Af, Af\rangle + \langle Af, if\rangle + \langle if, Af\rangle + \langle if, if\rangle \\
&= \|Af\|^2 - i\langle Af, f\rangle + i\langle f, Af\rangle + \|f\|^2 \\
&= \|Af\|^2 - i\langle Af, f\rangle + i\langle Af, f\rangle + \|f\|^2 \\
&= \|Af\|^2 + \|f\|^2 \\
&\geqq \|f\|^2.
\end{aligned}$$

This shows that for each $f \in D(A)$, $(A + i \cdot 1)f = 0$ implies $f = 0$. Thus $A + i \cdot 1$ is injective and hence $(A + i \cdot 1)^{-1}$ exists with the domain $\mathrm{Ran}(A + i \cdot 1)$ and the codomain $D(A)$. □

Definition 2.3 (Cayley transform).

Suppose $\mathcal{H} \neq \{0\}$, and let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator. The operator $C : \mathrm{Ran}(A + i \cdot 1) \to \mathrm{Ran}(A - i \cdot 1)$ defined by

$$C := (A - i \cdot 1)(A + i \cdot 1)^{-1}$$

is called the **Cayley transform** of A. This is well-defined because $(A + i \cdot 1)^{-1}$ exists with the domain $\mathrm{Ran}(A + i \cdot 1)$ and the codomain $D(A)$ by the lemma 2.2.

Proposition 2.4.

Suppose $\mathcal{H} \neq \{0\}$ and let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator. Then, C is unitary.

【Proof】

Actually, $\mathrm{Ran}(A + i \cdot 1) = \mathrm{Ran}(A - i \cdot 1) = \mathcal{H}$ holds by the self-adjointness of A, so we get $D(C) = \mathcal{H}$ and $\mathrm{Ran}(C) = \mathcal{H}$. I omit the proof of this because the argument is rather long. If you want to check it, see e.g. [9, Theorem 7.23].

First, we'll see that $\|Cf\| = \|f\|$ for all $f \in D(C) = \mathcal{H}$. Let $f \in \mathcal{H}$. For the viewability, put $g := (A + i \cdot 1)^{-1} f$.

$$\begin{aligned}
\|Cf\|^2 &= \|(A - i \cdot 1)(A + i \cdot 1)^{-1} f\|^2 \\
&= \|(A - i \cdot 1)g\|^2 \\
&= \|Ag - ig\|^2 \\
&= \langle Ag - ig, Ag - ig\rangle \\
&= \langle Ag, Ag\rangle + \langle Ag, -ig\rangle + \langle -ig, Ag\rangle + \langle ig, ig\rangle.
\end{aligned}$$

Now, we can see $\langle Ag, -ig\rangle = \langle g, A(-ig)\rangle = \langle g, -iAg\rangle = \langle ig, Ag\rangle$. Similarly, $\langle -ig, Ag\rangle = \langle Ag, ig\rangle$. Thus,

$$\begin{aligned}\|Cf\|^2 &= \langle Ag, Ag\rangle + \langle ig, Ag\rangle + \langle Ag, ig\rangle + \langle ig, ig\rangle \\ &= \langle Ag + ig, Ag + ig\rangle \\ &= \|Ag + ig\|^2 \\ &= \|(A + i\cdot 1)g\|^2 \\ &= \|(A + i\cdot 1)(A + i\cdot 1)^{-1} f\|^2 \\ &= \|f\|^2.\end{aligned}$$

This shows $\|Cf\| = \|f\|$.

Next, we will check that C is unitary. Since $\|Cf\| = \|f\|$ for all $f \in \mathcal{H}$, C is bounded and so is $C^*C - 1$. Now, for all $h \in \mathcal{H}$, we have

$$\begin{aligned}\langle (C^*C - 1)h, h\rangle &= \langle C^*Ch - h, h\rangle \\ &= \langle C^*Ch, h\rangle - \langle h, h\rangle \\ &= \langle Ch, Ch\rangle - \langle h, h\rangle \\ &= \|Ch\|^2 - \|h\|^2 \\ &= 0\end{aligned}$$

and hence $C^*C - 1 = 0$ by the lemma 2.1. Therefore $C^*C = 1$. Moreover, C is injective because $\|Cf\| = \|f\|$ for all $f \in \mathcal{H}$. The injectivity of C and $D(C) = \mathcal{H} = \mathrm{Ran}(C)$ indicate that C^{-1} exists with the domain $\mathcal{H}$ and the codomain $\mathcal{H}$. Hence $C^* = C^*CC^{-1} = 1\cdot C^{-1} = C^{-1}$. Therefore C is unitary. □

3 Parseval's Identity for a $\mathcal{H}$-valued Function and an Operator in $\mathcal{B}(\mathcal{H}, \mathcal{H})$

In this section, let $\mathcal{H}$ be a separable Hilbert space on $\mathbb{C}$, and we aim for **Parseval's identity for a $\mathcal{H}$-valued function and an operator in $\mathcal{B}(\mathcal{H}, \mathcal{H})$** :

$$\sum_{n\in\mathbb{Z}} \left\| B(\widehat{f}(n)) \right\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \|B(f(x))\|^2 \, dx.$$

Lemma 3.1.

Let $B \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $g : [0, 2\pi] \to \mathcal{H}$ be Bochner integrable, i.e., $\int_0^{2\pi} \|g(\theta)\| \, d\theta < \infty$ or $g \in \mathcal{L}^1([0, 2\pi], \mathcal{H})$. Then, $B \circ g$ is Bochner integrable and

$$\int_0^{2\pi} B \circ g(\theta) \, d\theta = B\left(\int_0^{2\pi} g(\theta) \, d\theta \right).$$

This formula can be rewritten to $\int_0^{2\pi} B(g(\theta)) \, d\theta = B\left(\int_0^{2\pi} g(\theta) \, d\theta \right)$, so this lemma says we can interchange an operator and an integral sign.

【Proof】

Bochner integrability of $B \circ g$ is seen by

$$\int_0^{2\pi} \|B \circ g(\theta)\| \, d\theta = \int_0^{2\pi} \|B(g(\theta))\| \, d\theta \leqq \int_0^{2\pi} \|B\| \, \|g(\theta)\| \, d\theta = \|B\| \int_0^{2\pi} \|g(\theta)\| \, d\theta < \infty.$$

Bochner integral can be regarded as the limit of Riemann sum. Setting $0 = c_0 < c_1 < \cdots < c_n = 2\pi$ as a partition of $[0, 2\pi]$ and taking $\gamma_k \in [c_{k-1}, c_k]$ for each k, we get

$$\begin{aligned}\int_0^{2\pi} B \circ g(\theta) \, d\theta &= \lim_{n\to\infty} \sum_{k=1}^{n} B \circ g(\gamma_k)(c_k - c_{k-1}) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} B(g(\gamma_k))(c_k - c_{k-1}) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} B(g(\gamma_k)(c_k - c_{k-1})) \\ &= B\left(\lim_{n\to\infty} \sum_{k=1}^{n} g(\gamma_k)(c_k - c_{k-1})\right) \\ &= B\left(\int_0^{2\pi} g(\theta) \, d\theta\right),\end{aligned}$$

where we put $\sum$ and lim under B by the linearity and the continuity of B. □

Lemma 3.2 (Parseval's identity for $\mathcal{H}$-valued functions).

For $h \in \mathcal{L}^2([0, 2\pi], \mathcal{H})$,

$$\sum_{n\in\mathbb{Z}} \|\widehat{h}(n)\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \|h(x)\|^2 \, dx$$

holds, where $\widehat{h}(n)$ is the n-th Fourier coefficient of h.

【Proof】

Since $\mathcal{H}$ is separable, we can choose $\{e_k\}_{k=1}^{\infty}$ as a countable orthonormal basis of $\mathcal{H}$. For each $k \in \mathbb{N}$, define the function $h_k : [0, 2\pi] \to \mathbb{C}$ by

$$h_k(x) = \langle h(x), e_k \rangle.$$

Each h_k is in $\mathcal{L}^2([0, 2\pi], \mathbb{C})$ because

$$\int_0^{2\pi} |h_k(x)|^2 \, dx \leqq \|e_k\|^2 \int_0^{2\pi} \|h(x)\|^2 \, dx < \infty$$

holds by the Cauchy-Schwarz inequality. Applying Parseval's identity for square integrable complex functions to each h_k, we get

$$\sum_{n\in\mathbb{Z}} |\widehat{h}_k(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |h_k(x)|^2 \, dx. \tag{3.1}$$

For each $n \in \mathbb{Z}$ and each $k \in \mathbb{N}$, noting that we can put the integral symbol into inner product, we can see

$$\begin{aligned}
\widehat{h}_k(n) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} h_k(\theta)\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} \langle h(\theta), e_k\rangle \, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \langle e^{-in\theta} h(\theta), e_k\rangle \, d\theta \\
&= \left\langle \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} h(\theta)\, d\theta, e_k \right\rangle \\
&= \langle \widehat{h}(n), e_k\rangle
\end{aligned}$$

so (3.1) is rewritten to

$$\sum_{n\in\mathbb{Z}} |\langle \widehat{h}(n), e_k\rangle|^2 = \frac{1}{2\pi}\int_0^{2\pi} |h_k(x)|^2\, dx.$$

Moreover, since $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$, we have

$$\left\| \widehat{h}(n) \right\|^2 = \sum_{k=1}^{\infty} |\langle \widehat{h}(n), e_k\rangle|^2$$

and

$$\sum_{k=1}^{\infty} |\langle h(x), e_k\rangle|^2 = \|h(x)\|^2 .$$

Using the formulas above, we get

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \left\| \widehat{h}(n) \right\|^2 &= \sum_{n\in\mathbb{Z}} \sum_{k=1}^{\infty} |\langle \widehat{h}(n), e_k\rangle|^2 \\
&= \sum_{k=1}^{\infty} \sum_{n\in\mathbb{Z}} |\langle \widehat{h}(n), e_k\rangle|^2 \\
&= \sum_{k=1}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} |h_k(x)|^2\, dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{\infty} |h_k(x)|^2\, dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{\infty} |\langle h(x), e_k\rangle|^2\, dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \|h(x)\|^2\, dx.
\end{aligned}$$

$\sum_{n\in\mathbb{Z}} \sum_{k=1}^{\infty} = \sum_{k=1}^{\infty} \sum_{n\in\mathbb{Z}}$ and $\sum_{k=1}^{\infty} \int_0^{2\pi} = \int_0^{2\pi} \sum_{k=1}^{\infty}$ are justified by Tonelli's theorem. $\square$

Theorem 3.3 (Parseval's identity for a $\mathcal{H}$-valued function and an operator in $\mathcal{B}(\mathcal{H}, \mathcal{H})$).

If $f \in \mathcal{L}^2([0,2\pi],\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H},\mathcal{H})$, then $B \circ f \in \mathcal{L}^2([0,2\pi],\mathcal{H})$ and

$$\sum_{n\in\mathbb{Z}} \left\| B(\widehat{f}(n)) \right\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \|B(f(x))\|^2\,dx,$$

where $\widehat{f}(n)$ is the n-th Fourier coefficient of f.

【Proof】

We can see $B \circ f \in \mathcal{L}^2([0,2\pi],\mathcal{H})$ by

$$\int_0^{2\pi} \|B\circ f(x)\|^2\,dx = \int_0^{2\pi} \|B(f(x))\|^2\,dx \leqq \|B\|^2 \int_0^{2\pi} \|f(x)\|^2\,dx < \infty.$$

Applying the lemma 3.2 to $B \circ f$, we get

$$\sum_{n\in\mathbb{Z}} \left\| (B\circ f)\widehat{\ }(n) \right\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \|B\circ f(x)\|^2\,dx. \tag{3.2}$$

Now, we have

$$\begin{aligned}
(B\circ f)\widehat{\ }(n) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}B(f(\theta))\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} B(e^{-in\theta}f(\theta))\,d\theta \\
&= \frac{1}{2\pi}B\left(\int_0^{2\pi} e^{-in\theta}f(\theta)\,d\theta\right) \\
&= B\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}f(\theta)\,d\theta\right) \\
&= B(\widehat{f}(n)).
\end{aligned}$$

The interchange of B and $\displaystyle\int$ is justified by $f \in \mathcal{L}^2([0,2\pi],\mathcal{H}) \subset \mathcal{L}^1([0,2\pi],\mathcal{H})$ and the lemma 3.1. Therefore (3.2) implies

$$\sum_{n\in\mathbb{Z}} \left\| B(\widehat{f}(n)) \right\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \|B(f(x))\|^2\,dx.$$

□

4 Vitali's Convergence Theorem

We'll see about **Vitali's convergence theorem** we used in the middle of the lecture. In this section, assume that $(X, \sum, \mu)$ is a (positive) measure space.

Definition 4.1 (uniform integrability).

Let $\{f_n : X \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of measurable functions. We say $\{f_n\}_{n=1}^{\infty}$ is **uniformly integrable** if for all $\epsilon > 0$, there exists $\delta > 0$ s.t.

$$E \in \sum \text{ and } \mu(E) < \delta$$

imply

$$\int_E |f_n(x)| d\mu(x) < \epsilon \text{ for each } n \in \mathbb{N}.$$

Lemma 4.2.

For a sequence of measurable functions $\{f_n : X \to \mathbb{C}\}_{n=1}^{\infty}$ and a measurable function $f : X \to \mathbb{C}$, if $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable and $\{f_n\}_{n=1}^{\infty}$ converges to f pointwise, then $\{f_n - f\}_{n=1}^{\infty}$ is also uniformly integrable.

【Proof】

Let $\epsilon > 0$. By the uniform integrability of $\{f_n\}_{n=1}^{\infty}$, there is $\delta > 0$ s.t. $E \in \sum$ and $\mu(E) < \delta$ imply

$$\int_E |f_n(x)| d\mu(x) < \frac{\epsilon}{2} \text{ for each } n \in \mathbb{N}. \tag{4.1}$$

Assume $E \in \sum$ and $\mu(E) < \delta$. Then (4.1) holds so we get

$$\int_E |f(x)| d\mu(x) \leqq \varliminf_{n\to\infty} \int_E |f_n(x)| d\mu(x) \leqq \frac{\epsilon}{2}$$

by Fatou's lemma. Thereupon, for each $n \in \mathbb{N}$, we have

$$\int_E |f_n(x) - f(x)| d\mu(x) \leqq \int_E |f_n(x)| d\mu(x) + \int_E |f(x)| d\mu(x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\{f_n - f\}_{n=1}^{\infty}$ is uniformly integrable. □

Lemma 4.3.

Suppose $\mu(X) < \infty$. For a sequence of measurable functions $\{f_n : X \to \mathbb{C}\}_{n=1}^{\infty}$ and a measurable function $f : X \to \mathbb{C}$, if $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable, $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^1(X)$, and $\{f_n\}_{n=1}^{\infty}$ converges to f pointwise, then $f \in \mathcal{L}^1(X)$.

【Proof】

By the lemma 4.2, $\{f_n - f\}_{n=1}^{\infty}$ is uniformly integrable. Thus there is $\delta > 0$ s.t. $E \in \sum$ and $\mu(E) < \delta$ imply $\int_E |f_n(x) - f(x)| d\mu(x) < 1$ for each $n \in \mathbb{N}$. Egorov's theorem guarantees that there exists $A \in \sum$ s.t. $\mu(X \backslash A) < \delta$ and $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly on A. Hereupon there is $N \in \mathbb{N}$ s.t. $|f_N(x) - f(x)| < 1$ for all $x \in A$. Hence we can see that

$$\begin{aligned}
&\int_X |f(x)| d\mu(x) \\
&\leqq \int_X |f(x) - f_N(x)| d\mu(x) + \int_X |f_N(x)| d\mu(x) \\
&= \int_A |f(x) - f_N(x)| d\mu(x) + \int_{X\backslash A} |f(x) - f_N(x)| d\mu(x) + \int_X |f_N(x)| d\mu(x) \\
&\leqq \mu(A) + 1 + \int_X |f_N(x)| d\mu(x) \\
&< \infty,
\end{aligned}$$

where the last line follows by $\mu(X) < \infty$ and $f_N \in \mathcal{L}^1$. Therefore $f \in \mathcal{L}^1$. □

Theorem 4.4 (**Vitali's convergence theorem**).

Suppose $\mu(X) < \infty$, a sequence of measurable functions $\{f_n : X \to \mathbb{C}\}_{n=1}^{\infty} \subset \mathcal{L}^1(X)$ is uniformly integrable, and $\{f_n\}_{n=1}^{\infty}$ converges to a measurable function $f : X \to \mathbb{C}$. Then, $f \in \mathcal{L}^1(X)$ and $\lim_{n\to\infty} \int_X |f_n(x) - f(x)| d\mu(x) = 0$, and eventually, $\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) = \int_X f(x)\, d\mu(x)$.

The argument of the proof is similar to that of the proof of the lemma 4.3.

【Proof】

$f \in \mathcal{L}^1$ follows by the lemma 4.3, so let us show $\lim_{n\to\infty} \int_X |f_n(x) - f(x)| d\mu(x) = 0$.

Let $\epsilon > 0$. Since $\{f_n - f\}_{n=1}^{\infty}$ is uniformly integrable by the lemma 4.2, there is $\delta > 0$ s.t. $E \in \sum$ and $\mu(E) < \delta$ imply $\int_E |f_n(x) - f(x)| d\mu(x) < \frac{\epsilon}{2}$ for each $n \in \mathbb{N}$. Egorov's theorem justifies the existence of $A \in \sum$ s.t. $\mu(X \setminus A) < \delta$ and $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly on A. Thus there is $N \in \mathbb{N}$ s.t.

$$n \geqq N \implies |f_n(x) - f(x)| < \frac{\epsilon}{2(1+\mu(X))} \text{ for all } x \in A.$$

If $n \geqq N$, then we have

$$\begin{aligned}\int_X |f_n(x) - f(x)| d\mu(x) &= \int_A |f_n(x) - f(x)| d\mu(x) + \int_{X\setminus A} |f_n(x) - f(x)| d\mu(x) \\ &< \mu(A) \cdot \frac{\epsilon}{2(1+\mu(X))} + \frac{\epsilon}{2} \\ &\leqq \mu(X) \cdot \frac{\epsilon}{2(1+\mu(X))} + \frac{\epsilon}{2} \\ &< \epsilon.\end{aligned}$$

Therefore $\lim_{n\to\infty} \int_X |f_n(x) - f(x)| d\mu(x) = 0$.

Finally, $\lim_{n\to\infty} \int_X f_n(x) d\mu(x) = \int_X f(x) d\mu(x)$ follows by

$$\left| \int_X f_n(x) d\mu(x) - \int_X f(x) d\mu(x) \right| \leqq \int_X |f_n(x) - f(x)| d\mu(x) \to 0 \text{ as } n \to \infty.$$

□

The next proposition states that under the condition $\mu(X) < \infty$, the supposition of Vitali's convergence theorem is weaker than that of the dominated convergence theorem and hence Vitali's convergence theorem is stronger than the dominated convergence theorem.

Proposition 4.5.

Assume $\mu(X) < \infty$. If a sequence of measurable functions $\{f_n : X \to \mathbb{C}\}_{n=1}^{\infty}$ converges to $f : X \to \mathbb{C}$ pointwise and there exists $g \in \mathcal{L}^1(X)$ s.t. $|f_n| \leqq g$ for each $n \in \mathbb{N}$, then $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^1(X)$ and $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable.

【Proof】

$\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^1(X)$ follows immediately by $|f_n| \leqq g$. To show the uniform integrability of $\{f_n\}_{n=1}^{\infty}$, let $\epsilon > 0$. By the dominated convergence theorem, we get $\lim_{M\to\infty} \int_{g \geqq M} g(x)\, d\mu(x) =$

0. Thus there is $M > 0$ s.t. $\displaystyle\int_{g \geqq M} g(x)\,d\mu(x) < \frac{\epsilon}{2}$. Setting $\delta := \dfrac{\epsilon}{2M}$ and supposing $E \in \sum$ and $\mu(E) < \delta$, we can see that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\int_E |f_n(x)| d\mu(x) &\leqq \int_{E \cap \{g \geqq M\}} g(x)\,d\mu(x) + \int_{E \cap \{g < M\}} g(x)\,d\mu(x) \\
&\leqq \int_{g \geqq M} g(x)\,d\mu(x) + \int_E M\,d\mu(x) \\
&< \frac{\epsilon}{2} + M\mu(E) \\
&< \frac{\epsilon}{2} + M\delta \\
&= \epsilon.
\end{aligned}$$

Therefore $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable. □

5 Appendix

5.1 Two Definitions of Uniform Integrability

In this subsection, suppose $(X, \sum, \mu)$ is a (positive) measure space and $\{f_n : X \to \mathbb{C}\}_{n=1}^{\infty}$ is a sequence of measurable functions.

In the section 4, we decided to say $\{f_n\}_{n=1}^{\infty}$ was uniformly integrable if $\{f_n\}_{n=1}^{\infty}$ satisfied the following condition :

$$\forall \epsilon > 0,\ \exists\,\delta > 0 \text{ s.t. } E \in \sum \text{ and } \mu(E) < \delta \implies \forall n \in \mathbb{N},\ \int_E |f_n(x)| d\mu(x) < \epsilon. \tag{5.1}$$

On the other hand, some mathematicians define uniform integrability by

$$\lim_{M \to \infty} \sup_{n \in \mathbb{N}} \int_{|f_n| \geqq M} |f_n(x)| d\mu(x) = 0. \tag{5.2}$$

Actually, Royden [2] defines the uniform integrability by (5.1), whilst Billingsley [4], Chung [3], and 舟木 [10] define that by (5.2).

Under the constraint that $\mu(X) < \infty$, the condition (5.2) is stronger than (5.1) due to the following proposition.

Proposition 5.1.

Suppose $\mu(X) < \infty$ and consider the condition

$$\sup_{n \in \mathbb{N}} \int_X |f_n(x)| d\mu(x) < \infty. \tag{5.3}$$

Then, we can see that

$$(5.1) \text{ and } (5.3) \iff (5.2).$$

【Proof】

$(\Longrightarrow)$

Put $\lambda := \sup_{n\in\mathbb{N}} \int_X |f_n(x)|d\mu(x) (< \infty)$. Let $\epsilon > 0$. By the supposition (5.1), there is $\delta > 0$ s.t.

$$E \in \sum \text{ and } \mu(E) < \delta \implies \forall n \in \mathbb{N}, \int_E |f_n(x)|d\mu(x) < \epsilon. \tag{5.4}$$

Set $M_0 := \dfrac{\lambda}{\delta}$ and assume $M \geqq M_0$. We have

$$\mu(|f_n| \geqq M) = \int_{|f_n|\geqq M} d\mu = \frac{1}{M}\int_{|f_n|\geqq M} M d\mu \leqq \frac{1}{M}\int_{|f_n|\geqq M} |f_n(x)|d\mu(x) \leqq \frac{\lambda}{M} \leqq \frac{\lambda}{M_0} = \delta$$

and hence by (5.4),

$$\forall n \in \mathbb{N}, \int_{|f_n|\geqq M} |f_n(x)|d\mu(x) < \epsilon$$

holds and this shows $\sup_{n\in\mathbb{N}} \int_{|f_n|\geqq M} |f_n(x)|d\mu(x) \leqq \epsilon$. Thus (5.2) holds.

$(\Longleftarrow)$

By (5.2), there exists $M > 0$ s.t. $\sup_{n\in\mathbb{N}} \int_{|f_n|\geqq M} |f_n(x)|d\mu(x) \leqq 1$. Thus

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \int_X |f_n(x)|d\mu(x) &\leqq \sup_{n\in\mathbb{N}} \int_{|f_n|\geqq M} |f_n(x)|d\mu(x) + \sup_{n\in\mathbb{N}} \int_{|f_n|<M} |f_n(x)|d\mu(x) \\
&\leqq 1 + \sup_{n\in\mathbb{N}} \int_{|f_n|<M} M d\mu \\
&\leqq 1 + \sup_{n\in\mathbb{N}} \int_X M d\mu \\
&= 1 + M\mu(X) \\
&< \infty.
\end{aligned}$$

Thus (5.3) has been confirmed.

Next, in order to check (5.1), fix $\epsilon > 0$ arbitrarily. By (5.2), there exists $L > 0$ s.t. $\sup_{n\in\mathbb{N}} \int_{|f_n|\geqq L} |f_n(x)|d\mu(x) < \dfrac{\epsilon}{2}$. Now, set $\delta := \dfrac{\epsilon}{2L}$. Then $E \in \sum$ and $\mu(E) < \delta$ imply that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\int_E |f_n(x)|d\mu(x) &= \int_{E\cap\{|f_n|\geqq L\}} |f_n(x)|d\mu(x) + \int_{E\cap\{|f_n|<L\}} |f_n(x)|d\mu(x) \\
&\leqq \int_{|f_n|\geqq L} |f_n(x)|d\mu(x) + \int_E L\, d\mu \\
&\leqq \sup_{n\in\mathbb{N}} \int_{|f_n|\geqq L} |f_n(x)|d\mu(x) + L\mu(E) \\
&< \frac{\epsilon}{2} + L\delta \\
&= \epsilon.
\end{aligned}$$

This completes the proof of (5.1). □

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