Exercises on the orthocomplement of a subspace

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Exercise 3.1.10

Check that $\mathcal{M}^\perp$ is a closed subspace of $\mathcal{H}$. The subspace $\mathcal{M}^\perp$ is called the orthocomplement of $\mathcal{M}$ in $\mathcal{H}$.

$$\mathcal{M}^\perp := \{f \in \mathcal{H} | \langle f, g\rangle = 0, \forall g \in \mathcal{M}\}$$

Solution:
$\forall f_1, f_2 \in \mathcal{M}^\perp$, $\forall \lambda \in \mathbb{C}$, $\forall g \in \mathcal{M}$, $\langle f_1 + \lambda f_2, g\rangle = \langle f_1, g\rangle + \lambda \langle f_2, g\rangle = 0$, so $\mathcal{M}^\perp$ is a subspace of $\mathcal{H}$.

If $(f_j)_{j\in\mathbb{N}} \subset \mathcal{M}$, then $s - \lim_{j\to\infty} f_j = f_\infty \in \mathcal{H}$. According to Lemma 3.1.7, we derive $w - \lim_{j\to\infty} f_j = f_\infty$, namely $\forall h \in \mathcal{H}$,

$$\lim_{j\to\infty} \langle h, f_j - f_\infty\rangle = \lim_{j\to\infty} \langle h, f_j\rangle - \langle h, f_\infty\rangle = 0.$$

Thus, $\forall h \in \mathcal{H}$, $\langle h, f_\infty\rangle = \lim_{j\to\infty} \langle h, f_j\rangle$. No doubt that $\forall g \in \mathcal{M}$, $\langle g, f_\infty\rangle = \lim_{j\to\infty} \langle g, f_j\rangle$.
From the definition of $\mathcal{M}^\perp$, we can say that $\forall j \in \mathbb{N}$, $\langle f_j, g\rangle = 0$, and thus $\langle f_\infty, g\rangle = 0$, satisfying $f_\infty \in \mathcal{M}^\perp$.

In conlusion, $\mathcal{M}^\perp$ is a closed subspace of $\mathcal{H}$.

Exercise 3.1.11

Check that a subspace $\mathcal{M} \subset \mathcal{H}$ is dense in $\mathcal{H}$ if and only if $\mathcal{M}^\perp = \{0\}$.

Solution:

1. $\mathcal{M} \subset \mathcal{H}$ is dense in $\mathcal{H} \Rightarrow \mathcal{M}^{\perp} = \{0\}$

   $\forall f \in \mathcal{H}$, $\forall \epsilon > 0$, $\exists g \in \mathcal{M}$ with $\|f-g\| \leq \epsilon$. Meanwhile, for any $h \in \mathcal{M}^{\perp}$, since $\langle h, g\rangle = 0$, we have

   $$\langle h, f\rangle = \langle h, f-g+g\rangle = \langle h, f-g\rangle + \langle h, g\rangle = \langle h, f-g\rangle .$$

   From Schwartz inequality, one gets

   $$|\langle h, f-g\rangle| \leq \|h\|\|f-g\| \leq \epsilon\|h\|.$$

   Since ϵ is arbitrary, one infers that

   $$|\langle h, f-g\rangle| = 0,$$

   namely

   $$\langle h, f\rangle = 0.$$

   This applies to any $f \in \mathcal{H}$ and $h \in \mathcal{M}^{\perp}$, so for each $h \in \mathcal{M}^{\perp}$ there must be

   $$\langle h, h\rangle = 0$$

   when $f = h$, leading to the fact that $h = 0$. $\mathcal{M}^{\perp} = \{0\}$ is therefore proved.

2. $\mathcal{M} \subset \mathcal{H}$ is dense in $\mathcal{H} \Leftarrow \mathcal{M}^{\perp} = \{0\}$

   $\overline{\mathcal{M}} \subset \mathcal{H}$ is a closed subspace of $\mathcal{H}$. According to Projection Theorem, $\forall f \in \mathcal{H}$, there exist a unique $f_1 \in \overline{\mathcal{M}}$ and a unique $f_2 \in \overline{\mathcal{M}}^{\perp}$ such that $f = f_1 + f_2$.
   Since $\mathcal{M}^{\perp} = \{0\}$, $\overline{\mathcal{M}}^{\perp} = \{0\}$ is easily deduced, and so $f_2 = 0$, $f = f_1$, which leads to $\overline{\mathcal{M}} = \mathcal{H}$. Then $\mathcal{M}$ is dense in $\mathcal{H}$.

Reference

Amrein, W 2009, *Hilbert Space Methods in Quantum Mechanics*, EPFL Press, Laussane.