Sequence of functions converging to the 0 function whose derivatives do not converge to 0

——from a special example to discussion

on the importance of derivative-convergence in certain situations [1]

by Zhang Zhiyang

This report is based on a sequence of functions set on an interval [0,1]:

$$f_j(x) = \frac{x}{j}\sin\frac{j}{x}, \qquad j \in \mathbb{N}^*, x \in [0,1]$$

It is proved that the sequence converges to

$$f_\infty(x) = 0, \qquad x \in [0,1]$$

as j tends to infinity, while the derivatives of the functions

$$f_j'(x) = \frac{1}{j}\sin\frac{j}{x} + \frac{x}{j}\left(-\frac{\mathrm{j}}{\mathrm{x}^2}\right)\cos\frac{j}{x}$$

$$= \frac{1}{j}\sin\frac{j}{x} - \frac{1}{\mathrm{x}}\cos\frac{j}{x}, \qquad x \in [0,1]$$

do not converge to

$$f_\infty'(x) = 0, \qquad x \in [0,1]$$

as j tends to infinity.

1. Proof that $\lim_{j\to\infty}\left|f_j(x) - f_\infty(x)\right| = 0,\ with\ x \in [0,1]$.

$$\left|f_j(x) - f_\infty(x)\right| = \left|f_j(x)\right| = \left|\frac{x}{j}\right|\left|\sin\frac{j}{x}\right| \le \left|\frac{x}{j}\right|$$

Since $x \in [0,1]$, we have

$$\left|f_j(x) - f_\infty(x)\right| \le \frac{1}{j}$$

[1] This report is inspired and has been commented by Professor Richard. Some of the methods inspired by: 1) report *"Three standard distributions"* by Yuu Hiramatsu, similar methods also found in relevant reports *"About some distributions"*, by Yat Ming Luk, *"Proofs on some distributions"*, by Firdaus Rafi Rizqy, Hadiko Rifqi Aufa Sholih, Sekiya Emika, *"On regular distributions"*, by Haruka Yajima, *"On various distributions"*, by Zhang Jiabin, and in other reports that I might not have noticed; 2) *"Distributions: characterisation, support, and order"*, by Pratham Dhomne and Vic Austen.

$$\sup_{x\in[0,1]}\left|f_j(x)-f_\infty(x)\right|\le\frac{1}{j}$$

$\forall\varepsilon>0,\ \exists j>1/\varepsilon,\ s.t.$

$$\sup_{x\in[0,1]}\left|f_j(x)-f_\infty(x)\right|\le\frac{1}{j}<\varepsilon$$

Thus, it has been proved that

$$\lim_{j\to\infty}\left|f_j(x)-f_\infty(x)\right|=0,\qquad with\ x\in[0,1]$$

2. Proof that $f_j'(x)$ does not converge to $f_\infty'(x)$ as $j\to\infty$, with $x\in[0,1]$.

$$\left|f_j'(x)-f_\infty'(x)\right|=\left|\frac{1}{j}\sin\frac{j}{x}-\frac{1}{\mathrm{x}}\cos\frac{j}{x}\right|$$

$$\ge\left|\frac{1}{x}\cos\frac{j}{x}\right|-\left|\frac{1}{j}\sin\frac{j}{x}\right|$$

$$\ge\left|\frac{1}{x}\cos\frac{j}{x}\right|-\frac{1}{j}$$

Since $x\in[0,1]$, we have

$$\left|\frac{1}{x}\cos\frac{j}{x}\right|-\frac{1}{j}\ge\left|\cos\frac{j}{x}\right|-\frac{1}{j}$$

Thus,

$$\left|f_j'(x)-f_\infty'(x)\right|\ge\left|\cos\frac{j}{x}\right|-\frac{1}{j}$$

Since $\frac{1}{\pi}\in[0,1]$,

$$\sup_{x\in[0,1]}\left|f_j'(x)-f_\infty'(x)\right|\ge\left|f_j'\left(\frac{1}{\pi}\right)-f_\infty'\left(\frac{1}{\pi}\right)\right|$$

$$\ge|\cos j\pi|-\frac{1}{j}$$

$$=1-\frac{1}{j}$$

$\forall j>2$, we have

$$\sup_{x\in[0,1]}\left|f_j'(x)-f_\infty'(x)\right|\ge\frac{1}{2}$$

Therefore, $f_j'(x)$ does not converge to $f_\infty'(x)$ as $j \to \infty$, with $x \in [0,1]$.

3. Discussion

The fact that $\lim_{j\to\infty} f_j(x) = f_\infty(x)$ does not necessarily mean $\lim_{j\to\infty} f_j'(x) = f_\infty'(x)$. It gives some hints on why convergence of all derivatives is specially needed for "convergence" in $D(\mathbb{R}^n)$:

$$\sup_{X\in\mathbb{R}^n} \left|\partial^\alpha f_j(X) - \partial^\alpha f_\infty(X)\right| \to 0 \quad as \quad j \to \infty, \qquad \forall \alpha \in \mathbb{N}^n \quad ①$$

in order that more can be deduced from the condition of "convergence".

For example, to prove that $\partial^\beta \delta_Y(f)$ is a distribution, we need

$$\sup_{X\in\mathbb{R}^n} \left|\partial^\beta f_j(X) - \partial^\beta f_\infty(X)\right| \to 0 \quad as \quad j \to \infty$$

if $(f_j)_{j\in\mathbb{N}} \subset D(\mathbb{R}^n)$ is convergent.[2]

And to prove that $P_V \frac{1}{x}(f)$ is a distribution, we need

$$\sup_{x\in\mathbb{R}} \left|f_j'(x) - f_\infty'(x)\right| \to 0 \quad as \quad j \to \infty$$

if $(f_j)_{j\in\mathbb{N}} \subset D(\mathbb{R})$ is convergent.[3]

Without requirement ① for convergence of $(f_j)_{j\in\mathbb{N}} \subset D(\mathbb{R}^n)$, we can only talk about distributions with order 0. For example, with the absence of the requirement, we can say T_h, δ_Y (whose orders are both 0) are distributions, but we cannot prove that δ_Y^α (whose order is $|\alpha|$) is also a distribution.[4] I will try to give a proof why requirement ① is needed. However it is not so rigorous, only a way of understanding.

Consider a map T of order m ($m > 0$ and is independent of Y and r). It

$$\forall Y \in \mathbb{R}^n, r > 0, \exists c > 0,$$

$$\forall (f_j) \in D(\mathbb{R}^n)\ with\ supp(f_j) \subset \overline{\mathcal{B}_r(Y)}\ for\ all\ j \in \mathbb{N}^*$$

$$and\ with\ f_j\ converging\ to\ f_\infty \in D(\mathbb{R}^n)\ as\ j\ tends\ to\ infinity, supp(f_\infty) \subset \overline{\mathcal{B}_r(Y)},$$

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[2] Please refer to report *"Proofs on some distributions"*, by Firdaus Rafi Rizqy, Hadiko Rifqi Aufa Sholih, Sekiya Emika or other reports focusing on the same topic.

[3] Please refer to report *"On various distributions"*, by Zhang Jiabin.

[4] Please refer to report *"Proofs on some distributions"*, by Firdaus Rafi Rizqy, Hadiko Rifqi Aufa Sholih, Sekiya Emika or other reports focusing on same topics.

$$\left|T(f_j - f_\infty)\right| \leq c \sum_{|\alpha|\leq m} \left\|\partial^\alpha f_j - \partial^\alpha f_\infty\right\|_\infty$$

as $g_j := f_j - f_\infty$ is also a test function.

T being a distribution also requires that

$$\lim_{j\to\infty}\left|T\left(f_j - f_\infty\right)\right| = 0$$

since f_j converges to f_∞, and a distribution is a vector space.

Remember we can refer to T as a distribution only if it's linear and it satisfies:

$$\forall Y \in \mathbb{R}^n, r > 0, \exists c > 0,$$

$$\forall g \in D(\mathbb{R}^n)\ with\ supp(g) \subset \overline{\mathcal{B}_r(Y)},$$

$$\left|T(g_j)\right| \leq c \sum_{|\alpha|\leq m} \left\|\partial^\alpha g_j\right\|_\infty$$

For $g_j = f_j - f_\infty$, if there really exists a β satisfying $|\beta| \leq m$, with $\lim_{j\to\infty}\left\|\partial^\beta f_j - \partial^\beta f_\infty\right\|_\infty = k > 0$, then the requirement

$$\left|T(f_j - f_\infty)\right| \leq c \sum_{|\alpha|\leq m} \left\|\partial^\alpha f_j - \partial^\alpha f_\infty\right\|_\infty$$

might be too weak to conclude that

$$\lim_{j\to\infty}\left|T\left(f_j - f_\infty\right)\right| = 0$$

since

$$\lim_{j\to\infty}\left|T(f_j - f_\infty)\right| \leq ck + c \sum_{|\alpha|\leq m, \alpha\neq\beta} \lim_{j\to\infty}\left\|\partial^\alpha f_j - \partial^\alpha f_\infty\right\|_\infty$$

is far from enough to make the conclusion.

But if there is no such β,

$$\lim_{j\to\infty}\left|T(f_j - f_\infty)\right| \leq c \sum_{|\alpha|\leq m} \lim_{j\to\infty}\left\|\partial^\alpha f_j - \partial^\alpha f_\infty\right\|_\infty = 0$$

makes a lot of sense.

Anyway, this proof is not so rigorous. It would be better if a counter-example could be given.[5]

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[5] Inspired by report *"Distributions: characterisation, support, and order"*, by Pratham Dhomne and Vic Austen. And please refer to this report for further relevant information that I haven't mentioned if needed.