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Exercize 1.1.9 show that T_h, δ_y, $\partial^\alpha \delta_y$ belongs to $\mathcal{D}'(\mathbb{R}^n)$ by proving that these 3 meet two conditions at Definition 1.1.7

$\boxed{Th}$ 1. $T_h(f+\lambda g) = \int_{\mathbb{R}^n} h(x)\{f+\lambda g\}(x)dx$

$$= \int_{\mathbb{R}^n} h(x)f(x)dx + \lambda \int_{\mathbb{R}^n} h(x)g(x)dx$$

$$= T_h(f) + \lambda T_h(g) \quad \square$$

2. Suppose $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ converges to $f_\infty \subset \mathcal{D}(\mathbb{R}^n)$ as $j \to \infty$

$\Rightarrow \forall_{\varepsilon>0}: \exists N \in \mathbb{N} : \|f_j - f_\infty\| \leq \varepsilon \quad \forall j \geq N$

then $|T_h(f_j) - T_h(f_\infty)| = \left| \int_{\mathbb{R}^n} h(x)f_j(x)dx - \int h(x)f_\infty(x)dx \right| \cdots ⊛$

set $\exists\, 0<r\in\mathbb{R}$ large enough s.t. $\mathrm{supp}(f_j) \subset B_r(0) \ \forall j \in \mathbb{N}$

since $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ and $f_\infty \subset \mathcal{D}(\mathbb{R}^n)$,

$$⊛ = \left| \int_{B_r(0)} h(x)\{f_j(x) - f_\infty(x)\}dx \right|$$

$$\leq \int_{B_r(0)} \left| h(x)\{f_j(x) - f_\infty(x)\} \right| dx$$

$$= \int_{B_r(0)} |h(x)||f_j(x) - f_\infty(x)| dx$$

$$\leq \int_{B_r(0)} |h(x)| \sup_{y\in\mathbb{R}^n} |f_j(y) - f_\infty(y)| dx$$

$$= \int_{B_r(0)} |h(x)| \, \|f_j - f_\infty\|_\infty dx$$

$$\leq \int_{B_r(0)} |h(x)| \varepsilon \, dx = \varepsilon \int_{B_r(0)} |h(x)| dx = \varepsilon'$$

Therefore, $T_h(f_j)$ converges to $T_h(f_\infty)$ as $j \to \infty$ $\square$

Now that condition 1. and 2 are proven, $T_h \in \mathcal{D}'(\mathbb{R}^n)$

$\boxed{\delta_Y}$ 1. $\delta_Y(f+\lambda g) = f+\lambda g(Y)$

$$= f(Y) + \lambda g(Y)$$

$$= \delta_Y(f) + \lambda \delta_Y(g) \quad \square$$

2. Suppose $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ converges to f_∞ as $j \to \infty$

$\Rightarrow \forall_{\varepsilon > 0} : \exists N \in \mathbb{N}\ \|f_j - f_\infty\|_\infty \leq \varepsilon \quad \forall j \geq N$

then $|\delta_Y(f_j) - \delta_Y(f_\infty)| = |f_j(Y) - f_\infty(Y)|$

$$\leq \sup_{Y\in\mathbb{R}^n} |f_j(Y) - f_\infty(Y)|$$

$$= \|f_j - f_\infty\|_\infty \leq \varepsilon$$

Therefore, $\delta_Y(f_j)$ converges to $\delta_Y(f_\infty)$ as $j \to \infty$ $\square$

Now that condition 1 and 2 are proven, $\delta_Y \in \mathcal{D}'(\mathbb{R}^n)$

$\boxed{\delta_Y^\alpha}$ 1. $\delta_Y^\alpha(f+\lambda g) = (-1)^{|\alpha|}[\partial^\alpha(f+\lambda g)](Y)$

$$= (-1)^{|\alpha|}[\partial^\alpha f](Y) + \lambda(-1)^{|\alpha|}[\partial^\alpha g](Y)$$

$$= \delta_Y(f) + \lambda \delta_Y(g) \quad \square$$

2. suppose the sequence $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ converges to $f_\infty \in \mathcal{D}(\mathbb{R}^n)$ as $j \to \infty$

$\Rightarrow \forall \alpha \in \mathbb{N}^n, \forall \varepsilon > 0, \exists N \subset \mathbb{N} \quad \| \partial^\alpha f_j - \partial^\alpha f_\infty \| \leq \varepsilon, \forall j > N$

then

$$
\begin{aligned}
|\delta_y^\alpha(f_j) - \delta_y^\alpha(f_\infty)| &= |(-1)^{|\alpha|}[\partial^\alpha f_j](y) - (-1)^{|\alpha|}[\partial^\alpha f_\infty](y)| \\
&= |[\partial^\alpha f_j](y) - [\partial^\alpha f_j](y)| \\
&\leq \sup_{x \in \mathbb{R}^n} |\partial^\alpha f_j(x) - \partial^\alpha f_\infty(x)| \\
&= \| \partial^\alpha f_j - \partial^\alpha f_\infty \|_\infty \leq \varepsilon
\end{aligned}
$$

Therefore, $\delta_y^\alpha(f_j)$ converges to $\delta_y^\alpha(f_\infty)$ as $j \to \infty$ □

Now that condition 1 and 2 are proven, $\delta_y^\alpha \in \mathcal{D}'(\mathbb{R}^n)$