exercise 3.1.6 For any $f, g \in H$, and $\lambda \in \mathbb{C}$ (Yuta)

from definition, $\langle f, g\rangle = \overline{\langle g, f\rangle}$ — ①

$\langle f, g+\lambda h\rangle = \langle f, g\rangle + \bar{\lambda}\langle f, h\rangle$ — ②

$\langle g+\lambda h, f\rangle = \langle g, f\rangle + \lambda\langle h, f\rangle$ — ③

$\left\{ \begin{array}{l} \|f\|^2 = \langle f, f\rangle \geq 0 \text{ and } \|f\| = 0 \\ \text{if and only if } f = 0 \end{array} \right.$ — ④

Now, we prove $|\langle f, g\rangle| \leq \|f\|\|g\|$ — Ⓐ

$\|f+g\| \leq \|f\| + \|g\|$ — Ⓑ

$\|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$ — Ⓒ

$\big|\|f\| - \|g\|\big| \leq \|f-g\|$ — Ⓓ

to prove Ⓐ, we think $\langle (f+\lambda\langle f, g\rangle g), (f+\lambda\langle f, g\rangle g)\rangle$ ($\lambda \in \mathbb{R}$)

from ① ② ③ ④, $\langle (f+\lambda\langle f, g\rangle g), (f+\lambda\langle f, g\rangle g)\rangle$

$= \langle (f+\lambda\langle f, g\rangle g), f\rangle + \lambda\overline{\langle f, g\rangle}\langle (f+\lambda\langle f, g\rangle g), g\rangle$

$= \|f\|^2 + \lambda\langle f, g\rangle\langle g, f\rangle + \lambda\overline{\langle f, g\rangle}\langle f, g\rangle$

$+ \lambda^2\overline{\langle f, g\rangle}\langle f, g\rangle\|g\|^2$

$= \lambda^2|\langle f, g\rangle|^2\|g\|^2 + 2|\langle f, g\rangle|^2\lambda + \|f\|^2 \geq 0$

it means the solution of

$\lambda^2|\langle f,g\rangle|^2\|g\|^2+2|\langle f,g\rangle|^2\lambda+\|f\|^2=0$ have an imaginary part. therefore

$$(|\langle f,g\rangle|^2)^2-\|f\|^2\|g\|^2|\langle f,g\rangle|^2\leqq 0$$

$$\therefore |\langle f,g\rangle|\leqq\|f\|\|g\| . \text{ — } Ⓐ$$

Next, we prove Ⓑ, $(\|f\|+\|g\|)^2-(\|f+g\|)^2$

$$=\|f\|^2+2\|f\|^2\|g\|^2+\|g\|^2-\|f\|^2-\langle f,g\rangle-\langle g,f\rangle-\|g\|^2$$

$$\geqq 2(\|f\|^2\|g\|^2-|\langle f,g\rangle|)\geqq 0 \ \therefore \|f+g\|\leqq\|f\|+\|g\| \text{ — } Ⓑ$$

then, we prove Ⓒ, from $2\|f\|^2+2\|g\|^2-\|f+g\|^2=\|f-g\|^2\geqq 0$

$$\therefore 2\|f\|^2+2\|g\|^2\geqq\|f+g\|^2 \text{ — } Ⓒ$$

Finally, we prove Ⓓ

from Ⓑ, $\|f-g\|+\|g\|\geqq\|f\| \ \therefore \|f-g\|\geqq|\|f\|-\|g\||$