The diameter of a bounded set is equal to its boundary in a normed space

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Let $(E, \|\cdot\|)$ be a normed vector space, and let A be a non-empty bounded subset of E. The diameter of A is defined as the quantity

$$\operatorname{diam}(A) = \sup_{(x,y)\in A^2} d(x,y) = \sup_{(x,y)\in A^2} \|x-y\| \in \mathbb{R}^+ \cup \{+\infty\}$$

We try to prove that $\operatorname{diam}(\partial A) = \operatorname{diam}(A)$.

Solution

We start by proving that $\overline{A}$ and ∂A are bounded.

Since A is bounded, there exists $M > 0$ such that $\forall x \in A, \|x\| \leq M$.

Let $x \in \overline{A}$. There exists a sequence $(x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ such that $x_n \to x$ when $n \to \infty$. We then have, for all $n \in \mathbb{N}$:

$$\|x\| \leq \|x_n - x\| + \|x_n\| \quad \text{where} \quad \|x_n - x\| \to 0 \quad \text{when} \quad n \to \infty$$

Thus $\|x\| \leq M$. Hence $\overline{A}$ is bounded.

Since ∂A is included in $\overline{A}$, then ∂A is bounded.

Then we are going to show that $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$.

Since $A \subset \overline{A}$, we have $\operatorname{diam}(A) \leq \operatorname{diam}(\overline{A})$. Let $(x,y) \in (\overline{A})^2$. There exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements of A such that $x_n \to x$ and $y_n \to y$. We then have, for all $n \in N$,

$$d(x,y) \leq d(x,x_n) + d(x_n,y_n) + d(y_n,y).$$

Here, when $n \to \infty$, we have $d(x,x_n) \to 0, d(x_n,y_n) \leq \operatorname{diam}(A)$ and $d(y_n,y) \to 0$. Thus $d(x,y) \leq \operatorname{diam}(A)$. Finally, $\operatorname{diam}(\overline{A}) \leq \operatorname{diam}(A)$ and $\operatorname{diam}(\overline{A}) = \operatorname{diam}(A)$.

Since $\partial A \subset \overline{A}$ and $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$, we have $\operatorname{diam}(\partial A) \leq \operatorname{diam}(A)$. Now the only thing left is to prove that $\operatorname{diam}(A) \leq \operatorname{diam}(\partial A)$.

We start our last step by setting $x \in A$ and $u \in E \setminus \{0\}$. Consider the set $X_u = \{t \geq 0, x + t.u \in A\}$. Since A is bounded, there exists $M > 0$ such that $\forall y \in A, \|y\| \leq M$.

Let $t \in X_u$. We have

$$|t|\|u\| = \|t.u\| \leq \|x + t.u\| + \|x\| \leq M + \|x\|.$$

Thus $t \leq \frac{M+\|x\|}{\|u\|}$. Hence, X_u is a bounded subset of $\mathbb{R}$.

Moreover, X_u is non-empty because $0 \in X_u$. Thus $t_{x,u} = \sup X_u \in \mathbb{R}$ is well-defined.

How to show that $x + t_{x,u}.u \in \partial A$?

For $n \in \mathbb{N}$, let $y_n = x + (t_{x,u} + \frac{1}{n})u$. By definition of $t_{x,u}$, $y_n \notin A$. We also have $y_n \to x + t_{x,u}.u$, so $x + t_{x,u}.u \in \overline{E \setminus A}$.

By definition of $t_{x,u}$, there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \to t_{x,u}$ and $x + t_n.u \in A$ for all $n \in \mathbb{N}$. We then have $x + t_n.u \to x + t_{x,u}.u$, so $x + t_{x,u}.u \in A$. $x + t_{x,u}.u \in \partial A$.

Let $(x, y) \in A^2$. We try to prove that there exist x' and y' related with x and y such that $x' \in \partial A$, $y' \in \partial A$, and $\|x' - y'\| \geq \|x - y\|$.

Let $u = y - x$. We then have $1 \in X_u$, so $t_{x,y-x} \geq 1$. We set $x' = x + t_{x,y-x}(y - x) \in \partial A$.

Similarly, for y and $u = x - y$, we have $t_{y,x-y} \geq 1$ and we set $y' = y + t_{y,x-y}(x - y) \in \partial A$.

We then have

$$\|y' - x'\| = \|y + t_{y,x-y}(x - y) - x - t_{x,y-x}(y - x)\| = |t_{y,x-y} + t_{x,y-x} - 1|\|x - y\| \geq \|x - y\|.$$

Finally, we have almost arrived, We finished the proof by proving that $\text{diam}(\partial A) = \text{diam}(A)$.

By definition of the diameter, there exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements of A such that $d(x_n, y_n) \to \text{diam}(A)$ when $n \to \infty$.

According to the previous process, there also exist sequences $(x'_n)_{n\in\mathbb{N}}$ and $(y'_n)_{n\in\mathbb{N}}$ of elements of ∂A such that for all $n \in \mathbb{N}$, $d(x'_n, y'_n) \geq d(x_n, y_n)$.

For all $n \in \mathbb{N}$, we therefore have $d(x_n, y_n) \leq \text{diam}(\partial A)$. By taking the limit, we obtain $\text{diam}(A) \leq \text{diam}(\partial A)$.

As $\text{diam}(\partial A) \leq \text{diam}(A)$, we finally obtain that $\text{diam}(\partial A) = \text{diam}(A)$.