A Proof of the Hölder inequality in n-dimensional Euclidean Space with Counting Measure

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Theorem (Hölder inequality in n-dimensional euclidean space with counting measure). For the n-dimensional Euclidean space, when the set S is $\{1, \dots, n\}$ with the counting measure, we have

$$\Rightarrow \sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}} \text{ for all } (a_1, \dots, a_n), (b_1, \dots, b_n) \in (\mathbb{R}^+)^{2n}.$$

We start our proof by introducing a lemma.

Lemma (Inequality of Discrete Convexity). Let $f : I \to \mathbb{R}$ be a convex function, and let $(n_1, \dots, n_n) \in I^n$ and $(\lambda_1, \dots, \lambda_n) \in (\mathbb{R}^+)^n$ such that $\sum_{k=1}^{n} \lambda_k > 0$. Then, $f\left(\frac{1}{\sum_{k=1}^{n} \lambda_k}(\sum_{k=1}^{n} \lambda_k x_k)\right) \leq \frac{1}{\sum_{k=1}^{n} \lambda_k}\left(\sum_{k=1}^{n} \lambda_k f(x_k)\right)$

We can also say that, the image of a barycenter with positive coefficients is less than or equal to the barycenter of the images.

We now prove this lemma by recurrence on n and associativity of the barycenter.

Initialization: For $n = 2$, it is the definition of convexity [Definition: for any $(a, b) \in I^2$ and any $t \in [0, 1]$, $f\bigg((1-t)a + tb\bigg) \leq (1-t)f(a) + tf(b)$] where $t = \frac{\lambda_2}{\lambda_1 + \lambda_2}$.

Induction: Assume the proposition holds for rank n and take $n+1$ values $(x_1, \dots, x_{n+1})$ in I and $n+1$ positive real numbers $\lambda_1, \dots, \lambda_{n+1}$ such that $\sum_{k=1}^{n+1} \lambda_k > 0$. If $\sum_{k=1}^{n} \lambda_k = 0$, then all λ_k are zero except λ_{n+1}, and the result is immediate. Otherwise, we have:

$$\frac{1}{\sum_{k=1}^{n+1} \lambda_k}(\sum_{k=1}^{n+1} \lambda_k x_k) = (1 - \frac{\lambda_{n+1}}{\sum_{k=1}^{n+1} \lambda_k})(\frac{1}{\sum_{k=1}^{n} \lambda_k} \sum_{k=1}^{n} \lambda_k x_k) + \frac{\lambda_{n+1}}{\sum_{k=1}^{n+1} \lambda_k} x_{n+1}$$

The convexity of f gives us:

$$f\left(\frac{1}{\sum_{k=1}^{n+1} \lambda_k}(\sum_{k=1}^{n+1} \lambda_k x_k)\right) \leq \left(1 - \frac{\lambda_{n+1}}{\sum_{k=1}^{n+1} \lambda_k}\right) f\left(\frac{1}{\sum_{k=1}^{n} \lambda_k} \sum_{k=1}^{n} \lambda_k x_k\right) + \frac{\lambda_{n+1}}{\sum_{k=1}^{n+1} \lambda_k} f(x_{n+1})$$

It remains to use the hypothesis of recurrence to bound the term $f\left(\frac{1}{\sum_{k=1}^{n} \lambda_k}(\sum_{k=1}^{n} \lambda_k x_k)\right)$ by $\frac{1}{\sum_{k=1}^{n} \lambda_k}\left(\sum_{k=1}^{n} \lambda_k f(x_k)\right)$. Therefore, the lemma is correct.

Proof of Hölder inequality in n-dimensional euclidean space with counting measure:

We have now proven the inequality of discrete convexity. Now, based on the inequality of discrete convexity, we use the convexity of the function $r \mapsto r^p$ and take $\lambda_k x_k = a_k b_k$, $\lambda_k = b_k^q$ and $\lambda_k x_k^p = a_k^p$.

This gives us, when $b_k \neq 0$:

$$\lambda_k = b_k^q \quad \text{and} \quad x_k = \frac{a_k b_k}{b_k^q}$$

As $\frac{1}{p} + \frac{1}{q} = 1$, we thus easily verify the third relation:

$$\lambda_k x_k^p = b_k^q (\frac{a_k b_k}{b_k^q})^p = b_k^{q+p-pq} a_k^p = a_k^p$$

By substituting these items into the inequality of discrete convexity [Recall: $f\left(\frac{1}{\sum_{k=1}^n \lambda_k}(\sum_{k=1}^n \lambda_k x_k)\right) \leq \frac{1}{\sum_{k=1}^n \lambda_k}\left(\sum_{k=1}^n \lambda_k f(x_k)\right)$]:

$$f\left(\frac{1}{\sum_{k=1}^n \lambda_k}(\sum_{k=1}^n \lambda_k x_k)\right) \leq \frac{1}{\sum_{k=1}^n \lambda_k}\left(\sum_{k=1}^n \lambda_k f(x_k)\right)$$

$$\Rightarrow \left(\frac{1}{\sum_{k=1}^n b_k^q}(\sum_{k=1}^n a_k b_k)\right)^p \leq \frac{1}{\sum_{k=1}^n b_k^q}\left(\sum_{k=1}^n a_k^p\right)$$

$$\Rightarrow \left(\sum_{k=1}^n b_k^q\right)^{-p} \left(\sum_{k=1}^n a_k b_k\right)^p \leq \left(\sum_{k=1}^n b_k^q\right)^{-1} \left(\sum_{k=1}^n a_k^p\right)$$

$$\Rightarrow \left(\sum_{k=1}^n a_k b_k\right)^p \leq \left(\sum_{k=1}^n b_k^q\right)^{p-1} \left(\sum_{k=1}^n a_k^p\right)$$

$$\Rightarrow \sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n b_k^q\right)^{1-\frac{1}{p}} \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}}$$

As $1 - \frac{1}{p} = \frac{1}{q}$, we finally have:

$$\Rightarrow \sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n b_k^q\right)^{\frac{1}{q}}$$

This ends our proof.