

Dirichlet Function

1	RATIONALS
0	IRRATIONALS

Yam

Let us consider the Dirichlet function defined by:

$$f(x) = \begin{cases} 1, & x \in [0, 1] \cap \mathbb{Q} \\ 0, & x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

Consider any Riemann Partition. Then, since \mathbb{Q} is dense in \mathbb{R} , in any $[x_i, x_{i+1}]$ with $x_{i+1} > x_i$, $\sup_{x \in [x_i, x_{i+1}]} f(x) = 1$, and $\inf_{x \in [x_i, x_{i+1}]} f(x) = 0 \forall i$. It is apparent that $U(f, P) \neq L(f, P)$: not R.I.

Instead, we would like to take partitions such that the supremum and infimum of $f(x)$ match. We simply separate the rationals and irrationals.

• $\Omega_{\mathbb{Q}} = [0, 1] \cap \mathbb{Q} = f^{-1}(1)$ 1 $\Omega_{\mathbb{Q}}$

• $\Omega_{\mathbb{Q}}^c = [0, 1] \setminus \mathbb{Q} = f^{-1}(0)$ 0 $\Omega_{\mathbb{Q}}^c$

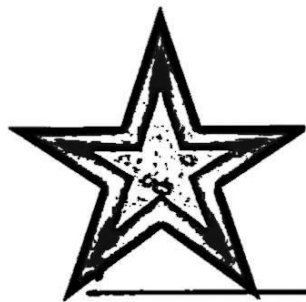
Clearly, $\Omega_{\mathbb{Q}} \cup \Omega_{\mathbb{Q}}^c = [0, 1]$ and $m(\Omega_{\mathbb{Q}} \cap \Omega_{\mathbb{Q}}^c) = 0$.

One can show that $m(\Omega_{\mathbb{Q}}) = 0$ and $m(\Omega_{\mathbb{Q}}^c) = 1$ (*)

Then $\sup_{\Omega_{\mathbb{Q}}} f(x) m(\Omega_{\mathbb{Q}}) = \inf_{\Omega_{\mathbb{Q}}} f(x) m(\Omega_{\mathbb{Q}}) = 1 \cdot 0$

and $\sup_{\Omega_{\mathbb{Q}}^c} f(x) m(\Omega_{\mathbb{Q}}^c) = \inf_{\Omega_{\mathbb{Q}}^c} f(x) m(\Omega_{\mathbb{Q}}^c) = 0 \cdot 1$

Clearly $U(f, \mathcal{P}) = L(f, \mathcal{P}) \Rightarrow \int_{[0,1]} f(x) \, dm = 0$ ■



Countable union of null measure sets is null measure.

Recall that a set has 0 measure if one can find a covering of boxes that is arbitrarily (ε) small. Let us write the union as $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where $m(\Omega_i) = 0$. Then, for each Ω_i , there exists a covering $\{I_{i,j}\}_j$ such that $\sigma(\{I_{i,j}\}_j) < 2^{-i}\varepsilon$, $\varepsilon > 0$.

Then we have:

$$\begin{aligned} m(\Omega) &= \sum_{i=1}^{\infty} m(\Omega_i) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} m(I_{i,j}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \text{vol}(I_{i,j}) = \\ &= \sum_{i=1}^{\infty} \sigma(\{I_{i,j}\}_j) < \sum_{i=1}^{\infty} 2^{-i} \varepsilon = \varepsilon. \end{aligned}$$

Since we can make ε arbitrarily small, $m(\Omega) = 0$ ■