Dirichlet Function

1 — RATIONALS

0 — IRRATIONALS

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Let us consider the dirichlet function defined by:

$$f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

Consider any Riemann Partition. Then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, in any $[x_i, x_{i+1}]$ with $x_{i+1} > x_i$, $\sup\limits_{[x_i, x_{i+1}]} f(x) = 1$, and $\inf\limits_{[x_i, x_{i+1}]} f(x) = 0 \ \forall i$. It is apparent that $U(f,P) \neq L(f,P)$: not R.I.

Instead, we would like to take partitions such that the supremum and infimum of $f(x)$ match. We simply separate the rationals and irrationals.

- $\Omega_{\mathbb{Q}} = [0,1] \cap \mathbb{Q} = f^{-1}(1)$ 1 — $\Omega_{\mathbb{Q}}$
- $\Omega_{\mathbb{Q}}^c = [0,1] \setminus \mathbb{Q} = f^{-1}(0)$ 0 — $\Omega_{\mathbb{Q}}^c$

Clearly, $\Omega_{\mathbb{Q}} \cup \Omega_{\mathbb{Q}}^c = [0,1]$ and $m(\Omega_{\mathbb{Q}} \cap \Omega_{\mathbb{Q}}^c) = 0$.

One can show that $m(\Omega_{\mathbb{Q}}) = 0$ and $m(\Omega_{\mathbb{Q}}^c) = 1$ $(\star)$

Then $\sup\limits_{\Omega_{\mathbb{Q}}} f(x) m(\Omega_{\mathbb{Q}}) = \inf\limits_{\Omega_{\mathbb{Q}}} f(x) m(\Omega_{\mathbb{Q}}) = 1 \cdot 0$

and $\sup\limits_{\Omega_{\mathbb{Q}}^c} f(x) m(\Omega_{\mathbb{Q}}^c) = \inf\limits_{\Omega_{\mathbb{Q}}^c} f(x) m(\Omega_{\mathbb{Q}}^c) = 0 \cdot 1$

Clearly $U(f,P)=L(f,P) \Rightarrow \int_{[0,1]} f(x)\, dm = 0$ $\blacksquare$

Countable union of null measure sets is null measure.

Recall that a set has 0 measure if one can find a covering of boxes that is arbitrarily (ε) small. Let us write the union as $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where $m(\Omega_i)=0$. Then, for each Ω_i, there exists a covering $\{I_{i,j}\}_j$ such that $\sigma(\{I_{i,j}\}_j) < 2^{-i}\varepsilon$. $\varepsilon>0$

Then we have:

$$m(\Omega) = \sum_{i=1}^{\infty} m(\Omega_i) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} m(I_{i,j}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathrm{vol}(I_{i,j}) =$$
$$= \sum_{i=1}^{\infty} \sigma(\{I_{i,j}\}_j) < \sum_{i=1}^{\infty} 2^{-i}\varepsilon = \varepsilon.$$

Since we can make ε arbitrarily small, $m(\Omega)=0$ $\blacksquare$