Equivalence Relation defined by Equality Almost Everywhere

Yam

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1 Introduction

In this report, I show that $f = g \quad a.e$ is an equivalence relation, namely it is reflexive, symmetric, and transitive.

2 Proof

1. **Reflexivity** Trivially, $f = f$ everywhere, and so

$$m(\{x \,|\, f(x) \neq f(x)\}) = m(\emptyset) = 0$$

2. **Symmetry** We assume that $f \sim g$ which means

$$m(\{x \,|\, f(x) \neq g(x)\}) = 0$$

   but by the symmetric property of the normal equality, we have

$$m(\{x \,|\, g(x) \neq f(x)\}) = 0$$

   and this implies $g \sim f$.

3. **Transitivity** We assume that $f \sim g$ and $g \sim h$.
   It is difficult to work with the inequation, so instead we use the complement.

$$\begin{aligned}
\{x \,|\, f(x) \neq h(x)\} &= \{x \,|\, f(x) = h(x)\}^C \\
&\subseteq \{x \,|\, f(x) = g(x) \wedge g(x) = h(x)\}^C \\
&= \{x \,|\, f(x) \neq g(x) \vee g(x) \neq h(x)\} \\
&= \{x \,|\, f(x) \neq g(x)\} \cup \{x \,|\, g(x) \neq h(x)\}
\end{aligned}$$

   Then from this, we have that

$$m(\{x \,|\, f(x) \neq h(x)\}) \leq m(\{x \,|\, f(x) \neq g(x)\}) + m(\{x \,|\, g(x) \neq h(x)\}) = 0$$

   and this implies $f \sim h$. **Q E D**