Properties of Fourier Transform

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DEFINITIONS

$$\hat{f}(k) = \tau^{-n/2} \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\, dx \qquad (\tau = 2\pi)$$

$$\check{f}(x) = \tau^{-n/2} \int_{\mathbb{R}^n} e^{ik\cdot x} f(k)\, dk \quad \text{for } f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

Claim 1 (linearity): $\mathcal{F} = \hat{\cdot}$ is a linear map on $L^1(\mathbb{R}^n)$

Proof: For $f, g \in L^1(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$:

$$\widehat{[f+\lambda g]}(k) = \tau^{-n/2} \int_{\mathbb{R}^n} e^{-ik\cdot x} [f+\lambda g](x)\, dx$$

$$= \tau^{-n/2} \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\, dx + \tau^{-n/2} \int_{\mathbb{R}^n} e^{-ik\cdot x} (\lambda g)(x)\, dx$$

$$= \hat{f}(k) + \lambda \hat{g}(k) \quad \blacksquare$$

Claim 2 (boundedness): $|\hat{f}(k)| \leq \tau^{-n/2}\int_{\mathbb{R}^n}|f(x)|\,dx$

Proof: Since $k\cdot x$ is real, $|e^{-ik\cdot x}| \leq 1$ $\forall x \in \mathbb{R}^n$.

$$|\hat{f}(k)| = \left|\tau^{-n/2}\int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\,dx\right| \leq \tau^{-n/2}\int_{\mathbb{R}^n}\left|e^{-ik\cdot x} f(x)\right| dx$$

$$\leq \tau^{-n/2}\int_{\mathbb{R}^n}|f(x)|\,dx \quad \blacksquare$$

Claim 3 (Rieman-Lebesgue Lemma): $\mathcal{F}$ maps $L^1(\mathbb{R}^n)$ to $C_0(\mathbb{R}^n)$

Proof: Let us first show that the characteristic function

$\chi_I(x) = \begin{cases} 1, & x \in I \\ 0, & x \notin I \end{cases}$, where I is an n-box $= \prod_{j=1}^{n} [a_j, b_j]$,

decays as $\|k\| \to \infty$. Consider $|\hat{\chi}_I(k)|$,

$$|\widehat{\chi_I}(k)| = \left| \prod_{j=1}^{n} \int_{a_j}^{b_j} e^{-ik_jx_j}\, dx_j \right|$$

Then there are two cases, $k_j = 0$ and $k_j \neq 0$

If $k_j = 0$, $\left|\int_{a_j}^{b_j} e^0\, dx_j\right| = |b_j - a_j|$

If $k_j \neq 0$, $\left|\int_{a_j}^{b_j} e^{-ik_jx_j}\, dx_j\right| = \left| \frac{i}{k_j} \left(e^{-ik_jb_j} - e^{-ik_ja_j}\right)\right| \leq \frac{2}{|k_j|}$

If $\|k\| \to \infty$, then for some $k_j \neq 0$, $k_j \to \infty$, so

$|\hat{\chi}_I(k)| \leq \prod_{j=1}^{m} |b_j - a_j| \prod_{j=m}^{n} \frac{2}{|k_j|} \longrightarrow 0$, as $\|k\| \to \infty$.

Since F is linear, and $\chi_I \in L^1(\mathbb{R}^n)$, any simple function

$g = \sum_k c_k \chi_{I_k}$

will also decay as $\|k\| \to \infty$. Furthermore, the space of simple functions is dense in L^1. If we let f be an arbitrary integrable function and $\varepsilon > 0$ be given, there exists a simple function g such that $\|f-g\|_1 < \varepsilon$. We also know that by Claim 2,

$$|\widehat{[f-g]}(k)| = |\hat{f}(k) - \hat{g}(k)| \leq \tau^{-n/2} \|f-g\|_1 < \varepsilon$$

Since $\lim_{\|k\| \to \infty} \hat{g}(k) = 0$, $\exists N \in \mathbb{N}$ such that when $\|k\| > N$, $|\hat{g}(k)| < \varepsilon$

$$\Rightarrow |\hat{f}(k)| \leq |\hat{f}(k) - \hat{g}(k)| + |\hat{g}(k)| < \varepsilon + \varepsilon = 2\varepsilon$$

Since ε is arbitrary, it follows that $\lim_{\|k\| \to \infty} \hat{f}(k) = 0$

We now prove that $\hat{f}(k)$ is continuous in $\mathbb{R}^n$. Fix $\varepsilon>0$.

Then we consider $\|\hat{f}(k)-\hat{f}(k')\|$ for some k'.

By definition:

$$\|\hat{f}(k)-\hat{f}(k')\| = \tau^{-n/2}\left|\int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\,dx - \int_{\mathbb{R}^n} e^{-ik'\cdot x} f(x)\,dx\right|$$
$$= \tau^{-n/2}\left|\int_{\mathbb{R}^n} (e^{-ik\cdot x} - e^{-ik'\cdot x}) f(x)\,dx\right|$$
$$\leq \tau^{-n/2}\int_{\mathbb{R}^n} |e^{-ik\cdot x} - e^{-ik'\cdot x}||f(x)|\,dx$$
$$= \tau^{-n/2}\left[\int_{\mathbb{R}^n\setminus B_r(0)} |e^{-ik\cdot x} - e^{-ik'\cdot x}||f(x)|\,dx + \int_{B_r(0)} |e^{-ik\cdot x} - e^{-ik'\cdot x}||f(x)|\,dx\right]$$

Since f is globally integrable, it's integrable in $\mathbb{R}^n\setminus B_r(0)$, so we can find an $r>0$ such that $\int_{\mathbb{R}^n\setminus B_r(0)} |f(x)|\,dx \leq \frac{\varepsilon}{4\tau^{n/2}}$.

Then $\tau^{-n/2}\int_{\mathbb{R}^n\setminus B_r(0)} |e^{-ik\cdot x} - e^{-ik'\cdot x}||f(x)|\,dx < 2\tau^{-n/2}\int_{\mathbb{R}^n\setminus B_r(0)} |f(x)|\,dx$

$\leq \frac{\varepsilon}{2}$, $\forall k'$.

Then consider $\int_{B_r(0)} |e^{-ik\cdot x} - e^{-ik'\cdot x}||f(x)|\,dx$

Since $|e^{-ia}|\leq 1\ \forall a$, it is less than $\|f(x)\|_{L^1(\mathbb{R}^n)}$, but more than that, it is bounded by supremum in $B_r(0)$, so

$$\|\hat{f}(k)-\hat{f}(k')\| \leq \frac{\varepsilon}{2} + \tau^{-n/2} \sup_{y\in B_r(0)} |e^{-ik\cdot y} - e^{-ik'\cdot y}| \|f(x)\|_{L^1(\mathbb{R}^n)}$$

However, we can make $|e^{-ik\cdot y} - e^{-ik'\cdot y}|$ arbitrarily small, precisely we can say that there exists δ so that for $k' \in B_\delta(k)$,

$$\sup_{y \in B_r(0)} \| e^{-ik\cdot y} - e^{-ik'\cdot y} \| \, \| f(x) \|_{L^1(\mathbb{R}^n)} \leq \frac{\varepsilon}{2\tau^{-n/2}}$$

Therefore, we have that for $k' \in B_\delta(k)$,

$\| \hat{f}(k) - \hat{f}(k') \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$, so $\hat{f}$ is continuous. ■

DEFINITION

$$\boxed{[f * g](X) = \tau^{-n/2} \int_{\mathbb{R}^n} f(X-Y)\, g(Y)\, dY}$$

Claim 4 (convolution): $\widehat{[f*g]} = \hat{f}\hat{g}$.

Proof: $\widehat{[f*g]}(k) = \tau^{-n} \int_{\mathbb{R}^n} e^{-ik\cdot x} \int_{\mathbb{R}^n} f(x-Y)\, g(Y)\, dY\, dx$

$= \tau^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x-Y)\, g(Y)\, dY\, dx$

Since $f, g \in L^1(\mathbb{R}^n)$, $\|f\|_1 \|g\|_1 < \infty$, but

$\int |[f*g](x)|\, dx \leq \iint |f(x-Y)\, g(Y)|\, dx\, dY = \iint |f(x-Y)|\, dx\, g(Y)\, dY$

$= \|f\|_1 \int |g(Y)|\, dY = \|f\|_1 \|g\|_1 < \infty \implies f*g \in L^1(\mathbb{R}^n)$

Therefore, $\iint |e^{-ik\cdot x}| |f(x-y)\, g(y)|\, dy\, dx < \infty$, and we may switch the order of integration.

$$\Rightarrow \tau^{-n} \int_{\mathbb{R}^n} g(y) \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x-y)\, dx\, dy$$

Also, $e^{-ik\cdot x} = e^{-ik\cdot(x-y) - ik\cdot y} = e^{-ik\cdot(x-y)} e^{-ik\cdot y}$

$$\Rightarrow \tau^{-n} \int_{\mathbb{R}^n} e^{-ik\cdot y} g(y) \underbrace{\left(\int_{\mathbb{R}^n} e^{-ik(x-y)} f(x-y)\, dx \right)}_{= \hat{f}(k)} dy$$

$$= \tau^{-n/2} \int_{\mathbb{R}^n} e^{-ik\cdot y} g(y)\, dy\ \hat{f}(k) = \hat{f}(k)\, \hat{g}(k)$$

So we have shown that $\widehat{[f * g]} = \hat{f}\, \hat{g}$ ■

Claim 5 (derivatives): If $\partial_j f \in L^1(\mathbb{R}^n) \cap L^0(\mathbb{R}^n)$ exists, then

$$\widehat{[-i\partial_j f]}(k) = k_j \hat{f}(k)$$

Proof: Consider $\hat{f}(k)$. Then $f(x) = \check{\hat{f}}(x)$.

$$\Rightarrow f(x) = \tau^{-n/2} \int_{\mathbb{R}^n} e^{ik\cdot x} \hat{f}(k)\, dk$$

$$\Rightarrow \partial_j f(x) = \tau^{-n/2} \frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} e^{ik\cdot x} \hat{f}(k)\, dk = \tau^{-n/2} \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} e^{+ik\cdot x}\right) \hat{f}(k)\, dk$$

And $\frac{\partial}{\partial x_j} e^{+ik\cdot x} = e^{+ik\cdot x}\left(\frac{\partial}{\partial x_j}(+ik\cdot x)\right)$

But $ik\cdot x = i \sum_{l} k_l x_l \Rightarrow \partial_j (ik\cdot x) = i \sum_{l} k_l \frac{\partial x_l}{\partial x_j}$

$= i \sum_{l} k_l \delta_{lj} = i k_j$, where we have used that $\partial x_l / \partial x_j = \delta_{lj}$

$$\Rightarrow -i\partial_j f(x) = \tau^{-n/2} \int_{\mathbb{R}^n} e^{ik\cdot x} (k_j \hat{f}(k))\, dk$$

$$\Rightarrow \widehat{[-i\partial_j f]}(k) = k_j \hat{f}(k) \quad \blacksquare$$