Supports of regular and dirac delta distributions

Yat Ming Luk

April 21, 2023

Preliminary Notions

As we know from the lectures, we can understand that a support is the closure of the domain for a function where it is non-zero. In particular, we consider an open set $\Omega \in \mathbb{R}^n$ and set $\mathcal{D}(\Omega)$ as the set of smooth functions with compact support in Ω. The support can be written set-theoretically as below:

$$\text{supp}(f) = \text{cl}_\Omega(\{x \in \Omega : f(x) \neq 0\})$$

We would like to generalize this notion to distributions. Specifically, we want to understand what a support means in $\mathcal{D}'(\Omega)$.

In order to do this, we need to establish the following notions:

1. Nullity, or the "zero distribution"
2. Set union and complement

These notions are important because if we are to generalize the notion of support, we need a notion of a zero distribution. We also want to construct the set such that given any test function, the output of the distribution is not zero somewhere in the domain. This is rather inconvenient, as there are an infinite number of test functions. However, we can simply consider the distributions that are zero *everywhere* on the domain, and take the complement.

Nullity

Recall that distributions are maps $T : \mathcal{D}(\Omega) \to \mathbb{K}$. The only appropriate definition for the nullity is when the kernel of the distribution is equal to the domain of the distribution. However, in this case, we can define nullity on an open set $K \subset \Omega$. Therefore, we write that $T = 0$, on K if

$$\forall f \in \mathcal{D}(\Omega) \text{ with supp(f)} \subset K,\ \langle T, f\rangle \equiv T(f) = 0$$

Set union and complement

We need to know that the union of open sets is open. The important fact is that we can take any finite union of open sets to attain another open set. We also know that the complement of an open set is a closed set. These are fundamental results from topology.

Supports in $\mathcal{D}'(\Omega)$

Definition

Let us suppose that there is an arbitrary family of open sets $K_i \subset \Omega$ that nullify the distribution $T \in \mathcal{D}'(\Omega)$. Let us consider the union of such a family and call it K. K is called an open annihilation set, as it is an open set (union of open sets) that "annihilates" the distribution. If we have the infinitary union over all open subsets $K_i \in \Omega$ that nullify the distribution, then clearly the complement of such a set will give a closed subset of Ω where the distribution is *not* annihilated.

Therefore, we define the support of a distribution $T \in \mathcal{D}'(\Omega)$ as the complement of the open annihilation set. Written set-theoretically below, we have:

$$\text{supp}(T) := \left(\bigcup K_{T=0}\right)^{\complement}$$

Remark: If we recall our definition of nullity, we defined the nullity of a distribution such that for any test function with its support in an open set K. However, in general, for two open sets K_1, K_2, that nullify T, the set of test functions $\{f\}_1$ with support in K_1 will not be the same as the test functions $\{f\}_2$ with support in K_2. Can we say that the union $K_1 \cup K_2$ will also nullify the distribution T? Fortunately, the answer is actually yes, but proving this requires an argument using partitions of unity and compactness, which is beyond the scope of the course. If this is unsatisfactory, we can simply replace the union of the arbitrarily open sets by choosing the largest open set that nullifies the distribution.

Regular Distributions

In this section, we consider the support of regular distributions.
Recall that any function $h \in L^1_{loc}(\Omega)$ can be identified with a distribution $T_h \in \mathcal{D}'(\Omega)$ by the formula

$$\langle T_h,\ f\rangle = \int_\Omega h(X)f(X)dX = \int_{\text{supp}(h)} h(X)f(X)dX$$

for any test function f.
Since $h(X') = 0, \forall X' \notin \text{supp}(h)$, we can limit the domain of integration to $\text{supp}(h)$. We also note that f is totally arbitrary. To find the largest open set $K \subset \Omega$ such that $\langle T_h,\ f\rangle = 0, \forall f$ with $\text{supp}(f) \subset K$, this will correspond to the open set for which $h(K) = 0$. This is exactly the complement of $\text{supp}(h)$, so we find that for regular distributions:

$$\text{supp}(T_h) = \text{supp}(h) = \text{cl}_\Omega(\{X \in \Omega : h(X) \neq 0\})$$

This is very nice. As soon as we have the support of a locally integrable function, we also have the support of its corresponding regular distribution.

Delta Distributions

Dirac delta distribution

Now we consider the dirac delta distribution δ_Y. Hopefully you can already imagine that the support of this distribution will be the set $\{Y\}$, based on the heuristic definition, however, we want to use our definition to show this. Recall that $\forall f \in \mathcal{D}(\Omega)$, the dirac delta is defined by the following relation:

$$\langle \delta_Y,\ f\rangle = f(Y)$$

We are trying to find the largest set $K \in \Omega$ for which $\delta_Y = 0,$ on K. We must remember that the test functions are arbitrary, so we have freedom to choose test functions with any arbitrary support. Let us assume that there is a set K such that the distribution is nullified. Then on K we have,

$$\langle \delta_Y,\ f\rangle = f(Y) = 0$$

It is clear that if the support of a test function f includes the point Y, then $f(Y) \neq 0$. Therefore, we want a set such that the support of the test function can never include the point Y. Since an annihilation set is such that the support of the test functions must be a subset of the set K, then we can simply exclude the point Y.

To construct the largest open annihilation set, we then consider all the open sets $\{K\}$ that do not include the point Y, namely $K \subset \Omega - \{Y\}$. For any such K, and any test function $f \in \mathcal{D}(\Omega)$ with support in K, one has that $Y \notin \text{supp}(f)$, which implies that $\langle \delta_Y,\ f\rangle = f(Y) = 0$. This means K is a nullity. It is then not difficult to see that the reunion of all such K will give the set from which these K were selected from, namely,

$$\tilde{K} = \bigcup_{\substack{K \subset \Omega - \{Y\} \\ K \text{ open}}} K = \Omega - \{Y\}$$

Alternatively, we can argue that the largest open annihilation set in Ω is $\Omega - \{Y\}$, as any open set in Ω that excludes Y will be a nullity.

The last step is to take the complement of $\tilde{K}$, and so we conclude that

$$\text{supp}(\delta_Y) = \{Y\}$$

.

α-Derivative of dirac delta distribution

Now, we consider the α-derivative of dirac delta distribution δ_Y^α. This is defined by the following relation.

$$\langle \delta_Y^\alpha,\ f \rangle = (-1)^{|\alpha|}[\partial^\alpha f](Y)$$

We clearly do not care about the (-1) in the front, as we are looking for nullity.
Let us try the same strategy as before. We again assume there exists some open set K such that

$$\langle \delta_Y^\alpha,\ f \rangle = (-1)^{|\alpha|}[\partial^\alpha f](Y) = 0 \Longleftrightarrow [\partial^\alpha f](Y) = 0$$

You may think that this constraint is less constraining than the nullifying condition of the dirac delta distribution. However, this is not the case.

Let us try using the previous set $\tilde{K_1} = \Omega - \{Y\}$. If $\text{supp}(f) \subset \tilde{K_1}$, then $Y \notin \text{supp}(f)$. This means $f(Y) = 0$. However, this trivially implies $[\partial^\alpha f](Y) = 0$. This means that $\tilde{K_1}$ *already* annihilates δ_Y^α. All that we need to check now is that this is the largest open annihilation set.

Let us now assume that the only larger set which is $\tilde{K_2} = \Omega$ also nullifies the distribution. Now we do not know that $f(Y) = 0$, so we cannot trivially conclude $[\partial^\alpha f](Y) = 0$. It turns out that the possibility that $f(Y) \neq 0$ means that this set does not nullify δ_Y^α. Since nullification requires the distribution to give 0 for *any* test function with support in the open set, any test function that has support in the open set that does *not* get mapped to 0 by the distribution will contradict the assumption that the set nullifies the distribution.

Since test functions are smooth and therefore infinitely differentiable, there must exist some test functions not in the kernel of δ^α. This means that for those functions f, $[\delta^\alpha f](Y) \neq 0$, and so we cannot include the point Y, because those test functions do not evaluate to 0, implying that $\tilde{K_2}$ is not a nullity.

We can conclude that the support of δ_Y^α is the complement of the largest open annihilation set $\tilde{K_1}$, namely

$$\text{supp}(\delta_Y^\alpha) = \{Y\}$$