## 2.4 Lebesgue integral

Note that we use this notation below: for $f, g : [a,b] \to \mathbb{R}$ and $s \in \mathbb{R}$,

$$\{f > g\} = \{x \in [a,b] \mid f(x) > g(x)\}, \quad \{f > s\} = \{x \in [a,b] \mid f(x) > s\}.$$

In preparation for the second half of the proof of Theorem 2.4.3., we show Lemma 2.4.6..

### Proof of Lemma 2.4.6

**Lemma 2.4.6.** *Let $f \in \mathcal{L}^\infty([a,b])$ be Lebesgue measurable, and assume that $f \geq 0$ a.e. on $[a,b]$ and that $\int_a^b f(x)\,\mathrm{d}x = 0$. Then $f = 0$ a.e. on $[a,b]$.*

Since $f \geq 0$ a.e., $m(\{f<0\}) = 0$, so we only need to show $m(\{f>0\}) = 0$ for proving $f = 0$ a.e.. We can easily observe that $\{f>0\} = \bigcup_{k=1}^{\infty} E_k$, where $E_k = \{f \geq \frac{1}{k}\}$.

For each $k \in \mathbb{N}$, $\frac{1}{k}\chi_{E_k}(x) \leq \frac{1}{k} \leq f(x)$ for all $x \in E_k$. (here, $\chi$ is a characteristic function.)

We can easily observe that for any measurable set $\Omega \subset [a,b]$, $\int_a^b \chi_\Omega(x)\,dx = m(\Omega)$, by considering the partition $P = \{\Omega, [a,b] \setminus \Omega\}$.

So for each $k \in \mathbb{N}$, $\frac{1}{k} m(E_k) = \int_a^b \frac{1}{k}\chi_{E_k}(x)\,dx \leq \int_a^b f(x) = 0$, so $m(E_k) = 0$.

By the above, $m(\{f>0\}) = m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k) = 0$, which was the only thing we need to show.

### Proof of Theorem 2.4.3

**Theorem 2.4.3.** *For any $f \in \mathcal{L}^\infty([a,b])$, $f$ is a Lebesgue measurable function if and only if $f$ is a Lebesgue integrable function.*

We show that for any $f \in \mathcal{L}^\infty([a,b])$, $f$'s integrability implies its measurability, the proof of which was not given in the lecture.

Suppose $f \in \mathcal{L}^\infty([a,b])$ and $f$ is L-integrable. By its integrability and a property of supremum and infimum, for each $k \in \mathbb{N}$, there exists a partition $P_k = \{E_j^k\}_{j=1}^{n_k}$ of $[a,b]$ such that (i) $U[f, P_k] - L[f, P_k] < \frac{1}{k}$,

(ii) for each $j \in \{1, \cdots, n_{k+1}\}$, $E_j^{k+1} \subset E_l^k$ for some $l \in \{1, \cdots, n_k\}$ ($P_{k+1}$ is a refinement of $P_k$),

(iii) $E_{j_1}^k \cap E_{j_2}^k = \phi$ if $j_1 \neq j_2$, and (iv) each $E_j^k$ is L-measurable.

Now we set $g_k(x) = \sum_{j=1}^{n_k} m_j^k \chi_{E_j^k}(x)$, where $m_j^k = \inf_{x \in E_j^k} f(x)$.

For any $x \in E_{j_0}^{k+1}$, $x \in E_{l_0}^k \left(\supset E_{j_0}^{k+1}\right)$ for some $l_0$ and so

$$g_{k+1}(x) = \sum_{j=1}^{n_{k+1}} m_j^{k+1} \chi_{E_j^{k+1}}(x) = \inf_{x \in E_{j_0}^{k+1}} f(x) \geq \inf_{x \in E_{l_0}^k} f(x) = \sum_{l=1}^{n_k} m_l^k \chi_{E_l^k}(x) = g_k(x).$$

Therefore, for all $x \in [a,b] \left(= \bigcup_{j=1}^{n_k} E_j^k\right)$, $\{g_k(x)\}_{k=1}^{\infty}$ is an increasing sequence.

Moreover, for each $k$ and all $x$, $g_k(x) = \sum_{j=1}^{n_k} m_j^k \chi_{E_j^k}(x) \leq \max_j m_j^k = \max_j \left(\inf_{x \in E_j^k} f(x)\right) \leq \sup_{x \in [a,b]} f(x) < \infty$,

so $\{g_k(x)\}_k$ is bounded. By monotone convergence theorem, the sequences have their limits and we can consider the function $g(x) := \lim_{k\to\infty} g_k(x)$.
Since $\{E_j^k\}_{j=1}^{n_k}$ are L-measurable sets for each $k$ and so each $g_k$ is L-measurable, the pointwise limit $g$ is also L-measurable by Theorem 2.3.11.
We can easily observe that $\int_a^b g_k(x)dx = L[f, P_k]$ and $g_k(x) \le g(x) \le f(x)$ for each simple function $g_k$. Thus, $L[f, P_k] = \int_a^b g_k(x)dx \le \int_a^b g(x)dx \le \int_a^b f(x)dx$.
On the other hand, we set $h_k(x) := \sum_{j=1}^{n_k} M_j^k \chi_{E_j^k}(x)$ where $M_j^k = \sup_{x\in E_j^k} f(x)$.
Then we get the pointwise limit function $h$, which is L-measurable and satisfies $\int_a^b f \le \int_a^b h \le \int_a^b h_k = U[f, P_k]$, by almost the same argument. Now we know $g \le f \le h$ and so $h - g \ge 0$, then
$0 \le \int_a^b (h-g) = \int_a^b h - \int_a^b g = U[f, P_k] - L[f, P_k] < \frac{1}{k}$ for any $k \in \mathbb{N}$.
Therefore, we get $\int_a^b (h-g) = 0$ and so $h = g$ a.e. by Lemma 2.4.6 that we proved above.
Since $g \le f \le h$, $\{g \ne f\} \subset \{g \ne h\}$ and $m(\{g\ne f\}) \le m(\{g \ne h\}) = 0$ so $g = f$ a.e.
By Exercise 2.3.8 (I provided the proof of it in my previous report), $f$ is L-measurable from the measurability of $g$.

Reference: [Ne]