Exercise 2.3.8 Show that if $f : [a,b] \to \mathbb{R}$ is Lebesgue measurable and $f = g$ a.e., then g is also Lebesgue measurable.

Let us use the following notations; $\{f \neq g\} = \{x \in [a,b] \mid f(x) \neq g(x)\}$, $\{f < s\} = \{x \in [a,b] \mid f(x) < s\}$.
For any $s \in \mathbb{R}$ and for any $x \in \{g > s\} (= \{x \in [a,b] \mid g(x) > s\})$, $x \notin \{f \neq g\} \iff f(x) = g(x) \Rightarrow x \in \{f > s\}$, so $\{g > s\} = (\{g > s\} \cap \{f \neq g\}) \cup (\{f > s\} \cap \{f \neq g\}^c)$. Since $f = g$ a.e., $\{f \neq g\}$ has measure 0, and so $\{g > s\} \cap \{f \neq g\}$ has out measure 0, which means it's Lebesgue measurable. Indeed, $m^*(\{g > s\} \cap \{f \neq g\}) \leq m^*(\{f \neq g\}) = m(\{f \neq g\}) = 0$. The Complement $\{f \neq g\}^c$ is also Lebesgue measurable. In addition, $\{f > s\}$ is Lebesgue measurable due to f's Lebesgue measurablity. Therefore, $\{g > s\}$ is Lebesgue measurable for any $s \in \mathbb{R}$, so we conclude g is Lebesgue measurable.

Exercise 2.3.10

(i) Show that $\limsup_{j \to \infty} f_j(x)$ and $\liminf_{j \to \infty} f_j(x)$ always exist.

Recall $f^*(x) = \limsup_{j \to \infty} f_j(x) := \lim_{j \to \infty} \{\sup_{k \geq j} f_k(x)\}$ and $f_*(x) = \liminf_{j \to \infty} f_j(x) := \lim_{j \to \infty} \{\inf_{k \geq j} f_k(x)\}$.
For any $x \in \mathbb{R}$, $\{\sup_{k \geq j} f_k(x)\}_{j=1}^{\infty}$ and $\{\inf_{k \geq j} f_k(x)\}_{j=1}^{\infty}$ are a monotonically decreasing and increasing sequence respectively. If $\sup_{k \geq j} f_k(x) = \pm\infty$ for any $j \in \mathbb{N}$, $\limsup_{j \to \infty} f_j(x) = \pm\infty$. If not, there exists $N \in \mathbb{N}$ and $M \in \mathbb{R}$ such that $|f_j(x)| < M$ for any $j \geq N$, namely $\{\sup_{k \geq j} f_k(x)\}_{j=N}^{\infty}$ is bounded. Since a bounded monotone sequence always converges (monotone convergence theorem), the limit of the sequence exists. The same argument can be applied to $\liminf_{j \to \infty} f_j(x)$.

(ii) Show that $f_*(x) \leq f^*(x)$

For any $j \in \mathbb{N}$, $\inf_{k \geq j} f_k(x) \leq \sup_{k \geq j} f_k(x)$ from the definition of supremum and infimum. The inequality (with the equal sign) remains true as j goes to ∞.

(iii) Show that $f_*(x) = f^*(x)$ if and only if $\lim_{j \to \infty} f_j(x)$ exists.

Firstly, we suppose $f_*(x) = f^*(x)$. Let $\alpha := f^*(x) (= f_*(x))$. We observe that $\inf_{k \geq j} f_k(x) \leq f_j(x) \leq \sup_{k \geq j} f_k(x)$. for any $j \in \mathbb{N}$, Since $\sup_{k \geq j} f_k(x)$ and $\inf_{k \geq j} f_k(x)$ Converge to α,

for any $\varepsilon > 0$, $\alpha - \varepsilon \leq \inf_{k \geq j} f_k(x) \leq f_j(x) \leq \sup_{k \geq j} f_k(x) \leq \alpha + \varepsilon$, for any $j > {}^{\exists}N$.

This indicates $\lim_{j \to \infty} f_j(x)$ exists with its value α.

Conversely, if we suppose $\lim_{j \to \infty} f_j(x)$ exists and $\beta := \lim_{j \to \infty} f_j(x)$, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $j > N$, $|\beta - f_j(x)| < \varepsilon$, namely $\beta - \varepsilon < f_j(x) < \beta + \varepsilon$.
So we observe $\beta - \varepsilon \leq \inf_{k \geq j} f_k(x) \leq f_j(x) \leq \sup_{k \geq j} f_k(x) \leq \beta + \varepsilon$ for any $j > N$.
Thus we get $|\beta - \inf_{k \geq j} f_k(x)|, |\beta - \sup_{k \geq j} f_k(x)| \leq \varepsilon$, which indicates both $\{\inf_{k \geq j} f_k(x)\}_{j=1}^{\infty}$ and $\{\sup_{k \geq j} f_k(x)\}_{j=1}^{\infty}$ converge to β and leads the conclusion.

Proof of theorem 2.3.11

Theorem 2.3.11. *Let $f_j : [a,b] \to \mathbb{R}$ be Lebesgue measurable functions, and assume that $(f_j)_{j\in\mathbb{N}}$ is a pointwise bounded sequence. Then the two functions f^* and f_* are also Lebesgue measurable on $[a,b]$.*

For each $j \in \mathbb{N}$, we set functions $\sup_{f_j}$ and $\inf_{f_j}$ as $\sup_{f_j}(x) := \sup_{k\geq j} f_k(x)$ and $\inf_{f_j}(x) := \inf_{k\geq j} f_k(x)$, $x\in[a,b]$.
If we observe that $\sup_{f_j}$ and $\inf_{f_j}$ are Lebesgue measurable functions, then it can be concluded that

$$f^*(x) = \lim_{j\to\infty}\left\{\sup_{k\geq j} f_k(x)\right\} = \inf_{j\geq 1}\left\{\sup_{k\geq j} f_k(x)\right\} = \inf_{\sup_{f_j}}(x) \text{ and } f_*(x) = \lim_{j\to\infty}\left\{\inf_{k\geq j} f_k(x)\right\} = \sup_{j\geq 1}\left\{\inf_{k\geq j} f_k(x)\right\} = \sup_{\inf_{f_j}}(x)$$

are Lebesgue measurable functions.
(Note that the second steps of the two equalities come from the property of supremum and infimum that $\{\sup_{k\geq j} f_k(x)\}_{j=1}^{\infty}$ and $\{\inf_{k\geq j} f_k(x)\}_{j=1}^{\infty}$ are a monotonically decreasing and increasing sequence respectively.)
So we only need to show the Lebesgue measurability of $\sup_{f_j}$ and $\inf_{f_j}$.
Let us use the same notations we used in Exercise 2.3.8. For any $s\in\mathbb{R}$,

$$x\in\{\sup_{f_j} > s\} \iff \sup_{k\geq j} f_k(x) > s \iff \exists k_0 \geq j \text{ s.t. } \left(\sup_{k\geq j} f_k(x) \geq\right) f_{k_0}(x) > s \iff \exists k_0\geq j \text{ s.t. } x\in\{f_{k_0} > s\} \iff x \in \bigcup_{k\geq j}^{\infty}\{f_k > s\},$$

so $\{\sup_{f_j} > s\} = \bigcup_{k\geq j}^{\infty}\{f_k > s\}$. Since each f_k is a Lebesgue measurable function, $\{f_k > s\}_{k=1}^{\infty}$ are Lebesgue measurable sets. From a property of Lebesgue measurable sets (in particular, the property that the class of Lebesgue measurable sets is a σ-algebra or a completely additive class), the union of Lebesgue measurable sets $\{\sup_{f_j} > s\} = \bigcup_{k\geq j}^{\infty}\{f_k > s\}$ is a Lebesgue measurable set.
Therefore, $\sup_{f_j}$ is a Lebesgue measurable function.
We can now easily observe that $\inf_{f_j}$ is also a Lebesgue measurable function since $\inf_{f_j}(x) = \inf_{k\geq j} f_k(x) = -\sup_{k\geq j}\{-f_k(x)\}$ and it can be applied to the above.

Proof of corollary 2.3.12

Corollary 2.3.12. *Let $f_j : [a,b] \to \mathbb{R}$ be Lebesgue measurable functions, and assume that $(f_j)_{j\in\mathbb{N}}$ is a pointwise bounded sequence. Assume that for each $x \in [a,b]$, the functions f_j have a limit as $j\to\infty$, namely $\exists f : [a,b] \to \mathbb{R}$ such that $\lim_{j\to\infty} f_j(x) = f(x)$ for all $x\in[a,b]$. Then the function f is a Lebesgue measurable function on $[a,b]$.*

From the result of Exercise 2.3.10. (iii) that we showed above, $\lim_{j\to\infty} f_j(x) = f^*(x)$ for all $x\in[a,b]$ if it exists (and it really does here).
Since f^* is Lebesgue measurable from Theorem 2.3.11., $f(x) = \lim_{j\to\infty} f_j(x)$ is also Lebesgue measurable.

Extra

Lebesgue measurability of composite functions

Let $f: \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function, and $g: \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the composite function $g \circ f$ is also a Lebesgue measurable function.

Let us take any $s > 0$. Since g is continuous, $\{g > s\}$ is an open set on $\mathbb{R}$.
Indeed, for any $\alpha \in \{g > s\}$, there exists $\delta > 0$ such that $|x - \alpha| < \delta \Rightarrow |g(x) - g(\alpha)| < \frac{g(\alpha) - s}{2}$, namely $s < \frac{g(\alpha) + s}{2} < g(x) < \frac{3g(\alpha) - s}{2}$ for any $x \in [\alpha - \delta, \alpha + \delta]$, which means an open ball $B_\delta(\alpha) \subset \{g > s\}$.
We now set $I_j := (j-1, j+1) \cap \{g > s\}$ for each $j \in \mathbb{Z}$, where we can take intervals $\{I_j\}$ by dividing them if necessary and reordering $\{I_j\}$ while skipping empty sets.
Then $\{I_j\}_{j=-\infty}^{\infty}$ is a countable group of open intervals and $\{g > s\} = \bigcup_{j=-\infty}^{\infty} I_j$.

So $\{g \circ f > s\} = \{x \in \mathbb{R} \mid f(x) \in \{g > s\}\} = \bigcup_{j=1}^{\infty} \{x \in \mathbb{R} \mid f(x) \in I_j\} = \bigcup_{j=1}^{\infty} f^{-1}(I_j)$.
For each j, if we write $I_j = (a_j, b_j)$, $f^{-1}(I_j) = \{x \in \mathbb{R} \mid f(x) \in (a_j, b_j)\} = \{f > a_j\} \cap \{f < b_j\}$.
Since f is a Lebesgue measurable function, both $\{f > a_j\}, \{f < b_j\}$ are Lebesgue measurable sets, so is $f^{-1}(I_j)$, and so is the union $\{g \circ f > s\} = \bigcup_{j=1}^{\infty} f^{-1}(I_j)$.
Therefore, $g \circ f$ is a Lebesgue measurable function.