Exercise 2.2.5

1. If $\Omega_1 \subset \Omega_2$, then $m^*(\Omega_1) \le m^*(\Omega_2)$

For any covering $\{U_j\}$ of Ω_2, $\{U_j\}$ is also a covering of Ω_1 since $\Omega_1 \subset \Omega_2 \subset \bigcup_j U_j$.
In other words, the set of coverings of Ω_1 includes the set of coverings of Ω_2.
In general, when infimum is taken among a larger set, it gets smaller.
Therefore, $m^*(\Omega_1) = \inf\{\sigma(S) \mid S : \text{covering of } \Omega_1\} \le \inf\{\sigma(S) \mid S : \text{covering of } \Omega_2\} = m^*(\Omega_2)$ ■

2. $m^*(\Omega_1 \cup \Omega_2) \le m^*(\Omega_1) + m^*(\Omega_2)$

As proven above, if $m^*(\Omega_1)$ and/or $m^*(\Omega_2)$ is infinite, then $m^*(\Omega_1 \cup \Omega_2)$ is also infinite,
so the inequation $\infty = m^*(\Omega_1 \cup \Omega_2) \le m^*(\Omega_1) + m^*(\Omega_2) = \infty$ is true by convention. We assume both $m^*(\Omega_1)$ and $m^*(\Omega_2)$ are finite below.
Let us take $\forall \varepsilon > 0$. Since $m^*(\Omega)$ is the infimum of the volume of Ω's coverings, there exists sets of coverings $\{U_j\}$ and $\{V_j\}$ such that
$m^*(\Omega_1) \le \sigma(\{U_j\}) < m^*(\Omega_1) + \frac{\varepsilon}{2}$, $m^*(\Omega_2) \le \sigma(T) < m^*(\Omega_2) + \frac{\varepsilon}{2}$, by a property of infimum. Since $\{U_j\} \cup \{V_j\}$ is a set of $\Omega_1 \cup \Omega_2$'s coverings,
$m^*(\Omega_1 \cup \Omega_2) \le \sigma(\{U_j\} \cup \{V_j\}) \le \sigma(\{U_j\}) + \sigma(\{V_j\}) < m^*(\Omega_1) + m^*(\Omega_2) + \varepsilon$.
Due to the arbitrarity of ε, we conclude $m^*(\Omega_1 \cup \Omega_2) \le m^*(\Omega_1) + m^*(\Omega_2)$ ■

3. $m^*(\bigcup_i \Omega_i) \le \sum_i m^*(\Omega_i)$ for a finite or countable family.

This can be proven with induction using the result of the previous question.
Indeed, if for $N \in \mathbb{N}$, $m^*(\bigcup_{i=1}^N \Omega_i) \le \sum_{i=1}^N m^*(\Omega_i)$, then $m^*(\bigcup_{i=1}^{N+1} \Omega_i) = m^*\left(\left(\bigcup_{i=1}^N \Omega_i\right) \cup \Omega_{N+1}\right) \le m^*(\bigcup_{i=1}^N \Omega_i) + m^*(\Omega_{N+1})$
$\le \sum_{i=1}^N m^*(\Omega_i) + m^*(\Omega_i) = \sum_{i=1}^{N+1} m^*(\Omega_i)$. ■

Exercise 2.2.4

Show that $m^*(I) = v(I)$ for any closed box.

Since $I \subset I$, $m^*(I) = \inf\{\sigma(S) \mid S \text{ is a covering of } I\} \le \sigma(I) = v(I)$
To show the other direction of inequality, assume $\{I_j\}_{j=1}^\infty$ is a covering of I.
If $v(I) \le \sigma(\{I_j\})$ for any $\{I_j\}$, then $v(I) \le \inf\{\sigma(S) \mid S \text{ is a covering of } I\}$ is concluded.
When $\sigma(\{I_j\}) = \infty$, the inequlity $v(I) \le \sigma(\{I_j\})$ obviously holds, so suppose $\sigma(S) < \infty$.
For any $\varepsilon > 0$, and for each j, $\exists S_j \subset \mathbb{R}^n$, open box, s.t. $I_j \subset S_j$, $v(\overline{S_j}) \le (1+\varepsilon) v(I_j)$
(If $I_j = \{x \in \mathbb{R}^n \mid a_k \le x_k \le b_k \text{ for all } k \in \{1, \cdots, n\}\}$, take $S_j = \{x \in \mathbb{R}^n \mid a_k - l_k < x_k < b_k + l_k$ with $l_k = \frac{1}{2}((1+\varepsilon)^{\frac{1}{n}} - 1)(b_k - a_k)$ for all $k\}$ for example)
Observe that since $I \subset \bigcup_{j=1}^\infty I_j \subset \bigcup_{j=1}^\infty S_j$, $\{S_j\}_{j=1}^\infty$ is an open covering, and by the compactness of I,
there exists a subsequence of $\{S_j\}_{j=1}^\infty$, such that $I \subset \bigcup_{k=1}^N S_{j_k}$, and so $v(I) \le \sum_{k=1}^N v(\overline{S_{j_k}}) \le (1+\varepsilon) \sum_{k=1}^N v(I_{j_k}) \le (1+\varepsilon) \sum_{j=1}^\infty v(I_j) \cdots (*)$
Due to the arbitrarity of ε, $v(I) \le \sum_{j=1}^\infty v(I_j)$ for any covering $\{I_j\}_{j=1}^\infty$, so $v(I) \le m^*(I)$. Thus, $m^*(I) = v(I)$.
The first inequality of $(*)$ requires a little bit more argument, but it is easy to show if one conciders the intersection. ■

Show that $m([0,1] \cap \mathbb{Q}) = 0$.

Since $\mathbb{Q}$ is a countable set, $[0,1] \cap \mathbb{Q}$ can be written as $[0,1] \cap \mathbb{Q} = \{r_j\}_{j=1}^\infty$
For any $\varepsilon > 0$, assume $\Omega_j = [r_j - \frac{\varepsilon}{2^{j+1}}, r_j + \frac{\varepsilon}{2^{j+1}}]$, $j = 1, 2, \cdots$
Then $\{\Omega_j\}_{j=1}^\infty$ is a covering of $[0,1] \cap \mathbb{Q}$ because each point of $[0,1] \cap \mathbb{Q}$, r_j, belongs to Ω_j.
Because of this, $m^*([0,1] \cap \mathbb{Q}) \le m^*(\bigcup_{j=1}^\infty \Omega_j) \le \sum_{j=1}^\infty m^*(\Omega_j)$
Each Ω_j is an interval (a closed box in $\mathbb{R}$), so $m^*(\Omega_j) = v(\Omega_j) = \frac{\varepsilon}{2^j}$ (proved above),
and so $m^*([0,1] \cap \mathbb{Q}) \le \sum_{j=1}^\infty m^*(\Omega_j) = \sum_{j=1}^\infty \frac{\varepsilon}{2^j} = \varepsilon$
Dut to the arbitrarity of ε, $m^*([0,1] \cap \mathbb{Q}) = 0$ ■