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Motivation for introducing distributions

In the lecture, we learned the concept of distributions, which followed that of test function and was followed by that of differentiation, convergence, summability of distribution and so on.
But in the first place, why did we need to introduce these concepts? What can we use them for? For one, one may need distributions in Partial Differential Equations (PDE) theory.

For $u \in C^1(\mathbb{R}_x \times \mathbb{R}_t ; \mathbb{R})$, let us consider $u_t + [f(u)]_x = 0$, $u(x,0) = \phi(x)$ $\cdots$ $(*)$, where f is a C^1 function on $C^1(\mathbb{R}_x \times \mathbb{R}_t ; \mathbb{R})$ to $C^1(\mathbb{R}_x \times \mathbb{R}_t ; \mathbb{R})$, which means if $f(u) = 3u^2 + 1$ then $f(u)$ is a function defined by $(f(u))(x,t) = 3u^2(x,t) + 1$, and we suppose $f(0) = 0$ (e.g. $f(u) = u^2 + \frac{u}{2}$) and ϕ is given.
The second equality of $(*)$ gives an initial condition to the first equation of $(*)$, which is a PDE and the main part of $(*)$.
$(*)$ is called a conservation law. This is because for any C^1 function u, which decays as $|x| \to \infty$, if u solves the PDE, the total "mass" (it's mass if we regard u as density) does not change over t.

Indeed, $\frac{d}{dt} \int_{\mathbb{R}} u(x,t)\,dt = \int_{\mathbb{R}} u_t(x,t)\,dt = -\int_{\mathbb{R}} [f(u)]_x\,dx = 0$

Notice that the second equality is from $(*)$ and the last one is because u decays as x goes to infinity. Most importantly, the first equality requires an explanation and it's not easy to show in a general sense.
(Actually I'm not sure if it's possible.)

Now let us try to solve $(*)$. Firstly,
$(*)$ is a kind of "first order quasilinear PDEs", which are of the form

$$a(x,y,u(x,y))\,u_x(x,y) + b(x,y,u(x,y))\,u_y(x,y) = c(x,y,u(x,y)) \quad (**)$$

where we suppose $a, b, c \in C^1(\mathbb{R}^3 ; \mathbb{R})$.

We get (*) if we assign $a = f'(u)$, $b = 1$, $c = 0$ (notice $[f(u)]_x = f'(u)\, u_x$).
So we can solve (*) by applying the way to solve first order quasilinear PDEs. The logic is a bit technical, but in short, (**) can be reduced to some ODEs (Ordinary Differential Equation), which we can solve (relatively) easily, by introducing parameters s and r. Namely, for (*),

$$\begin{cases} \partial_s x(r,s) = f'(z(r,s)),\ x(r,0) = r. & (1) \\ \partial_s t(r,s) = 1,\ t(r,0) = 0, & (2) \\ \partial_s z(r,s) = 0,\ z(r,0) = \phi(r). & (3) \end{cases}$$

where $z(r,s) = u(x(r,s), t(r,s))$.

(2) and (3) give $z(r,s) = \phi(r)$, $t(r,s) = s$

We substitute this $z(r,s)$ into (1) and get $x(r,s) = f'(\phi(r))\, s + r$.
Since $s = t$, if r is fixed, $x = f'(\phi(r))\, t + r$ gives a straight line in xt-plane. Also, while r is fixed,
$u(x(r,s), t(r,s)) = z(r,S) = \phi(r)$ is constant.
So if we suppose $\exists r_1, \exists r_2$ s.t $f'(\phi(r_1)) \neq f'(\phi(r_2))$ (so also $\phi(r_1) \neq \phi(r_2)$), two straight lines in xt-plane intersect at some point s_0,
so $\phi(r_1) = z(r_1, s_0) = z(r_2, s_0) = \phi(r_2)$, which means singularities occur,
and therefore any (classical) continuous solution of (*) does not exist.
We need the idea of distribution here. In other words, the existence of discontinuous solutions of (*) can be proven with distributions.

Let us think briefly about why distribution can offer a solution to the problem. As we learned in the lecture, for any $h \in L^1_{loc}(\mathbb{R}^n)$, T_h (defined as $T_h(f) = \int_{\mathbb{R}^n} h(X) f(X)\, dX$, $f \in D(\mathbb{R}^n)$) gives an example of a distribution. This map $T : h \mapsto T_h$ is injective, and therefore $D(\mathbb{R}^n)$ has a one-to-one correspondence with the image $T(D(\mathbb{R}^n))$.

(Recall $D(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$)

Also, as a matter of fact, $T(\mathcal{D}(\mathbb{R}^n))$ is dense in $\mathcal{D}'(\mathbb{R}^n)$, which means any element of $\mathcal{D}'(\mathbb{R}^n)$ can be approximated by T_{h_j} ($h_j \in \mathcal{D}(\mathbb{R}^n)$). (This is actually why elements of $\mathcal{D}(\mathbb{R}^n)$ are called "test" functions!)

Furthermore, we can rationally define many properties of functions such as convergence, differentiation, multiplication with elements of $C^\infty(\mathbb{R}^n)$.

Given that $\mathcal{D}'(\mathbb{R}^n)$ is a strictly larger space than $T(\mathcal{D}(\mathbb{R}^n))$, and taking the above into consideration, we conclude that $\mathcal{D}'(\mathbb{R}^n)$ can be considered to be an extension of $T(\mathcal{D}(\mathbb{R}^n))$.

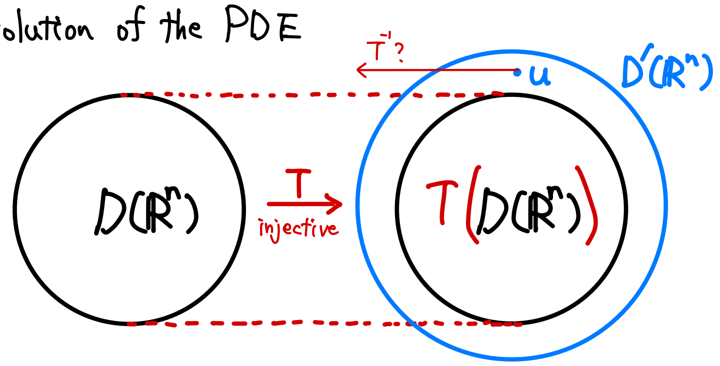
So we can say if some u satisfies a PDE in the sense of distributions, which means, in the case of (*), $u_t + [f(u)]_x = 0$, $T_u^{-1}(x,0) = \phi(x)$, $u \in \mathcal{D}'(\mathbb{R}^n)$, then T_u^{-1} is a discontinuous solution of the PDE.

[To be precise, h and T_h (or, u and T_u^{-1}) should be totally identified, so we just write $T_u^{-1}(x)$ as $u(x)$ because T^{-1} does not generally exist, and we call $u \in \mathcal{D}'(\mathbb{R}^n)$ a weak solution.

More explanation;

If $h \in \mathcal{D}(\mathbb{R}^n)$ solves a PDE $\mathcal{L}h = f$ ($\mathcal{L}$: partial differential operator), then T_h satisfies the corresponding PDE $\mathcal{L}T_h = T_f$. In this analogue, since we can consider $\mathcal{D}'(\mathbb{R}^n)$ to be an extension of $\mathcal{D}(\mathbb{R}^n)$, if $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfies $\mathcal{L}u = T_f$ in the same sense as $\mathcal{L}T_h = T_f$, we can say the function that corresponds to u by T is a solution of the original PDE.

However, $h \in \mathcal{D}(\mathbb{R}^n)$ satisfying $u = T_h$ does not exist if $u \in \mathcal{D}'(\mathbb{R}^n) \setminus T(\mathcal{D}(\mathbb{R}^n))$, so we directly call $u \in \mathcal{D}'(\mathbb{R}^n)$ a weak solution of the PDE



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This is how distributions play a key role in analysis.

I hope this promotes your understanding of distribution theory.