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Schwarz Inequality in Hilbert space

$$|\langle f,g\rangle| \leq \|f\|\,\|g\|$$

From definition , $\langle f|f\rangle = \|f\|^2$

$$|\langle f,g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle$$

Since $\|f-\lambda g\| \geq 0$

Let $I = \langle f-\lambda g\,|\,f-\lambda g\rangle \geq 0$ for $\lambda \in \mathbb{C}$

$$\langle f|f\rangle - \langle f|\lambda g\rangle - \langle \lambda g|f\rangle + \langle \lambda g|\lambda g\rangle \geq 0$$

$$\langle f|f\rangle - \lambda\langle f|g\rangle - \lambda^*\langle g|f\rangle + |\lambda|^2\langle g|g\rangle \geq 0$$

Given $\lambda = \dfrac{\langle g|f\rangle}{\langle g|g\rangle}$, $\lambda^* = \dfrac{\langle f|g\rangle}{\langle g|g\rangle}$

$$|\lambda|^2 = \lambda\lambda^* = \frac{\langle g|f\rangle\langle f|g\rangle}{(\langle g|g\rangle)^2}$$

$$\Rightarrow \langle f|f\rangle - \frac{\langle g|f\rangle}{\langle g|g\rangle}\langle f|g\rangle - \cancel{\frac{\langle f|g\rangle}{\langle g|g\rangle}\langle g|f\rangle} + \cancel{\frac{\langle g|f\rangle\langle f|g\rangle}{(\langle g|g\rangle)^2}}\,\cancel{\langle g|g\rangle} \geq 0$$

$$\langle f|f\rangle\langle g|g\rangle - \langle f|g\rangle^*\langle f|g\rangle \geq 0$$

$$\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2 \quad \square$$