Ex 2.2.8

Proof on 5 properties of L.m.

1. If Ω is an open set, then Ω is L.m.

if we set $\Lambda \subset \mathbb{R}^n$ such that $\Lambda = \Omega$

Then, $m^*(\Lambda \setminus \Omega) = m^*(\Omega \setminus \Omega)$

$= 0$ ($\because \Omega$ is open)

If we set $\varepsilon > 0$

we can say that $m^*(\Lambda \setminus \Omega) \leq \varepsilon$, $^\forall \varepsilon > 0$

Finally Ω is L.m. //

2. If $m^*(\Omega) = 0$, then Ω is L.m.

If we set

$\forall \varepsilon > 0$, $\exists \Lambda$ open and $\Omega \subset \Lambda$

such that $m^*(\Lambda) \leq m^*(\Omega) + \varepsilon$ ($\because$ Theorem 2.2.6) $\cdots$ ①

So, $m^*(\Lambda \setminus \Omega) \leq m^*(\Lambda)$

$\leq m^*(\Omega) + \varepsilon$

$\leq \varepsilon$ ($\because m^*(\Omega) = 0$)

Finally Ω is L.m //

3. If $\Omega := \bigcup_j \Omega_j$ is a finite or countable union of L.m sets, then Ω is L.m. and $m(\Omega) \leq \sum_j m(\Omega_j)$ ⑤

$$\Omega := (\Omega_1 \cup \Omega_2 \cup \Omega_3 \cdots)$$

$$m^*(\Omega)$$

$$= m^*(\bigcup_j \Omega_j) \leq m^*(\Omega_1) + m^*(\Omega_2) + m^*(\Omega_3) \cdots$$

($\because$ outer measure property (3) in my Report 4)

Ω_j is L.m.

So, if we set $\Lambda_j \subset \mathbb{R}^n$, Λ_j is open, and $\Lambda_j \supset \Omega_j$ and $\forall \varepsilon > 0$, $j \in \mathbb{N}$

$$m^*(\Lambda_j \setminus \Omega_j) \leq \frac{\varepsilon}{(j+1)^2} \quad - ②$$

Also we set $\Lambda := \bigcup_j \Lambda_j$

We set $\Lambda := \bigcup_j \Lambda_j$ and $\Lambda_j \supset \Omega_j$, so

$$\Lambda \supset \Omega$$

and $\Lambda \setminus \Omega \subseteq \bigcup_j (\Lambda_j \setminus \Omega_j)$

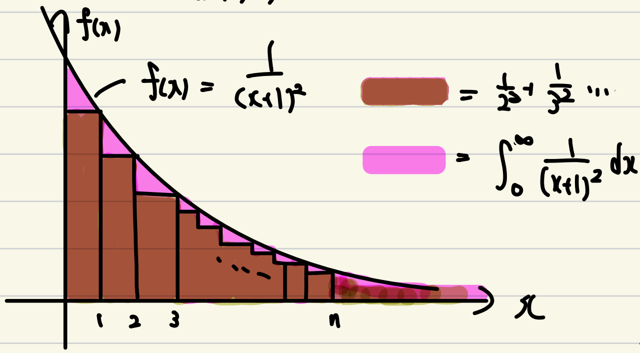
$$\therefore m^*(\Lambda \setminus \Omega) \leq m^*(\bigcup_j (\Lambda_j \setminus \Omega_j)) \quad (\because \text{outer measure property (1) in my Report 4})$$

$$\leq m^*(\Lambda_1 \setminus \Omega_1) + m^*(\Lambda_2 \setminus \Omega_2) \cdots$$

$$= \frac{\varepsilon}{2^2} + \frac{\varepsilon}{3^2} + \frac{\varepsilon}{4^2} \cdots \qquad (\because ②)$$

$$\leq \varepsilon \left(\int_0^\infty \frac{1}{(x+1)^2} \right)$$

$(\because$



$)$

$$= \varepsilon \left[-\frac{1}{(x+1)} \right]_0^{\infty}$$

Finally, Ω is L.m.

And about the proof on ⑤,

we show that $m^*(\bigcup_j \Omega_j) \leq \sum_j m^*(\Omega_j)$

in my Report 4

In this point.

For any L.m. set Ω, we define $m(\Omega) \doteq m^*(\Omega)$.

Therefore,

$$m(\Omega) = m(\bigcup_j \Omega_j)$$

$$\leq \sum_j m(\Omega_j) \,//$$

4. If $\Omega := \bigcap_j \Omega_j$ is a finite or countable intersection of Lebesgue measurable sets, then Ω is L.m.

$\bigcap_j \Omega_j = \Omega_1 \cap \Omega_2 \cap \Omega_3 \cdots$

$(\bigcap_j \Omega_j)^c = \Omega_1^c \cup \Omega_2^c \cup \Omega_3^c \cdots$ (we set $A \in \mathbb{R}$ such that $A^c = \mathbb{R}/A$)

Ω_j is L.m.

so, Ω_j^c is L.m. ($\because$ Theorem 2.2-(0)) $\cdots$ (1)

Also using 3rd property in this report and (1),
$(\bigcap_j \Omega_j)^c$ is L.m.

$\therefore$ $\bigcap_j \Omega_j$ is L.m. ($\because$ Theorem 2.2-(0))

finally Ω is L.m.//

5 Any closed set is L.m

As 1st property proof,

any open set Ω is L.m. — ①

And according to theorem 2.2.10.

if we set Ω is L.m, Ω^c is also L.m. — ②

So, we set any open set A,

thanks to ①. ②.

A^c is L.m.

A is any open set, so A^c is any close set

Therefore any closed set is L.m. //