Ex 2.2.5

Proof on following properties

1. If $\Omega_1 \subset \Omega_2$, then $m^*(\Omega_1) \leq m^*(\Omega_2)$

We set S is a covering of Ω_2

$\therefore$ S is also a covering of Ω_1 ($\because \Omega_1 \subset \Omega_2$)

$\therefore m^*(\Omega_1) \leq \sigma(S)$

$\therefore m^*(\Omega_1) \leq \inf\{\sigma(S) \mid S \text{ covering of } \Omega_2\}$

$\therefore m^*(\Omega_1) \leq m^*(\Omega_2)$ //

2. $m^*(\Omega_1 \cup \Omega_2) \leq m^*(\Omega_1) + m^*(\Omega_2)$

If we think the set of S, T and $\forall \varepsilon > 0$

such that S is a covering of Ω_1, so $m^*(\Omega_1) + \varepsilon \geq \sigma(S)$

T is a covering of Ω_2, so $m^*(\Omega_2) + \varepsilon \geq \sigma(T)$ $\cdots$ ①

Then $S \cup T$ means a covering of Ω_1 & Ω_2

$\therefore m^*(\Omega_1 \cup \Omega_2) \leq \sigma(S \cup T)$

By the way $\sigma(S \cup T) \leq \sigma(S) + \sigma(T)$

So, $m^*(\Omega_1 \cup \Omega_2) \leq \sigma(S \cup T) \leq \sigma(S) + \sigma(T)$ $\cdots$ ②

Using ①, ② $m^*(\Omega_1 \cup \Omega_2) \leq m^*(\Omega_1) + m^*(\Omega_2) + 2\varepsilon$ $\cdots$ ③

$\forall \varepsilon > 0$ satisfies ③, so. if we take ε arbitrary small

$$m^*(\Omega_1 \cup \Omega_2) \le m^*(\Omega_1) + m^*(\Omega_2)$$

3. $m^*(\bigcup_j \Omega_j) \le \sum_j m^*(\Omega_j)$ for a finite or countable family

we can use mathmatic induction method if $m(\Omega_j)$ is finite

But if that is infinite, we can't apply the induction rule ...

Let me consider infinite case.

If we suppose $m^*(\Omega_j)$ is infinite, $m^*(\bigcup_j \Omega_j)$ is also infinite

that means. the result is true by the convention that $\infty \le \infty$

So, infinite case is solved.

Then let me focus finite case

We set S_j being a covering of Ω_j $\qquad 1-r > 1$

$\forall \varepsilon > 0$

such that $m^*(\Omega_j) \le \sigma(S_j) \le m^*(\Omega_j) + \frac{\varepsilon}{2^j}$ ... ①

Because S_j being a covering of Ω_i, by closed intervals

$\bigcup S_j$ is a covering of $\bigcup_j \Omega_j$ by closed intervals

Therefore, $m^*(\bigcup_j \Omega_j) \le \sigma(\bigcup S_j)$

$\le \sum_j \sigma(S_j)$ ($\because$ the same closed interval is contained more in $\sum \sigma(S_j)$)

$$\leq \sum_j \left(m^*(\Omega_j) + \frac{\varepsilon}{2^j} \right)$$

$$\leq \sum_j m^*(\Omega_j) + \varepsilon \quad \cdots ②$$

$\forall \varepsilon > 0$ satisfies ②

So, if we take ε arbitrary small,

$$m^*(\cup_j \Omega_j) \leq \sum_j m^*(\Omega_j).$$