Ex 1.3.8

1) Proof ①

$$\boxed{\begin{array}{l}\text{Set } h: \mathbb{R}^n \to \mathbb{K} \text{ satisfying } \underbrace{\int_{\mathbb{R}^n} h(x)\,dx = 1}_{S_1}, \\ j \in \mathbb{N},\ \underbrace{h_j(x) := (j)^n h(jx)}_{S_2} \text{ with:} \\ T_{h_j} \longrightarrow \delta_0 \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ as } j \to \infty\end{array}}$$

Set $j \in \mathbb{N}$, $h_j(x) := j^n h(jx)$, $f \in \mathcal{D}(\mathbb{R}^n)$ with:

$$T_{h_j}(f) = \int_{\mathbb{R}^n} h_j(x) \cdot f(x)\,dx$$

$$= \int_{\mathbb{R}^n} j^n h(jx) f(x)\,dx \quad — Ⓐ \quad (\because S_2)$$

Let me replace jx with y $(x, y \in \mathbb{R}^n)$

So, $jx = y$ — Ⓑ

$(j)^n dx = dy$ ($\because$ thinking in $\mathbb{R}^n$)

$\therefore dx = \frac{1}{(j)^n} dy$ — Ⓒ

Using Ⓐ, Ⓑ, Ⓒ,

$$T_{h_j}(f) = \int_{\mathbb{R}^n} j^n h(y) f\left(\frac{y}{j}\right) \frac{1}{(j)^n}\,dy$$

$$= \int_{\mathbb{R}^n} h(y) f\left(\frac{y}{j}\right) dy \quad \cdots Ⓓ \quad (*)$$

(*) By the way

Set : $f_j(y) = h(y) f(y/j)$

$f_j(y) \xrightarrow[j \to \infty]{} h(y) f(0)$ $\left(\because \begin{array}{c} f \text{ belongs to } D(\mathbb{R}^n), \text{so} \\ f \text{ is continuous} \end{array} \right)$ … (1)

; And f belongs to $D(\mathbb{R}^n)$, so $\exists M > 0$ $|f(y)| \leq M$, $\forall y \in \mathbb{R}^n$,

if we set $g(y) = M|h(y)|$,

$|f_j(y)| = |h(y) f(y/j)|$

$= |h(y)| \cdot |f(y/j)|$

$\leq M|h(y)| = g(y)$ … (2), $\forall j \geq 1$

Also $\int_{\mathbb{R}^n} g(y)\,dy = M \int_{\mathbb{R}^n} |h(y)|\,dy < \infty$, then $g \in L^1(\mathbb{R}^n)$ … (3)

lastly, h, f are L.m., so f_j is also L.m. — (4) $\left(\because \begin{array}{l} \int_{\mathbb{R}^n} h(x)\,dx = 1 \text{ and} \\ f \in D(\mathbb{R}^n). \end{array} \right)$

Using (1) ~ (4), by dominated convergence theorem, we have

f_j is L.i. and

$\lim_{j \to \infty} \int_{\mathbb{R}^n} f_j(y)\,dy = \int_{\mathbb{R}^n} \lim_{j \to \infty} f_j(y)\,dy = \int_{\mathbb{R}^n} h(y)\,dy \cdot f(0) = f(0) = \delta_0(f)$ — (5)

$(\because (S_1))$

Finally Using (D) and (5)

$\lim_{j \to \infty} T_{h_j}(f) = \delta_0(f)$, $\forall f \in D(\mathbb{R}^n)$

$\Rightarrow T_{h_j} \longrightarrow \delta_0$ as $j \to \infty$