Ex1.2.3

First of all, Let me set $f, g \in D(\mathbb{R})$ … (S)

$$\int_{\mathbb{R}} f'(x) g(x) = f(x)\cdot g(x)\Big|_{\mathbb{R}} - \int_{\mathbb{R}} f(x) g'(x)\,dx$$

$$= -\int_{\mathbb{R}} f(x) g'(x)\,dx \quad (\because \text{ using (S)},$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} g(x) = 0,$$

$$f(x)\cdot g(x)\Big|_{\mathbb{R}} = 0\cdot 0 - 0\cdot 0 = 0)$$

Likewise, If I set $f, g \in D(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} [\partial_j f](x) g(x) dx = -\int_{\mathbb{R}^n} f(x) [\partial_j g](x) dx$

if I do this α $(\in \mathbb{N}^n)$ times.

$$\int_{\mathbb{R}^n} [\partial^\alpha f](x) g(x) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)\cdot[\partial^\alpha g](x)\,dx \quad -(A)$$

Next, Let me prove. $\partial^\alpha T_h = T_{\partial^\alpha h}$ $(\alpha \in \mathbb{N}^n)$

$$\partial^\alpha T_h(f) = (-1)^{|\alpha|} T_h(\partial^\alpha f) \quad (\because \text{Def } 1.2.1)$$

$$= (-1)^{|\alpha|} \int_{\mathbb{R}^n} h(x)\,(\partial^\alpha f)(x)\,dx$$

$$= \int_{\mathbb{R}^n} [\partial^\alpha h](x)\cdot f(x)\,dx \quad (\because (A))$$

$$= T_{\partial^\alpha h}(f) \quad //$$

Ex 1.1.11

In this report, I admit theorem 1.1.10 is true.

$T: D(\mathbb{R}^n) \to \mathbb{K}$ belongs to $D'(\mathbb{R}^n)$

$$\begin{cases} \cdot\, T(f_1 + \lambda f_2) = T(f_1) + \lambda T(f_2) \ (f_1, f_2 \in D(\mathbb{R}^n), \lambda \in \mathbb{K}) & -(A) \\ \cdot\, \forall Y \in \mathbb{R}^n, \forall r > 0, \exists c > 0, \exists m \in \mathbb{N}, \\ \quad \forall f \in D(\mathbb{R}^n) \ (\operatorname{supp} f \subset \overline{B_r(Y)}) \text{ with :} \\ \qquad |T(f)| \le c \sum_{|\alpha|\le m} \|\partial^\alpha f\|_\infty & -(B) \end{cases}$$

Using the conditions above,

I'd like to think about each order $(T_h, \delta_Y, \delta_Y^\alpha)$

① the order of T_h

Because of my report 1, I proved $T_h \in D'(\mathbb{R}^n)$.

So using $\operatorname{supp} f \subset \overline{B_r(Y)}$ in (B),

$$T_h(f) = \int_{\overline{B_r(Y)}} h(x) f(x)\,dx \quad (\text{after this, I call } \overline{B_r(Y)}\ M)$$

$$\therefore |T_h(f)| = \left|\int_M h(x) f(x)\,dx\right|$$

$$\le \int_M |h(x) f(x)|\,dx$$

$$= \int_M |h(x)||f(x)|\,dx$$

$$\le \sup_{x\in\mathbb{R}} |f(x)| \cdot \int_M |h(x)|\,dx$$

$$= \|f\|_\infty \int_M |h(x)|\,dx \quad (\because \text{Def } 1.1.6$$

Actually,

$h(x) \in L^1_{loc}(\mathbb{R}^n)$ and $|h(x)| \in \mathbb{K}$

$f_j \xrightarrow[j\to\infty]{} f_\infty$

$f_j, f_\infty \in D(\mathbb{R}^n))$

so. I set $\int_M |h(x)|\,dx = H$ (H is constant)

$$\therefore |T_h(f)| \le \|f\|_\infty \cdot H$$

In this case, there are no derivative
therefore $|\alpha|=0$

$\therefore |T_h(f)| \le H \sum_{|\alpha|\le 0} \|f_\infty\|$

$\therefore$ T_h's order is 0 (smallest $m=0$)

② the order of δ_Υ

$$|\delta_\Upsilon(f)| = |f(\Upsilon)|$$
$$\le \sup_{\Upsilon\in\mathbb{R}} |f(\Upsilon)|$$
$$= \|f\|_\infty \quad (\because \text{Def } 1.1.6)$$
$$= 1\cdot \sum_{|\alpha|\le 0} \|f\|_\infty \quad (\text{Likewise ①})$$

$\therefore$ δ_Υ's order is also 0 (smallest $m=0$)

③ the order of δ_Υ^α

$$|\delta_\Upsilon^\alpha(f)| = |[\partial^\alpha f](\Upsilon)|$$
$$\le \sup_{\Upsilon\in\mathbb{R}} |[\partial^\alpha f](\Upsilon)|$$

according to Def 1.1.6,

$$= \|\partial^\alpha f\|_\infty \quad (\because \text{if I derivative } |\alpha| \text{ times on } f,\ \partial^\alpha f_j \xrightarrow[j\to\infty]{} \partial^\alpha f_\infty)$$
$$\le 1\cdot \sum_{|\beta|\le|\alpha|} \|\partial^\beta f\|_\infty$$

$\therefore$ δ_Υ^α's order is $|\alpha|$ // (But, I'm not sure…)

Reference;
Distributions: characterisation, support, and order,
by Pratham Dhomne and Vic Austen