Ryosuke Mizutani (1st y)

Ex 1.1.9

These two conditions are needed to be proved

(i) $T(f_1 + \lambda f_2) = T(f_1) + \lambda T(f_2)$ $\forall f_1, f_2 \in D(\mathbb{R}^n)$ and $\lambda \in \mathbb{K}$

(ii) whenever the sequence $(f_j)_{j\in\mathbb{N}} \subset D(\mathbb{R}^n)$ converges to $f_\infty \in D(\mathbb{R}^n)$, — Ⓢ

then, the sequence $(T(f_j))_{j\in\mathbb{N}}$ converges to $T(f_\infty)$ in $\mathbb{K}$ as $j \to \infty$

Then, I'd like to show why T_h, δ_Y, δ_Y^α belongs to $D'(\mathbb{R}^n)$ each.

① the proof that T_h belongs to $D'(\mathbb{R}^n)$

(i) $T_h(f) = \int_{\mathbb{R}^n} h(x) f(x)\,dx$ (set $h \in L^1_{loc}(\mathbb{R}^n)$ ⋯Ⓐ, $f \in D(\mathbb{R}^n)$)

because of properties of $D(\mathbb{R}^n)$,
if $f_1, f_2 \in D(\mathbb{R}^n)$ and $\lambda \in \mathbb{K}$,
then I can say that $f_1 + \lambda f_2 \in D(\mathbb{R}^n)$ ⋯Ⓑ

using Ⓐ and Ⓑ,

$$
\begin{aligned}
T_h(f_1+\lambda f_2) &= \int_{\mathbb{R}^n} h(x)(f_1+\lambda f_2)(x)\,dx \\
&= \int_{\mathbb{R}^n} \{h(x) f_1(x) + \lambda h(x) f_2(x)\}\,dx \\
&= \int_{\mathbb{R}^n} h(x) f_1(x)\,dx + \lambda \int_{\mathbb{R}^n} h(x) f_2(x)\,dx \\
&= T_h(f_1) + \lambda T_h(f_2) \quad (\because f_1, f_2 \in D(\mathbb{R}^n) \text{ and Ⓐ})
\end{aligned}
$$

(ii) Owing to Ⓢ, I can state two consequences

- $\forall \varepsilon_0 > 0$ $\exists n_0 \in \mathbb{N}$, $\forall j > n_0$, $f_j, f_\infty \in D(\mathbb{R}^n)$ with:
  $\|f_j - f_\infty\|_\infty < \varepsilon_0$ all times — Ⓒ
- $\exists r \in \mathbb{R}$ is so large enough : $\mathrm{supp}(f_j) \subset B_r(0)$ $\forall j \in \mathbb{N}$ — Ⓓ

My goal is to show

$$\boxed{\begin{array}{l}\forall \varepsilon_1 > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall j > n_0 \text{ with:} \\ |T_h(f_j) - T(f_\infty)| < \varepsilon_1 \text{ all times}\end{array}}$$

$$
\begin{aligned}
&|T_h(f_j) - T_h(f_\infty)| \\
&= \left| \int_{\mathbb{R}^n} h(x) f_j(x)\,dx - \int_{\mathbb{R}^n} h(x) f_\infty(x)\,dx \right| \quad (\because Ⓐ) \\
&= \left| \int_{\mathbb{R}^n} h(x)(f_j(x) - f_\infty(x))\,dx \right| \quad \text{— Ⓔ}
\end{aligned}
$$

by the way, thanks to Ⓓ, let me change integral interval.
So let's replace $\mathbb{R}^n$ with $B_r(0)$

$$
\begin{aligned}
Ⓔ &= \left| \int_{B_r(0)} h(x)(f_j(x) - f_\infty(x))\,dx \right| \\
&\le \int_{B_r(0)} |h(x)(f_j(x) - f_\infty(x))|\,dx \\
&\le \int_{B_r(0)} |h(x)| \cdot \|f_j - f_\infty\|_\infty\,dx \quad (\because |f_j(x) - f_\infty(x)| \le \sup_{x\in\mathbb{R}^n} |f_j(x) - f_\infty(x)| = \|f_j - f_\infty\|_\infty) \\
&< \int_{B_r(0)} |h(x)|\,dx \cdot \varepsilon_0 \text{ — Ⓕ} \quad (\because Ⓒ)
\end{aligned}
$$

Actually $\forall \varepsilon_0 > 0$, and I can set :

$$\varepsilon_0 = \frac{\varepsilon_1}{\int_{B_r(0)} |h(x)|\,dx} \quad (\because h(x) \in L^1_{loc}(\mathbb{R}^n), \text{ so } h(x) \in \mathbb{K})$$

All in all, Ⓕ $= \varepsilon_1$

$\therefore |T_h(f_j) - T_h(f_\infty)| < \varepsilon_1$

$\therefore$ Finally reaching to my goal in ① − (ii)

This is why $T_h(f)$ belongs to $D'(\mathbb{R}^n)$ //

② the proof that $\delta_Y \in D'(\mathbb{R}^n)$

(i) $\delta_Y(f) := f(Y) \quad (\underset{\text{set}}{Y} \in \mathbb{R}^n,\ f \in D(\mathbb{R}^n)) \ - Ⓐ$

Set: $f_1 . f_2 \in D(\mathbb{R}^n)$, $\lambda \in \mathbb{K}$

$$\delta_Y(f_1+\lambda f_2) = f_1(Y) + \lambda f_2(Y)$$
$$= \delta_Y(f_1) + \lambda \delta_Y(f_2) \quad (\because f_1 . f_2 \in D(\mathbb{R}^n), Ⓐ) \ //$$

(ii) Like ①–(ii), I'll use the same method.

My goal:

$$\boxed{\forall \varepsilon_2 > 0 \quad \exists n_0 \in \mathbb{N},\ \forall j > n_0 \text{ with: } |\delta_Y(f_j) - \delta_Y(f_\infty)| < \varepsilon_2 \text{ all times}}$$

Set $f_j . f_\infty \in D(\mathbb{R}^n)$, $Y \in \mathbb{R}^n$

$$|\delta_Y(f_j) - \delta_Y(f_\infty)| = |f_j(Y) - f_\infty(Y)|$$
$$\leq \sup_{Y \in \mathbb{R}^n} |f_j(Y) - f_\infty(Y)|$$
$$= \|f_j - f_\infty\|_\infty$$
$$< \varepsilon_0 \quad (\because ①-(ii)-Ⓒ) \ - Ⓑ$$

Actually $\forall \varepsilon_0 > 0$, $\forall \varepsilon_2 > 0$ so I can set

$\varepsilon_0 < \varepsilon_2 \ - Ⓒ$

All in all, $|\delta_Y(f_j) - \delta_Y(f_\infty)| < \varepsilon_0 < \varepsilon_2 \quad (\because Ⓑ, Ⓒ)$

Finally reaching my goal in ②–(ii) //

This is why δ_Y belongs to $D'(\mathbb{R}^n)$ //

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③ the proof that δ_Y^α belongs to $D'(\mathbb{R}^n)$

(i) Set $\alpha \in \mathbb{N}^n$, $Y \in \mathbb{R}^n$, $f \in D(\mathbb{R}^n)$.

$$\delta_Y^\alpha(f) := (-1)^{|\alpha|} [\partial^\alpha f](Y) \ - Ⓐ$$

Set $f_1 . f_2 \in D(\mathbb{R}^n)$, $\lambda \in \mathbb{K}$.

$$\delta_Y^\alpha(f_1+\lambda f_2) = (-1)^{|\alpha|}[\partial^\alpha f_1 + \lambda \partial^\alpha f_2](Y)$$
$$= (-1)^{|\alpha|}[\partial^\alpha f_1](Y) + \lambda (-1)^{|\alpha|}[\partial^\alpha f_2](Y)$$
$$= \delta_Y^\alpha(f_1) + \lambda \delta_Y^\alpha(f_2) \ // \quad (\because f_1 . f_2 \in D(\mathbb{R}^n), Ⓐ)$$

(ii) Like ①–(ii), I'll use the same method

My goal:

$$\boxed{\forall \varepsilon_3 > 0 \quad \exists n_0 \in \mathbb{N},\ \forall j > n_0 \text{ with: } |\delta_Y^\alpha(f_j) - \partial_Y^\alpha(f_\infty)| < \varepsilon_3 \text{ all times}}$$

Set $f_j, f_\infty \in D(\mathbb{R}^n)$,

$$|\partial_Y^\alpha(f_j) - \partial_Y^\alpha(f_\infty)| = |(-1)^{|\alpha|}[\partial^\alpha f_j](Y) - (-1)^{|\alpha|}[\partial^\alpha f_\infty](Y)|$$
$$= |(-1)^{|\alpha|}([\partial^\alpha f_j](Y) - [\partial^\alpha f_\infty](Y))|$$
$$= |[\partial^\alpha f_j](Y) - [\partial^\alpha f_\infty](Y)|$$
$$\leq \sup_{Y \in \mathbb{R}^n} |[\partial^\alpha f_j](Y) - [\partial^\alpha f_\infty](Y)|$$
$$= \|[\partial^\alpha f_j] - [\partial^\alpha f_\infty]\|_\infty \ - Ⓑ$$

By the way, remember Def 1.1.6.

it says 1. $\forall \alpha \in \mathbb{N}$ with: $\sup_{x \in \mathbb{R}^n} |[\partial^\alpha f_j](x) - [\partial^\alpha f_\infty](x)| \xrightarrow[j \to \infty]{} 0$

this means

$\forall \varepsilon_4 > 0 \quad \exists n_0 \in \mathbb{N}$, $\forall j > n_0$ with:

$\|[\partial^\alpha f_j] - [\partial^\alpha f_\infty]\|_\infty < \varepsilon_4$ all times $- Ⓒ$

Also $\forall \varepsilon_3 > 0$ $\forall \varepsilon_4 > 0$ I can set

$\varepsilon_4 < \varepsilon_3 \ - Ⓓ$

Using Ⓑ, Ⓒ and Ⓓ,

$|\delta_Y^\alpha(f_j) - \delta_Y^\alpha(f_\infty)| < \varepsilon_4 < \varepsilon_3$ // (reaching my goal in ③–(ii))

This is why δ_Y^α belongs to $D(\mathbb{R}^n)$

Reference;
Three standards distributions, by Yuu Hiramatsu

On regular distributions, by Haruka Yajima

Proofs on some distributions, by Firdaus Rafi Rizqy, Hadiko Rifqi Aufa Sholih, Sekiya Emika