Proving Riesz Lemma

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Lemma 3.1.13 (Riesz Lemma). *For any $\varphi \in \mathcal{H}^*$, there exists a unique $g \in \mathcal{H}$ such that for any $f \in \mathcal{H}$*

$$\varphi(f) = \langle g, f \rangle.$$

In addition, g satisfies $\|\varphi\|_{\mathcal{H}^*} = \|g\|$.

Proof:

For $\varphi \in \mathcal{H}^*$, let us define a set $\mathcal{M} := \{ f \in \mathcal{H} \mid \varphi(f) = 0 \}$. Since for $f_1, f_2 \in \mathcal{M}$ and $\lambda \in \mathbb{C}$ we have

$$\varphi(f_1 + \lambda f_2) = \varphi(f_1) + \lambda \varphi(f_2) = 0$$

then $f_1 + \lambda f_2 \in \mathcal{M}$, which indicate $\mathcal{M}$ is a subspace of $\mathcal{H}$.

If $(f_j)_{j \in \mathbb{N}} \subset \mathcal{M}$ such that $\lim_{j \to \infty} \| f_j - f_\infty \| = 0$, then the sequence has a limit $f_\infty \in \mathcal{H}$. We want to show that $f_\infty \in \mathcal{M}$. Consider that

$$\begin{aligned}
|\varphi(f_\infty)| &= |\varphi(f_\infty - f_j + f_j)| \\
&= |\varphi(f_\infty - f_j) + \varphi(f_j)| \\
&\leq |\varphi(f_\infty - f_j)| + |\varphi(f_j)| \\
&= \frac{|\varphi(f_\infty - f_j)|}{\| f_\infty - f_j \|} \| f_\infty - f_j \| \\
&\leq \|\varphi\|_{\mathcal{H}^*} \| f_\infty - f_j \|
\end{aligned}$$

$\varphi(f_j) = 0$

$\|\varphi\|_{\mathcal{H}^*} = \sup \dfrac{|\varphi(f_\infty - f_j)|}{\| f_\infty - f_j \|}$

Since $\lim_{j \to \infty} \| f_\infty - f_j \| = 0$ and φ is bounded (so $\|\varphi\|_{\mathcal{H}^*} < \infty$), then $|\varphi(f_\infty)| \leq 0$, which implies $\varphi(f_\infty) = 0$ and $f_\infty \in \mathcal{M}$. Hence, $\mathcal{M}$ is a closed subspace of $\mathcal{H}$.

For $\varphi = 0$, $g = 0$ satisfies $\varphi(f) = \langle g, f \rangle = 0 \quad \forall f \in \mathcal{H}$.

For $\varphi \neq 0$, since $\mathcal{M}$ is closed subspace, then by the projection theorem, we can decompose a vector $h \in \mathcal{H}$ into $h = h_1 + h_2$ with $h_1 \in \mathcal{M}$ and $h_2 \in \mathcal{M}^\perp$. Then, for $h \in \mathcal{H}$ such that $\varphi(h) \neq 0$, we get

$$\varphi(h) = \varphi(h_1 + h_2) = \varphi(h_1) + \varphi(h_2) = \varphi(h_2) \neq 0.$$

Since only $h_2 = 0$ gives $\varphi(h_2) = 0$ for $h_2 \in \mathcal{M}^\perp$, then $h_2 \neq 0$ for $\varphi(h) \neq 0$.

For arbitrary $f \in \mathcal{H}$, consider $f - \frac{\varphi(f)}{\varphi(h_2)} h_2 \in \mathcal{H}$ with $h_2 \in \mathcal{M}^\perp$ and $h_2 \neq 0$. Observe that

$$\varphi\left(f - \frac{\varphi(f)}{\varphi(h_2)} h_2\right) = \varphi(f) - \frac{\varphi(f)}{\varphi(h_2)} \varphi(h_2) = 0.$$

Hence, $f - \frac{\varphi(f)}{\varphi(h_2)} h_2 \in \mathcal{M}$. Then, since $\langle h_1, h_2 \rangle = 0$ for $h_1 \in \mathcal{M}$ and $h_2 \in \mathcal{M}^\perp$, we get

$$\langle h_2, f - \frac{\varphi(f)}{\varphi(h_2)} h_2 \rangle = 0$$

$$\Longleftrightarrow \quad \langle h_2, f \rangle - \frac{\varphi(f)}{\varphi(h_2)} \|h_2\|^2 = 0$$

$$\Longleftrightarrow \quad \varphi(f) = \frac{\varphi(h_2)}{\|h_2\|^2} \langle h_2, f \rangle = \left\langle \overline{\frac{\varphi(h_2)}{\|h_2\|^2}} h_2, f \right\rangle$$

Then, set $g := \frac{\overline{\varphi(h_2)}}{\|h_2\|^2} h_2$, we get

$$\varphi(f) = \left\langle \overline{\frac{\varphi(h_2)}{\|h_2\|^2}} h_2, f \right\rangle = \langle g, f \rangle.$$

Therefore, we obtain $\varphi(f) = \langle g, f \rangle \ \forall f \in \mathcal{H}$.

We want to check that g is unique. If $g_1, g_2 \in \mathcal{H}$ such that

$$\varphi(f) = \langle g_1, f \rangle = \langle g_2, f \rangle, \quad \forall f \in \mathcal{H}$$
$$\iff \langle g_1, f \rangle - \langle g_2, f \rangle = \langle g_1 - g_2, f \rangle = 0, \quad \forall f \in \mathcal{H}$$

Hence, $g_1 - g_2 = 0 \implies g_1 = g_2$ and g is unique.

Additionally, we have

$$\|\varphi\|_{\mathcal{H}^*} = \sup_{0 \neq f \in \mathcal{H}} \frac{|\varphi(f)|}{\|f\|} = \sup_{0 \neq f \in \mathcal{H}} \frac{\langle g, f \rangle}{\|f\|} = \frac{\|g\| \|f\|}{\|f\|}$$
$$= \|g\|. \qquad \square$$

As a consequence of the previous statement, one often identifies $\mathcal{H}^*$ with $\mathcal{H}$ itself.

Exercise 3.1.14. *Check that this identification is not linear but* anti-linear.

Proof:

For $\varphi_1, \varphi_2 \in \mathcal{H}^*$, $\lambda \in \mathbb{C}$, and $g_1, g_2 \in \mathcal{H}$ such that $\varphi_1(f) = \langle g_1, f \rangle$ and $\varphi_2(f) = \langle g_2, f \rangle$ $\forall f \in \mathcal{H}$, we have

$$(\varphi_1 + \lambda \varphi_2)(f) = \varphi_1(f) + \lambda \varphi_2(f) = \langle g_1, f \rangle + \lambda \langle g_2, f \rangle$$
$$= \langle g_1 + \bar{\lambda} g_2, f \rangle$$

Hence, the identification is anti-linear. $\square$