Exercise 2.6.12

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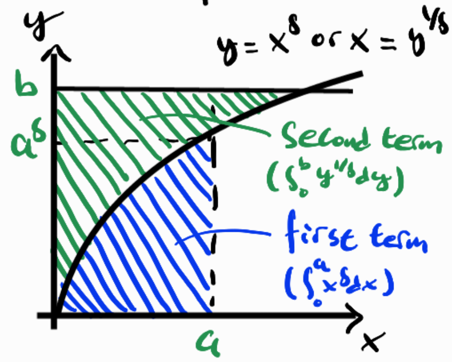
Lemma 2.6.9. *Let $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$. Then for any $a, b \geq 0$ one has*

$$ab \leq \alpha a^{\frac{1}{\alpha}} + \beta b^{\frac{1}{\beta}}.$$

Proof :

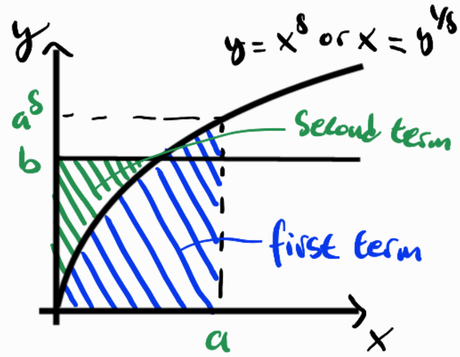
Let $\delta > 0$ and $a, b \geq 0$. Consider the graph of $y = x^\delta$ for $x > 0$. We want to examine bounded area in the graph.



Case 1 : $a^\delta \neq b$ ($a^\delta > b$ or $a^\delta < b$)

In this case, the area of rectangle formed by the axes, $x = a$, and $y = b$ is smaller than the area bounded by x-axis, $x = a$, and $y = x^\delta$ curve plus the area bounded by y-axis, $y = b$, and $y = x^\delta$ curve. Hence, we get

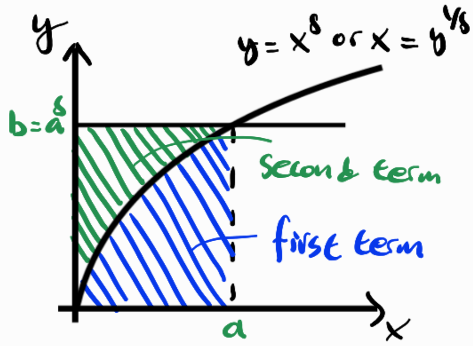
$$ab < \int_0^a x^\delta \, dx + \int_0^b y^{1/\delta} \, dy$$



Case 2 : $a^\delta = b$

In this case, the area of rectangle formed by the axes, $x = a$, and $y = b$ is equal to the area bounded by x-axis, $x = a$, and $y = x^\delta$ curve plus the area bounded by y-axis, $y = b$, and $y = x^\delta$ curve. Hence, we obtain

$$ab = \int_0^a x^\delta \, dx + \int_0^b y^{1/\delta} \, dy$$



From these cases, we get

$$ab \leq \int_0^a x^\delta \, dx + \int_0^b y^{1/\delta} \, dy$$
$$= \frac{1}{\delta+1} a^{\delta+1} + \frac{\delta}{\delta+1} b^{\frac{\delta+1}{\delta}}$$

Set $\frac{1}{\delta+1} = \alpha$ and $\frac{\delta}{\delta+1} = \beta$, then

$$ab \leq \alpha a^{1/\alpha} + \beta b^{1/\beta}$$

$\square$

Theorem 2.6.10 (Hölder inequality)**.** *Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, and consider $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then the product fg belongs to $L^1(\Omega)$ and the following inequality holds*

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q. \tag{2.6.1}$$

Proof:

Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and suppose $f \in L^p(\Omega)$ and $g \in L^p(\Omega)$.

Consider the special case where $\|f\|_p = 1$ and $\|g\|_q = 1$. Then, by using Lemma 2.6.9 with $\alpha = \frac{1}{p}$ and $\beta = \frac{1}{q}$, we obtain

$$\begin{aligned}
\|fg\|_1 &= \int_\Omega |f(x)g(x)|\,dx \\
&\leq \int_\Omega \left(\frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q\right)dx \\
&= \frac{1}{p}\int_\Omega |f(x)|^p\,dx + \frac{1}{q}\int_\Omega |g(x)|^q\,dx \\
&= \frac{1}{p}(\|f\|_p)^p + \frac{1}{q}(\|g\|_q)^q \qquad \text{by assumption} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1
\end{aligned}$$

$\Longrightarrow \quad \|fg\|_1 \leq 1$ for $\|f\|_p = 1$ and $\|g\|_q = 1$.

Then, let us consider $\tilde{f}(x) = \dfrac{f(x)}{\|f\|_p}$ and $\tilde{g}(x) = \dfrac{g(x)}{\|g\|_q}$ with $\|\tilde{f}\|_p = 1$ and $\|\tilde{g}\|_q = 1$. By applying the previous result on $\tilde{f}$ and $\tilde{g}$, we get

$$\|\tilde{f}\tilde{g}\|_1 \leq 1 \iff \frac{\|fg\|_1}{\|f\|_p\|g\|_q} \leq 1$$

$$\iff \|fg\|_1 \leq \|f\|_p \|g\|_q$$

Therefore, the Hölder inequality holds. $\square$

Theorem 2.6.11 (Minkowski's inequality). *For $p \ge 1$ and for any $f, g \in L^p(\Omega)$ one has*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Proof:

Let $f, g \in L^p(\Omega)$.

For $p=1$, by triangle inequality, we get

$$\begin{aligned} \|f+g\|_1 &= \int_\Omega |f(x)+g(x)|\,dx \\ &\le \int_\Omega |f(x)|\,dx + \int_\Omega |g(x)|\,dx \\ &= \|f\|_1 + \|g\|_1. \end{aligned}$$

Hence, Minkowski's inequality holds for $p=1$.

For $p>1$, set $q = \frac{p}{p-1}$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let us consider

$$\begin{aligned} \left(\|f+g\|_p\right)^{p-1} &= \left(\int_\Omega |f(x)+g(x)|^p\,dx\right)^{\frac{p-1}{p}} \\ &= \left(\int_\Omega \left(|f(x)+g(x)|^{p-1}\right)^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}} \\ &= \left(\int_\Omega \left(|f(x)+g(x)|^{p-1}\right)^{q}dx\right)^{1/q} \\ &= \|(|f+g|^{p-1})\|_q. \end{aligned}$$

Therefore, $\left(\|f+g\|_p\right)^{p-1} = \|(|f+g|^{p-1})\|_q$. ①

Also consider that

$$\begin{aligned} |f(x)+g(x)|^p &= |f(x)+g(x)|\,|f(x)+g(x)|^{p-1} \\ &\le (|f(x)|+|g(x)|)\,|f(x)+g(x)|^{p-1} \\ &= |f(x)|\,|f(x)+g(x)|^{p-1} + |g(x)|\,|f(x)+g(x)|^{p-1}. \end{aligned}$$ ②

Then, by using two results above and Hölder inequality, we obtain.

$$
\begin{aligned}
(\|f+g\|_p)^p &= \int_\Omega |f(x)+g(x)|^p \, dx \\
&\leqslant \int_\Omega |f(x)||f(x)+g(x)|^{p-1} dx + \int_\Omega |g(x)||f(x)+g(x)|^{p-1} dx \quad ②\\
&= \|f(|f+g|^{p-1})\|_1 + \|g(|f+g|^{p-1})\|_1 \\
&\leqslant \|f\|_p \|(|f+g|)^{p-1}\|_q + \|g\|_p \|(|f+g|)^{p-1}\|_q \quad \text{Hölder inequality} \\
&= \|f\|_p (\|f+g\|_p)^{p-1} + \|g\|_p (\|f+g\|_p)^{p-1}. \quad ①
\end{aligned}
$$

Dividing both sides by $(\|f+g\|_p)^{p-1}$, we obtain

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

For $p=\infty$, since $\|f\|_\infty = \inf\{M \,|\, |f| \leq M \text{ a.e}\}$, then $|f| \leq \|f\|_\infty$ a.e and $|g| \leq \|g\|_\infty$ a.e by definition. Hence we get

$$|f+g| \leq |f| + |g| \leq \|f\|_p + \|g\|_p \quad \text{a.e},$$

Which is equivalent statement with $\|f\|_\infty + \|g\|_\infty \in \{M \,|\, |f+g| \leq M \text{ a.e}\}$.
Since $\|f+g\|_\infty = \inf\{M \,|\, |f+g| \leq M \text{ a.e}\}$, then by the definition of infimum, we obtain

$$|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty .$$

Therefore, the Minkowski's inequality holds for $p \in [1, \infty]$. $\square$