Exercise 2.2.4

Show that $m^*(I) = v(I)$ for any closed box I

Proof Plan

1) show $m^*(I) \leq v(I)$
2) show $v(I) \leq m^*(I)$

3) from 1) and 2) $m^*(I) = v(I)$ will be proved

1)
From the definition 2.2.3 $m^*(\Omega)$ is defined by
$m^*(\Omega)$: $\inf\{\sigma(S) \mid S \text{ covering of } \Omega\}$.

And $\sigma(S)$ is defined in the definition 2.2.2 by
$\sigma(S) := \sum_j v(I_j) \in (0, \infty]$.

Then, now, $S = \{I\}$. This is a perfectly acceptable covering of I by closed intervals.

This fact gurantees that $m^*(I) \leq v(I)$

2)
Next, I need to show that $v(I) \leq m^*(I)$

to show $v(I) = m^*(I)$

To prove $v(I) \leq m^*(I)$, I need to prove that if $S = \{I_k\}$ is a countably infinite covering of I by closed intervals, then $v(I) \leq \sigma(S)$

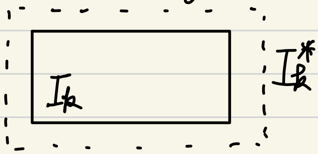
Let us set I_k^* and $\varepsilon > 0$ meeting

$$I_k \subseteq \operatorname{int}(I_k^*)$$

and

$$v(I_k^*) \leq (1+\varepsilon)\, v(I_k)$$

This inequation image



Then $\{\operatorname{int}(I_k^*)\}$ is an open cover of I.

That is

$$I \subseteq \bigcup_{k=1}^{\infty} \operatorname{int}(I_k^*)$$

Since I is closed and bounded, I can be covered by,

$$I \subseteq \bigcup_{k=1}^{M} \operatorname{int}(I_k^*) \subseteq \bigcup_{k=1}^{M} I_k^*$$

This means that $S' = \{I_k^*\}_{k=1}^{M}$ is a covering of I by a finite numbers of closed intervals.

Therefore, $v(I) \leq m^*(I)$ holds.

3) In conclusion, only the condition both condition 1) and 2) hold is when $m^*(I) = v(I)$ holds.

For the reason,

$m^*(I) = v(I)$ for any closed box.