Exercise 3.1.4

Show these 4 inequalities

1. $|\langle f,g\rangle| \leq \|f\|\|g\|$
2. $\|f+g\| \leq \|f\|+\|g\|$
3. $\|f+g\|^2 \leq 2\|f\|^2+2\|g\|^2$
4. $|\|f\|-\|g\|| \leq \|f\|-\|g\|$

($\langle \cdot,\cdot\rangle$ means a scalar product
$\|\cdot\|$ means a corresponding norm)

Proof 1

If $f=g$, $\langle f.g\rangle = \langle f.f\rangle = \|f\|^2$

If $f\neq g$, assume for example that $g\neq 0$ Then $\forall \alpha \in \mathbb{C}$

$$
\begin{aligned}
0 \leq \|f+\alpha g\|^2 &= \langle f+\alpha g, f+\alpha g\rangle \\
&= \langle f+\alpha g. f\rangle + \alpha \langle f+\alpha g, g\rangle \\
&= \langle f.f\rangle + \bar{\alpha}\langle g.f\rangle + \alpha\langle f.g\rangle + \alpha\bar{\alpha}\langle g.g\rangle \\
&= \|f\|^2 + \alpha\langle f.g\rangle + \bar{\alpha}\langle g.f\rangle + \alpha\bar{\alpha}\|g\|^2
\end{aligned}
$$

Then, I take

$$\alpha = \frac{-\overline{\langle f.g\rangle}}{\|g\|^2}$$

$$0 \leq \|f\|^2 - \frac{1}{\|g\|^2}|\langle f.g\rangle|^2$$

$$\Leftrightarrow \quad |\langle f.g\rangle|^2 \leq \|f\|^2\|g\|^2$$

for both $|\langle f.g\rangle| \geq 0$, $\|f\|\|g\| \geq 0$

$$|\langle f.g\rangle| \leq \|f\|\|g\|$$

Proof 2.

$\|f+g\| \leq \|f\|+\|g\|$

$\|f+g\|^2 = \|f\|^2 + \langle f.g \rangle + \langle g.f \rangle + \|g\|^2$

$(\|f\|+\|g\|)^2 = \|f\|^2 + 2\|f\|\|g\| + \|g\|^2$

from the $\boxed{\text{proof 1}}$ I mentioned before,

$|\langle f.g \rangle| \leq \|f\|\|g\|$ and $|\langle g.f \rangle| \leq \|g\|\|f\|$ hold in Hilbert space.

Now, let's consider about

$$(\|f+g\|)^2 = \|f\|^2 + |\langle f.g \rangle| + |\langle g.f \rangle| + \|g\|^2$$
$$= (\|f\|+\|g\|)^2 - 2\|f\|\cdot\|g\| + |\langle f.g \rangle| + |\langle g.f \rangle|$$

from the proof 1 I can change the equation into

$(\|f+g\|)^2 \leq (\|f\|+\|g\|)^2$

It is because $|\langle f.g \rangle| - \|f\|\cdot\|g\| \leq 0$ and $|\langle g.f \rangle| - \|f\|\cdot\|g\| \leq 0$

All of $\|f+g\|$, $\|f\|$ and $\|g\|$ are norm, $\|f+g\|, \|f\|+\|g\| \geq 0$ hold.

Therefore,

$\|f+g\| \leq \|f\|+\|g\|$ //

Proof 3

$$\|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$$

From the definition, left side of the equation can change into this form.

$$2\|f\|^2 + 2\|g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right)$$

$$= 2\left(\|f+g\|^2 - \langle f.g\rangle - \langle g f\rangle\right)$$

Then, $2\left(\|f+g\|^2 - \langle f.g\rangle - \langle g.f\rangle\right) - \|f+g\|^2$

$$= \|f+g\|^2 - 2\langle f.g\rangle - 2\langle g.f\rangle$$

$$= \|f\|^2 + \langle f.g\rangle + \langle g.f\rangle + \|g\|^2 - 2\langle f.g\rangle - 2\langle g.f\rangle$$

$$= \|f\|^2 - \langle f.g\rangle - \langle g.f\rangle + \|g\|^2$$

$$= \|f-g\|^2 \geq 0$$

If $f = g$, $\|f-g\| = 0$

Else, $\|f-g\| > 0$

For these reasons,

$\|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$ is proved

//

Proof 4

$$\big|\|f\|-\|g\|\big| \le \|f-g\|$$

(i) If $\|f\|=\|g\|$, the inequation above becomes an equation.

(ii) If $\|f\|>\|g\|$,

$$\begin{aligned}
\big|\|f\|-\|g\|\big| &= \|f\|-\|g\| \\
&= \|f-g+g\| - \|g\| \\
&\le \|f-g\|+\|g\|-\|g\| \\
&= \underline{\|f-g\|}
\end{aligned}$$

(iii) else if $\|g\|>\|f\|$,

$$\begin{aligned}
\big|\|f\|-\|g\|\big| &= \|g\|-\|f\| \\
&= \|g-f+f\| - \|f\| \\
&\le \|g-f\|+\|f\|-\|f\| \\
&= \|g-f\| \\
&= \|f-g\|
\end{aligned}$$

from (i) to (iii)

the inequation $\big|\|f\|-\|g\|\big| \le \|f-g\|$ always holds in Hilbert space