Introduction to Functional Analysis - Proofs of Some Relations for Orthogonal Projections

Zachary Kokot

June 2023

Some of the following proofs have been inspired by the methods laid out in the book *Quantum mechanics in Hilbert space*[1]. Below are the relations we wish to prove:

Let $\mathcal{H}$ be an arbitrary Hilbert Space and $\mathcal{M}, \mathcal{N}$ be closed sub-spaces of $\mathcal{H}$. Then denote the corresponding projections onto these sub-spaces as, $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$, respectively.

1. If $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}}$, then $P_{\mathcal{M}}P_{\mathcal{N}}$ is a projection and the associated closed sub-space is $\mathcal{M} \cap \mathcal{N}$,
2. If $\mathcal{M} \subset \mathcal{N}$, then $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}$,
3. If $\mathcal{M} \perp \mathcal{N}$, then $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = \mathbf{0}$, and $P_{\mathcal{M}\oplus\mathcal{N}} = P_{\mathcal{M}} + P_{\mathcal{N}}$,
4. If $P_{\mathcal{M}}P_{\mathcal{N}} = \mathbf{0}$, then $\mathcal{M} \perp \mathcal{N}$.

Proof of 1

Theorem: If $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}}$, then $P_{\mathcal{M}}P_{\mathcal{N}}$ is a projection and the associated closed sub-space is $\mathcal{M} \cap \mathcal{N}$.

Proof: In order for $P_{\mathcal{M}}P_{\mathcal{N}}$ to be a projection it must satisfy $P_{\mathcal{M}}P_{\mathcal{N}} = (P_{\mathcal{M}}P_{\mathcal{N}})^2 = (P_{\mathcal{M}}P_{\mathcal{N}})^*$.
Assuming $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}}$, we start by considering,

$$
\begin{aligned}
(P_{\mathcal{M}}P_{\mathcal{N}})^2 &= P_{\mathcal{M}}P_{\mathcal{N}}P_{\mathcal{M}}P_{\mathcal{N}} && (1)\\
&= P_{\mathcal{M}}P_{\mathcal{M}}P_{\mathcal{N}}P_{\mathcal{N}} && (2)\\
&= P_{\mathcal{M}}^2 P_{\mathcal{N}}^2 && (3)\\
&= P_{\mathcal{M}}P_{\mathcal{N}} && \text{By our initial assumption.} \quad (4)
\end{aligned}
$$

Next consider,

$$(P_{\mathcal{M}}P_{\mathcal{N}})^* = P_{\mathcal{N}}^* P_{\mathcal{M}}^*. \quad (5)$$

Which we know from the fact that for two bounded linear operators, $A, B \in \mathcal{B}(\mathcal{H})$, $(AB)^* = B^*A^*$.
Furthermore, as both $P_{\mathcal{N}}, P_{\mathcal{M}}$ are projections we have that $P_{\mathcal{N}} = P_{\mathcal{N}}^*, P_{\mathcal{M}} = P_{\mathcal{M}}^*$.
Therefore,

$$
\begin{aligned}
(P_{\mathcal{M}}P_{\mathcal{N}})^* &= P_{\mathcal{N}}P_{\mathcal{M}} && (6)\\
&= P_{\mathcal{M}}P_{\mathcal{N}}, && \text{By our initial assumption.} \quad (7)
\end{aligned}
$$

Hence, $P_{\mathcal{M}}P_{\mathcal{N}}$ is a projection.
Then, for an arbitrary element of Hilbert space, $f \in \mathcal{H}$, we can decompose f into $f = f_1 + f_2$ where $f_1 \in \mathcal{M}$ and $f_2 \in \mathcal{M}^{\perp}$ such that $P_{\mathcal{M}}f = f_1$. f_1 can be further decomposed into $f_1 = f_{11} + f_{12}$ where $f_{11} \in \mathcal{N}$ and $f_{12} \in \mathcal{N}^{\perp}$ such that $P_{\mathcal{N}}P_{\mathcal{M}}f = P_{\mathcal{N}}f_1 = f_{11}$. Therefore, after applying the projection $P_{\mathcal{M}}P_{\mathcal{N}}$ to an arbitrary element f in Hilbert space we obtain the element f_{11} such that $f_{11} \in \mathcal{M}$ and $f_{11} \in \mathcal{N}$. In other words the closed sub-space associated with the projection $P_{\mathcal{M}}P_{\mathcal{N}}$ is $\mathcal{M} \cap \mathcal{N}$.

■

Proof of 2

Theorem: If $\mathcal{M} \subset \mathcal{N}$, then $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}$.

Proof: Consider $P_{\mathcal{M}}f$ for any $f \in \mathcal{H}$ and observe that by assuming $\mathcal{M} \subset \mathcal{N}$ we can deduce that $P_{\mathcal{M}}f \in \mathcal{N}$. Furthermore, an equivalent definition for the sub-space $\mathcal{N}$ is the set of all $g \in \mathcal{H}$ such that $P_{\mathcal{N}}g = g$. Using these facts we obtain that $P_{\mathcal{N}}(P_{\mathcal{M}}f) = P_{\mathcal{M}}f$ for all $f \in \mathcal{H}$. Therefore, $P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}$, where $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are projections so,

$$\begin{aligned} P_{\mathcal{N}}P_{\mathcal{M}} &= P_{\mathcal{M}} && (8)\\ &= P_{\mathcal{M}}^* && (9)\\ &= (P_{\mathcal{N}}P_{\mathcal{M}})^* && (10)\\ &= P_{\mathcal{M}}^*P_{\mathcal{N}}^* && (11)\\ &= P_{\mathcal{M}}P_{\mathcal{N}}. && (12) \end{aligned}$$

Hence, $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}$.

■

Proof of 3

Theorem: If $\mathcal{M} \perp \mathcal{N}$, then $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = \mathbf{0}$, and $P_{\mathcal{M}\oplus\mathcal{N}} = P_{\mathcal{M}} + P_{\mathcal{N}}$.

Proof: $P_{\mathcal{M}}, P_{\mathcal{N}}$ are projections so $P_{\mathcal{M}} = P_{\mathcal{M}}^*, P_{\mathcal{N}} = P_{\mathcal{N}}^*$. Assuming $\mathcal{M} \perp \mathcal{N}$, consider $f, g \in \mathcal{H}$ such that,

$$\langle P_{\mathcal{M}}f, P_{\mathcal{N}}g\rangle = \langle P_{\mathcal{N}}P_{\mathcal{M}}f, g\rangle = \langle f, P_{\mathcal{M}}P_{\mathcal{N}}g\rangle \qquad \forall f, g \in \mathcal{H}. \quad (13)$$

From out initial assumption that $\mathcal{M} \perp \mathcal{N}$ and the fact $(P_{\mathcal{M}}f) \in \mathcal{M}$, $(P_{\mathcal{N}}g) \in \mathcal{N}$. We can deduce that the LHS of the above equality must be 0 as the inner product between elements of orthogonal sub-spaces is 0, by definition of the inner product. Therefore,

$$\langle P_{\mathcal{N}}P_{\mathcal{M}}f, g\rangle = \langle f, P_{\mathcal{M}}P_{\mathcal{N}}g\rangle = 0 \qquad \forall f, g \in \mathcal{H}, \quad (14)$$

which is only true for all f, g if $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = \mathbf{0}$.

Next, define $P_{\mathcal{L}} := P_{\mathcal{M}} + P_{\mathcal{N}}$, where if $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = \mathbf{0}$, we observe that

$$\begin{aligned} P_{\mathcal{L}}^2 &= (P_{\mathcal{M}} + P_{\mathcal{N}})^2 && (15)\\ &= P_{\mathcal{M}}^2 + P_{\mathcal{M}}P_{\mathcal{N}} + P_{\mathcal{M}}P_{\mathcal{N}} + P_{\mathcal{N}}^2 && (16)\\ &= P_{\mathcal{M}} + \mathbf{0} + \mathbf{0} + P_{\mathcal{N}} && (17)\\ &= P_{\mathcal{M}} + P_{\mathcal{N}} && (18)\\ &= P_{\mathcal{L}}. && (19) \end{aligned}$$

Also, in this case, it is easy to see that $P_{\mathcal{L}} = P_{\mathcal{L}}^*$ as the adjoint distributes over addition and the components of $P_{\mathcal{L}}$ are projections. Therefore, $P_{\mathcal{L}} := P_{\mathcal{M}} + P_{\mathcal{N}}$ is a projection, in the case of $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = \mathbf{0}$. Thus, for an arbitrary $f \in \mathcal{L}$ we observe that,

$$f = P_{\mathcal{L}}f = (P_{\mathcal{M}}f + P_{\mathcal{N}}f) \in \mathcal{M} \oplus \mathcal{N}, \quad (20)$$

which shows that $\mathcal{L} \subset \mathcal{M} \oplus \mathcal{N}$.

Then given an arbitrary $g \in \mathcal{M} \oplus \mathcal{N}$ observer that,

$$P_{\mathcal{L}}g = P_{\mathcal{M}}g + P_{\mathcal{N}}g, \quad (21)$$

where by our initial assumption, $\mathcal{M} \perp \mathcal{N}$, g can be decomposed into $g = g_1 + g_2$ where $g_1 \in \mathcal{M}$ and $g_2 \in \mathcal{N}$. Therefore,

$$\begin{aligned} P_{\mathcal{M}}g + P_{\mathcal{N}}g &= P_{\mathcal{M}}(g_1 + g_2) + P_{\mathcal{N}}(g_1 + g_2) && (22)\\ &= P_{\mathcal{M}}g_1 + P_{\mathcal{N}}g_1 + P_{\mathcal{M}}g_2 + P_{\mathcal{N}}g_2 && (23)\\ &= P_{\mathcal{M}}g_1 + \mathbf{0} + \mathbf{0} + P_{\mathcal{N}}g_2 && (24)\\ &= P_{\mathcal{M}}g_1 + P_{\mathcal{N}}g_2 && (25)\\ &= g_1 + g_2 && (26)\\ &= g. && (27) \end{aligned}$$

Therefore, $g \in \mathcal{L}$, proving that $\mathcal{L} \supset \mathcal{M} \oplus \mathcal{N}$. Hence, $\mathcal{L} = \mathcal{M} \oplus \mathcal{N}$ and therefore $P_{\mathcal{M}\oplus\mathcal{N}} = P_{\mathcal{M}} + P_{\mathcal{N}}$.

■

Proof of 4

Theorem: If $P_{\mathcal{M}}P_{\mathcal{N}} = \mathbf{0}$, then $\mathcal{M} \perp \mathcal{N}$.

Proof: Assuming $P_{\mathcal{M}}P_{\mathcal{N}} = \mathbf{0}$, start by considering an arbitrary pair f, g satisfying $f \in \mathcal{M}$ and $g \in \mathcal{N}$ such that $P_{\mathcal{M}}f = f$ and $P_{\mathcal{N}}g = g$. $P_{\mathcal{M}}, P_{\mathcal{N}}$ are projections so $P_{\mathcal{M}} = P^*_{\mathcal{M}}$, $P_{\mathcal{N}} = P^*_{\mathcal{N}}$ and it follows that,

$$\begin{aligned} \langle P_{\mathcal{M}}f, P_{\mathcal{N}}g\rangle &= \langle f, P_{\mathcal{M}}P_{\mathcal{N}}g\rangle && (28)\\ &= \langle f, \mathbf{0}g\rangle && (29)\\ &= \langle f, \mathbf{0}\rangle && (30)\\ &= 0. && (31)\end{aligned}$$

Therefore, $\langle P_{\mathcal{M}}f, P_{\mathcal{N}}g\rangle = \langle f, g\rangle = 0$ and $f \perp g$. With f and g arbitrary elements of the closed sub-spaces $\mathcal{M}$ and $\mathcal{N}$, respectively, we can conclude that $\mathcal{M} \perp \mathcal{N}$.

■

References

1. E. Prugovečki, *Quantum mechanics in Hilbert space*, eng (Acad. Press, New York, 1971), ISBN: 9780125660501.