Student number : 082250538
Name : YOSHIDA Koki

1. $|\langle f,g\rangle| \le \|f\|\,\|g\|$

Define $f_\perp$ which meets following two equalities.

$$\begin{cases} \cdot \langle g, f_\perp\rangle = 0 \\ \cdot f_\perp = f - \lambda g \end{cases}$$

$$\langle g, f_\perp\rangle = \langle g, f-\lambda g\rangle$$
$$= \langle g,f\rangle - \lambda\langle g,g\rangle$$
$$= \langle g,f\rangle - \lambda\|g\|^2 = 0.$$
$$\therefore \lambda = \frac{\langle g,f\rangle}{\|g\|^2}$$

$\|f-\lambda g\|^2$ is positive, so

$$0 \le \|f-\lambda g\|^2$$
$$= \left\langle f - \frac{\langle g,f\rangle}{\|g\|^2} g,\ f - \frac{\langle g,f\rangle}{\|g\|^2} g\right\rangle$$
$$= \left\langle f - \frac{\langle g,f\rangle}{\|g\|^2} g,\ f\right\rangle - \frac{\langle g,f\rangle}{\|g\|^2}\left\langle f - \frac{\langle g,f\rangle}{\|g\|^2} g,\ g\right\rangle$$
$$= \langle f,f\rangle - \frac{|\langle g,f\rangle|^2}{\|g\|^2} - \frac{|\langle g,f\rangle|^2}{\|g\|^2} + \frac{|\langle g,f\rangle|^2}{\|g\|^2}$$
$$= \|f\|^2 - \frac{|\langle g,f\rangle|^2}{\|g\|^2}$$

$$\therefore |\langle g,f\rangle|^2 \le \|f\|^2\|g\|^2$$

$$|\langle g,f\rangle|^2 = \langle g,f\rangle\overline{\langle g,f\rangle}$$
$$= \overline{\langle f,g\rangle}\langle f,g\rangle$$
$$= |\langle f,g\rangle|^2$$

$$\therefore |\langle f,g\rangle|^2 \le \|f\|^2\|g\|^2$$

Both $|\langle f,g\rangle|$ and $\|f\|\|g\|$ are positive, so $|\langle f,g\rangle| \le \|f\|\|g\|$ □

2. $\|f+g\| \le \|f\| + \|g\|$

$$(\|f+g\|)^2 = \langle f+g, f+g\rangle$$
$$= \langle f+g, f\rangle + \langle f+g, g\rangle$$
$$= \overline{\langle f, f+g\rangle} + \overline{\langle g, f+g\rangle}.$$
$$= \overline{\langle f,f\rangle} + \overline{\langle f,g\rangle} + \overline{\langle g,f\rangle} + \overline{\langle g,g\rangle}$$
$$= \langle f,f\rangle + \langle g,g\rangle + \langle f,g\rangle + \overline{\langle f,g\rangle}$$
$$= \langle f,f\rangle + \langle g,g\rangle + 2\mathrm{Re}\langle f,g\rangle$$

$$(\|f\|+\|g\|)^2 = \langle f,f\rangle + \langle g,g\rangle + 2\|f\|\|g\|$$

Here, $\mathrm{Re}\langle f,g\rangle \le |\langle f,g\rangle|$ ($\because$ For any $z = x+yi$ $(z\in\mathbb{C}, x\in\mathbb{R}, y\in\mathbb{R})$, $\mathrm{Re}\,z \le |z|$)

so $\mathrm{Re}\langle f,g\rangle \le |\langle f,g\rangle| \le \|f\|\|g\|$ (inequality 1)

Therefore $(\|f+g\|)^2 \le (\|f\|+\|g\|)^2$ $\quad\therefore \|f+g\| \le \|f\| + \|g\|$ (Both $\|f+g\|$ and $\|f\|+\|g\|$ are positive)

3. $\|f+g\|^2 \le 2\|f\|^2 + 2\|g\|^2$

$$2\|f\|^2 + 2\|g\|^2 - \|f+g\|^2 \ge 2\|f\|^2 + 2\|g\|^2 - (\|f\| + \|g\|)^2 \quad (\text{Inequality } 2)$$
$$= \|f\|^2 + \|g\|^2 - 2\|f\|\|g\|$$
$$= (\|f\| - \|g\|)^2$$
$$\ge 0$$

$$\therefore \|f+g\|^2 \le 2\|f\|^2 + 2\|g\|^2$$

4. $\big|\|f\| - \|g\|\big| \le \|f-g\|$

$$\big|\|f\| - \|g\|\big|^2 = \|f\|^2 - 2\|f\|\|g\| + \|g\|^2$$
$$= \langle f,f\rangle - 2\|f\|\|g\| + \langle g,g\rangle$$

$$\|f-g\|^2 = \langle f-g, f-g\rangle$$
$$= \langle f-g, f\rangle - \langle f-g, g\rangle$$
$$= \overline{\langle f, f-g\rangle} - \overline{\langle g, f-g\rangle}$$
$$= \overline{\langle f,f\rangle} - \overline{\langle f,g\rangle} - \overline{\langle g,f\rangle} + \overline{\langle g,g\rangle}$$
$$= \langle f,f\rangle - 2\mathrm{Re}\langle f,g\rangle + \langle g,g\rangle$$

$\mathrm{Re}\langle f,g\rangle \le |\langle f,g\rangle|$, so $-2\mathrm{Re}\langle f,g\rangle \ge -2|\langle f,g\rangle| \ge -2\|f\|\|g\|$

$$\therefore \big|\|f\| - \|g\|\big|^2 \le \|f-g\|^2$$

$$\therefore \big|\|f\| - \|g\|\big| \le \|f-g\|$$ (Both $\big|\|f\| - \|g\|\big|$ and $\|f-g\|$ are positive)