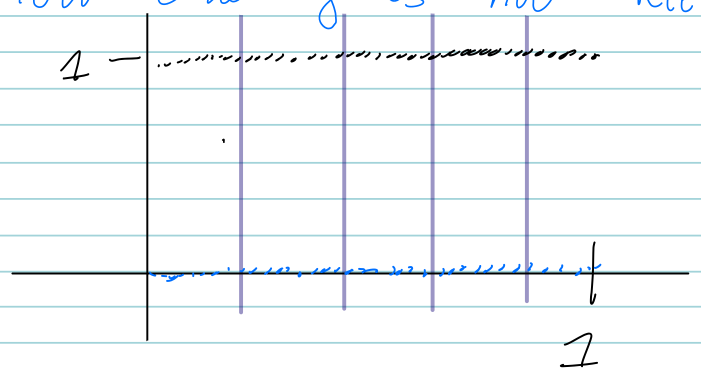
Alberto-John Thornton and Katayama Marin

Ex. 2.1.2

$$f:[0,1]\to\mathbb{R},\quad f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$

Show that f is not Riemann integrable



- $L(f,P)=\sum_{j=1}^{n}\left(\inf_{x\in[x_{j-1},x_j]} f(x)\right)(x_j-x_{j-1})$

for any partition P $\exists x\in(x_j,x_{j-1})$ s.t $x\in\mathbb{R}\setminus\mathbb{Q}\Rightarrow \inf_{x\in[x_{j-1},x_j]} f(x)=0$

$$L(f,P)=\sum_{j=1}^{n}(0)(x_j-x_{j-1})=0 \Rightarrow \sup_P L(f,P)=0$$

- $U(f,P)=\sum_{j=1}^{n}\left(\sup_{x\in[x_{j-1},x_j]} f(x)\right)(x_j-x_{j-1})$

for any partition P $\exists x\in(x_j,x_{j-1})$ s.t $x\in\mathbb{Q}\Rightarrow \sup_{x\in[x_{j-1},x_j]} f(x)=1$

We recall that Q is dense in R

$$U(f,P)=\sum_{j=1}^{n}(1)(x_j-x_{j-1})=x_n-x_0=1 \Rightarrow \inf_P U(f,P)=1$$

$\sup L(f,P)\neq\inf U(f,P)$ so f is not Riemann integrable.