Example 3.1.9 (2) $\overline{\mathcal{D}(\mathbb{R}^n)} = \overline{\mathcal{S}(\mathbb{R}^n)} = \mathcal{L}^2(\mathbb{R}^n)$.

Proof: We would claim 2 porperties.

① $\mathcal{L}^2(\mathbb{R}^n)$ is closed.

② $\forall f \in \mathcal{L}^2(\mathbb{R}^n)$, $\exists \{d_i\}_{i=1}^{+\infty} \subset \mathcal{D}(\mathbb{R}^n)$, $d_i \longrightarrow f$.

Then we would have

$$\overline{\mathcal{D}(\mathbb{R}^n)} = \overline{\mathcal{L}^2(\mathbb{R}^n)} = \mathcal{L}^2(\mathbb{R}^n)$$

And since $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{L}^2(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n) \supset \mathcal{D}(\mathbb{R}^n)$,

we are done.

proof of claim ①:

consider $\{f_j\}_{j=1}^{+\infty} \subset \mathcal{L}^2(\mathbb{R}^n)$, $f_j \to f$.

So for $\varepsilon = 1$, $\exists\ j_0$ s.t.

$\|f - f_{j_0}\| < 1$.

So
$$\|f - f_{j_0} + f_{j_0}\|^2 = \|f - f_{j_0}\|^2 + \|f_{j_0}\|^2 + 2\langle f - f_{j_0}, f_{j_0}\rangle$$
$$\leq 2(\|f - f_{j_0}\|^2 + \|f_{j_0}\|^2)$$
$$\leq 2 + 2\|f_{j_0}\|^2$$
$$< +\infty$$

i.e. $\|f\|^2 < +\infty$.

So $f \in \mathcal{L}^2(\mathbb{R}^n)$. $\mathcal{L}^2(\mathbb{R}^n)$ is closed.

proof of claim ②:

$$\eta(x) := \begin{cases} C e^{\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0 , & |x| \geq 1 \end{cases}, \text{ where } C = 1 \Big/ \left(\int_{\mathbb{R}^n} e^{\frac{1}{1-|x|^2}} dx\right)$$

so that $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$.

$\eta_\varepsilon(x) := \eta(x/\varepsilon)/\varepsilon^n$.

$\cancel{f(x)}\ f \in \mathcal{L}^2(\mathbb{R}^n)$, we define $f^\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) f(y)\,dy$.

then we would prove (a) $\boxed{f^\varepsilon \in \mathcal{D}(\mathbb{R}^n)}$ and (b) $\boxed{f^\varepsilon \xrightarrow{\mathcal{L}^2} f}$

(a) We denote $U_f = \operatorname{supp} f$.

then ① $f^\varepsilon \in C^\infty(\mathbb{R}^n)$.

We only prove the case $\frac{\partial}{\partial x_i} f^\varepsilon$.

other multiindex α would be similar.

$$\frac{f^\varepsilon(x+he_i) - f^\varepsilon(x)}{h}$$

$$= \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \frac{1}{h} f(y) \left(\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right) dy$$

So let $g(x) = \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} \eta^\varepsilon(x-y)\right) f(y)\, dy$

then $\left| \frac{f^\varepsilon(x+he_i) - f^\varepsilon(x)}{h} - g(x) \right|$

$$\leq \underbrace{\left| \int_{\mathbb{R}^n - B_r(0)} f(y) \frac{1}{\varepsilon^n} \frac{\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)}{h} dy \right|}_{I_1}$$

$\mathbb{R}^n - B_r(0) = \{x \mid |x| > r\}$.

$$+ \underbrace{\left| \int_{\mathbb{R}^n - B_r(0)} \frac{\partial}{\partial x_i} \eta^\varepsilon(x-y) f(y)\, dy \right|}_{I_2}$$

$$+ \underbrace{\left| \int_{B_r(0)} f(y) \left(\frac{\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)}{\varepsilon^n \cdot h} - \frac{\partial}{\partial x_i} \eta^\varepsilon(x-y)\right) dy \right|}_{I_3}$$

$I_3 \to 0$ as $h \to 0$,

I_1, I_2 would be arbitrary small

if we choose r is large enough.

So $\frac{\partial}{\partial x_i} f^\varepsilon(x)$ exists, and it happens to be

$$\int_{\mathbb{R}^n} f(y) \frac{\partial}{\partial x_i} \eta^\varepsilon(x-y)\, dy.$$

So do other multiindex α.

In fact we have $\frac{\partial^\alpha}{\partial x^\alpha} f^\varepsilon(x) = \int_{\mathbb{R}^n} f(y) \frac{\partial^\alpha}{\partial x^\alpha} \eta^\varepsilon(x-y)\, dy$

② f^ε have compact support set.

Since if $d(x, U_f) > \varepsilon$, we must have

$\eta^{\varepsilon}(x-y) = 0$ for $\forall\, y \in U_f$.

So $f^{\varepsilon}(x) = \int_{U_f} + \int_{\mathbb{R}^n - U_f} f(y)\, \eta^{\varepsilon}(x-y)\, dy$

$$= \int_{U_f} f(y) \underbrace{\eta^{\varepsilon}(x-y)}_{=0} dy + \int_{\mathbb{R}^n - U_f} \overset{\overset{0}{\shortparallel}}{\underline{f(y)}}\, \eta^{\varepsilon}(x-y)\, dy$$

$= 0$ if $d(x, U_f) > \varepsilon$.

So we know $f^{\varepsilon} \in \mathcal{D}(\mathbb{R}^n)$ by ①, ②.

(b) $f^{\varepsilon} \xrightarrow{L^2} f$.

We only need to prove

$f^{\varepsilon} \to f$ a.e., Since

$$\| f - f^{\varepsilon} \|^2 = \int_{\mathbb{R}^n} |f - f^{\varepsilon}|^2$$

$$= \underbrace{\int_{B_r(0)} |f - f^{\varepsilon}|^2}_{=J_1} + \underbrace{\int_{\mathbb{R}^n - B_r(0)} |f - f^{\varepsilon}|^2}_{J_2 =}.$$

we can choose large enough r

to make J_2 arbitrary small,

and if $f^{\varepsilon} \xrightarrow{a.e.} f$, we have

$J_1 \to 0$ as $\varepsilon \to 0$.

We claim $$\frac{\int_{B_r(x)} |f(y) - f(x)|\, dy}{|B_r(x)|} \xrightarrow{a.e.} 0 \quad \text{as } r \to 0$$

without proof. (which is called Lebesgue's Differentiation theorem).

So we have

$$\begin{aligned}
&|f^{\varepsilon}(x) - f(x)| \\
&= \Big| \int_{B_\varepsilon(x)} (\eta^{\varepsilon}(x-y) f(y) - \eta^{\varepsilon}(x-y) f(x))\, dy \Big| \\
&\le \int_{B_\varepsilon(x)} \eta^{\varepsilon}(x-y)\, |f(y) - f(x)|\, dy \\
&= \int_{B_\varepsilon(x)} \eta\left(\frac{x-y}{\varepsilon}\right) \cdot \frac{|f(y) - f(x)|}{\varepsilon^n}\, dy
\end{aligned}$$

we know $|B_\varepsilon(x)| \propto \varepsilon^n$,

i.e. $|B_\varepsilon(x)| = \alpha \varepsilon^n$, α is constant

$$\propto \frac{\int_{B_\varepsilon(x)} |f(y) - f(x)|\, dy}{|B_\varepsilon(x)|}$$

$\longrightarrow 0$ as $\varepsilon \to 0$.

So $f^{\varepsilon} \to f$ a.e.

So we are done by (a) and (b).