张嘉彬 Report 2.

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Exercise 1.5.1 Properties of Fourier transform.

For $f, g \in \mathcal{L}^1(\mathbb{R}^n)$, we have:

$\boxed{① \ \mathcal{F} \text{ is linear,}}$ since

$$[\mathcal{F}(f+\lambda g)](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} (f+\lambda g)(x)\, dx)$$

$$= \frac{1}{(2\pi)^{n/2}} \left(\int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx + \lambda \int_{\mathbb{R}^n} e^{-i\xi\cdot x} g(x)\, dx \right)$$

$$= ([\mathcal{F}f] + \lambda[\mathcal{F}g])(\xi) \quad \text{for } \forall \lambda \in \mathbb{R}.$$

② $$|\hat{f}(\xi)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |e^{-i\xi\cdot x}| \cdot |f(x)|\, dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_{\mathcal{L}^1} / (2\pi)^{n/2}$$

$\boxed{③ \ \hat{f}(\xi) \in C_0(\mathbb{R}^n),}$ as

(i) $\hat{f}(\xi)$ continuous. Since

$$|\hat{f}(x) - \hat{f}(y)| = \frac{1}{(2\pi)^{n/2}} \left| \int_{\mathbb{R}^n} (e^{-ix\cdot X} - e^{-iy\cdot X}) f(x)\, dx \right|$$

consider $d(t) = e^{-i(\cancel{ty}+(1-t)x)\cdot X}$

then $\exists\, \theta \in (0,1)$,

$$d'(\theta) = \frac{d(1)-d(0)}{1} = e^{-iy\cdot X} - e^{-ix\cdot X}$$

So $$e^{-ix\cdot X} - e^{-iy\cdot X} = -d'(\theta)$$

$$= +i\, e^{-i(\theta y + (1-\theta)x)\cdot X}\, X\cdot(x-y)$$

So $$|\hat{f}(x) - \hat{f}(y)| \leq \frac{1}{(2\pi)^{n/2}} \left(\int_{B_r(0)} |e^{-ix\cdot X} - e^{-iy\cdot X}| \cdot |f(x)|\, dx \right.$$

$$\left. \int_{\mathbb{R}^n - B_r(0)} (e^{-ix\cdot X} - e^{-iy\cdot X}) f(X)\, dX \right)$$

(Here $B_r(0) = \{x \in \mathbb{R}^n : \|x\| < r\}$. $\forall r \in \mathbb{R}_{>0}$)

$$\leq \frac{1}{(2\pi)^{n/2}} \int_{B_r(0)} |X\cdot(x-y)|\, |f(x)|\, dX + \frac{1}{(2\pi)^{n/2}} \cdot 2 \int_{\mathbb{R}^n - B_r(0)} |f(X)|\, dX$$

For any $\varepsilon > 0$, $\exists\, r$,

$$\left| \int_{B_r(0)} |f(x)|\, dx - \|f\|_1 \right| < \varepsilon \cdot \frac{(2\pi)^{n/2}}{2}$$

i.e. $$\frac{2}{(2\pi)^{n/2}} \cdot \int_{\mathbb{R}^n - B_r(0)} |f(x)|\, dx < \varepsilon .$$

let $y \to x$, the first term

$$\frac{1}{(2\pi)^{n/2}} \int_{B_r(0)} |x \cdot (x-y)|\, |f(x)|\, dx \to 0 .$$

So we have ~~$\hat{f}(x) - \hat{f}(y) \to 0$~~

$$|\hat{f}(x) - \hat{f}(y)| < 2\varepsilon \quad \text{if } |x-y| \text{ is small enough.}$$

ε is arbitrary. i.e. $\hat{f}(y) \to \hat{f}(x)$ as $y \to x$.

So $\hat{f}$ is continuous.

<u>(ii) $\hat{f}(\xi) \to 0$ as $\|\xi\| \to \infty$</u>

~~denote $g(x) = f(\frac{x}{t})$, $t \in \mathbb{R}^*$.~~

~~we have~~

$$\cancel{\hat{f}(t\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-it\xi \cdot x} f(x)\, dx}$$

$$\cancel{= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot (tx)} g(tx)\, \frac{d\,tx}{t^n}}$$

$$\cancel{= \frac{1}{t^n} \hat{g}(\xi)}.$$

~~Since~~ $$\cancel{|\hat{g}(\xi)| = \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} e^{-i\xi x} f(\tfrac{x}{t})\, dx \right|}$$

Sorry I made a mistake here.
I'm not sure how to show that.

④ $$[\mathcal{F}(f * g)](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x}\, dx \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \Big/ (2\pi)^{n/2}$$

$$\underline{\underline{\text{Fubini's thm}}} \quad \frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi y} f(y)\, dy \int_{\mathbb{R}^n} e^{-i\xi (x-y)} g(x-y)\, d(x-y)$$

$$= [\mathcal{F}(f)](\xi) \cdot [\mathcal{F}(g)](\xi).$$

⑤ $$[F(-i\partial_j f)](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}(-i\partial_j f(x))\, dx$$

$$= \underbrace{\text{boundary terms}}_{\substack{\shortparallel \\ 0 \\ \text{since } f\in t'(\mathbb{R}^n)}} + \frac{i}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f\cdot e^{-i\xi x}(-i\xi x)_j\, dx$$

$$= \int_{\mathbb{R}^n} f e^{-i\xi x}\left(\sum_{i=1}^{n}\xi_i x_i\right)_j dx$$

$$= \xi_j \hat{f}(\xi). \quad \square$$

Exercise 1.5.12.

Here $f \in \mathcal{S}(\mathbb{R}^n)$

① $F\delta_0 = T_{(2\pi)^{-n/2}}$.

First, δ_0 is a tempered distribution

since if $f_j \to f_\infty$ in $\mathcal{S}(\mathbb{R}^n)$,

then $\|f_j - f_\infty\|_\infty \to 0$

So $f_j(0) \to f_\infty(0)$, i.e. $\delta_0 f_j \to \delta_0 f_\infty$.

Then, $$F\delta_0(f) = \delta_0 F(f)$$

$$= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} e^{-i\xi\cdot x} f(x)\, dx \Big/_{\xi=0}$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\, dx = T_{\frac{1}{(2\pi)^{n/2}}}(f).$$

② $F T_1 = (2\pi)^{n/2}\delta_0$.

First, T_1 is a tempered distribution.

Since if $f_j \to f_\infty$ in $\mathcal{S}(\mathbb{R}^n)$,

then $$\int_{\mathbb{R}^n} |f_j - f_\infty|\, dx$$

$$\underset{B_r(0):=\{x:\|x\|<r\}}{=} \int_{B_r(0)} |f_j - f_\infty|\, dx + \int_{\mathbb{R}^n - B_r(0)} |f_j - f_\infty|\, dx$$

$$\leq \int_{B_r(0)} \|f_j - f_\infty\|_\infty\, dx + \|x_1^2 x_2^2 \cdots x_n^2 (f_j - f_\infty)\|_\infty \cdot \int_{\mathbb{R}^n - B_r(0)} \frac{dx_1\cdots dx_n}{|x_1^2\cdots x_n^2|}$$

$$\to 0 + 0 = 0 \quad \text{as } j \to \infty.$$

Then $\mathcal{F}(T_1)(f) = T_1 \mathcal{F}(f)$

$$= \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} e^{-i\xi\cdot x} f(x)\, dx$$

$$= \int_{\mathbb{R}^n} d\xi \quad \hat{f}(\xi)$$

$$= (2\pi)^{n/2} \int_{\mathbb{R}^n} e^{i\xi\cdot 0} \hat{f}(\xi)\, d\xi \Big/ (2\pi)^{n/2}$$

$$= (2\pi)^{n/2} \mathcal{F}^{-1} \circ \mathcal{F}(f)(0)$$

$$= (2\pi)^{n/2} \delta_0(f)$$

③ First, T_h is a tempered distribution for $h \in L^2(\mathbb{R}^n)$,

Since $\int_{\mathbb{R}^n} |h(f_j - f_\infty)|\, dx$

$$\leq \int_{B_r(0)} |h| \, \|f_j - f_\infty\|_\infty \, dx + \int_{\mathbb{R}^n \setminus B_r(0)} |h|\, |f_j - f_\infty|\, dx$$

$$\leq \int_{B_r(0)} |h| \, \|f_j - f_\infty\|_\infty \, dx \; \left(\int_{\mathbb{R}^n \setminus B_r(0)} |h|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^n \setminus B_r(0)} |f_j - f_\infty|^2 dx \right)^{1/2}$$

$|f_j - f_\infty|(x) \leq \|f_j - f_\infty\|_\infty$

so we choose j is large s.t. $\|f_j - f_\infty\|_\infty < 1$

$$\int_{B_r(0)} |h| \, \|f_j - f_\infty\|_\infty \, dx + \|h\|_2^{1/2} \underbrace{\left(\int_{\mathbb{R}^n - B_r(0)} |f_j - f_\infty| \right)^{1/2}}_{\to 0}$$

, which has been proved in ②.

$\longrightarrow 0$ as $j \to \infty$.

Exercise 2.3.4

We denote $A_s = \{x \mid f > s\}$. $B_s = \{x \mid f \geq s\}$,

$C_s = \{x \mid f < s\}$, $D_s = \{x \mid f \leq s\}$.

Notice that $B_s = \bigcap_{n=1}^{\infty} A_{s-1/n}$, $C_s = B_s^c$,

$D_s = \bigcap_{n=1}^{\infty} C_{s+1/n}$, $A_s = (D_s)^c$,

we are done. Since we have proved

$$\begin{array}{ccc} \forall s,\ A_s \text{ L.m.} & \Rightarrow & \forall s,\ B_s \text{ L.m.} \\ \Uparrow & & \Downarrow \\ \forall s,\ D_s \text{ L.m.} & \Leftarrow & \forall s,\ C_s \text{ L.m.} \end{array} \quad \boxtimes.$$

The proof of ~~statement~~ of lemma 2.4.4.

$f \in L^{\infty}([a,b])$, $f = 0$ a.e. then $\int_a^b f = 0$.

proof: $f = 0$ a.e.

So $\exists$ $E \subset [a,b]$ with $m(E) = 0$

s.t. $f|_{[a,b]-E} = 0$.

Now, $\forall s \geq 0$, $A := \{x : f > s\} \subset E$, so $m^*(A) = 0$

So $m(A) \overset{\exists}{=} 0$

$\forall s < 0$, $B := \{x : f > s\}$

$= (\{x : f \leq s\})^c \cap [a,b]$

Since $\{x : f \leq s\} \subset E$,

we know $\{x : f \leq s\}$ L.m.

so $\{x : f \leq s\}^c$ also L.m.

~~So~~

So f is L.m.

So f is Lebesgue integrable.

and $\int_{[a,b]} f = \underbrace{\int_{E \cap [a,b]} f}_{\substack{= 0 \text{ since} \\ m(E \cap [a,b]) = 0}} + \underbrace{\int_{[a,b]-E} f}_{=0} = 0$. $\boxtimes$.