张嘉彬 Report ~~HW~~ 1.

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Exercise 1.1.9

The linearity is obvious.

Just focus on the continuity. Suppose $f_j \to f$ as $j \to +\infty$.

① $$|T_h(f_j) - T_h(f)| = \left| \int_{\mathbb{R}^n} (h(x)(f_j - f)(x))\, dx \right|$$

$$\leq \int_{\mathbb{R}^n} |h|\,|f_j - f|\, dx$$

$$= \int_{B_r} |h|\,|f_j - f|\, dx$$

here we have $f_j = f = 0$ in B_r^c, $\forall x$.

$$\leq \|f_j - f\|_\infty \int_{B_r} |h|\, dx$$

$\to 0$ since $\int_{B_r} |h|\, dx$ is constant. as $j \to +\infty$

② $$|\delta_y(f_j) - \delta_y(f)| = |f_j(y) - f(y)|$$

$$\leq \|f_j - f\|_\infty \to 0$$

③ $$|\delta_y^\alpha(f_j) - \delta_y^\alpha(f)| = |(\partial^\alpha f_j - \partial^\alpha f)(y)|$$

$$\leq \|\partial^\alpha f_j - \partial^\alpha f\|_\infty \to 0$$

Exercise 1.1.11

They're $\underset{T_h}{0}$, $\underset{\delta_y}{0}$ and $\underset{\delta_y^\alpha}{|\alpha|}$ obviously by Exercise 1.1.9

Exercise 1.2.3

Since
$$\partial^\alpha T_h(f) = (-1)^{|\alpha|} T_h(\partial^\alpha f) = (-1)^{|\alpha|} \int_{\mathbb{R}} (\partial^\alpha f)\, h$$
$$= (-1)^{|\alpha|-1} \int_{\mathbb{R}} \partial^{(\alpha_1 - 1, \alpha_2, \dots, \alpha_n)} f \; \partial^{(1,0,\dots)} h$$
$$= \dots$$
$$= (-1)^{|\alpha| - |\alpha_1|} \int_{\mathbb{R}} \partial^{(0, \alpha_2, \dots, \alpha_n)} f \; \partial^{(\alpha_1, 0, \dots, 0)} h$$
$$= \dots$$
$$= \int_{\mathbb{R}} f\, \partial^\alpha h.$$
$$= T_{\partial^\alpha h}(f).$$

Exercise 1.2.5

$Pv\frac{1}{x}$ is a distribution as

1. $$Pv\frac{1}{x}(f_1 + \lambda f_2) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x}(f_1 + \lambda f_2)\,dx$$
$$= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x} f_1\,dx + \lambda \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x} f_2\,dx$$
$$= Pv\frac{1}{x}(f_1) + \lambda\, Pv\frac{1}{x}(f_2)$$

2. when
$$f_j \to f \quad \text{in} \quad D(\mathbb{R}^n) \quad \text{as} \quad j \to \infty$$

we have
$$\cancel{\lim_{j\to\infty}} \left| Pv\frac{1}{x}(f_j) - Pv\frac{1}{x}(f) \right|$$
$$= \left| \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x}(f_j - f) \right|$$
$$\cancel{\leq \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x}}$$
$$= \left| \int_{\mathbb{R}} \ln|x|\,(f_j' - f') \right|$$
$$= \left| \int_{\substack{C \\ \text{compact}}} \ln|x|\,(f_j' - f') \right|$$
$$\leq \underbrace{\left(\int_C |\ln|x|| \right)}_{<\infty} \|f_j' - f'\|_\infty$$
$$\to 0 \quad \text{as} \quad j \to +\infty.$$

and the $O(\varepsilon \ln |\varepsilon|)$ is in fact:

$$\int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \ln(|x|)' f(x)\, dx$$

$$= \ln |x| f(x) \Big|_{-\infty}^{-\varepsilon} + \ln |x| f(x) \Big|_{+\varepsilon}^{+\infty}$$

$$- \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \ln(|x|) f'(x)\, dx$$

So ~~they~~ it in fact is

$$f(-\varepsilon) \ln \varepsilon - f(\varepsilon) \ln \varepsilon$$

$$= \ln \varepsilon \left(f(-\varepsilon) - f(\varepsilon) \right)$$

$$= \ln \varepsilon \left(-2\varepsilon \cdot f'(0) + o(\varepsilon) \right)$$

$$= -2 f'(0)\, \varepsilon \ln \varepsilon + o(\varepsilon \ln \varepsilon).$$